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THE COLORED JONES POLYNOMIAL AND THE AJ CONJECTURE

by

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Abstract. – We present the basics of the colored Jones polynomial and discuss the AJ conjecture which relates the Jones polynomial and the A-polynomial of a knot.

***Résumé* (Le polynôme de Jones colorié et la conjecture AJ).** – Nous présentons le polynôme de Jones colorié et la conjecture AJ qui relie le polynôme de Jones et le A-polynôme d'un nœud.

1. Introduction

In this note we give a survey of the theory of the colored Jones polynomials and the AJ conjecture which relates the Jones polynomial and the A-polynomial.

The Jones polynomial was discovered in 1984. It came as a shocking surprise in low-dimensional topology and has since stimulated many new developments. The Jones polynomial also opened new connections between knot theory and many other branches of mathematics and theoretical physics, such as Lie theory, number theory, and statistical physics. New algebraic structures are constructed in the study of the Jones polynomials. Soon after the discovery of the Jones polynomial, many generalizations, known as quantum invariants of knots and 3-manifolds, were discovered. In particular, for every simple Lie algebra $\mathfrak{g}$ and every finite-dimensional irreducible $\mathfrak{g}$-module, the theory assigns to every knot in the 3-space an invariant, which is a Laurent polynomial in the quantum parameter. The colored Jones polynomial, which is an invariant of knots colored by integers, is among these generalizations; it is the invariant corresponding to the Lie algebra $sl_2(\mathbb{C})$ and its finite-dimensional irreducible modules.

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The Jones polynomial of a knot and its generalizations are defined through a diagram of the knot, an object essentially 2-dimensional. It is hard to understand the Jones polynomial in terms of classical invariants like the fundamental group, which is intrinsically 3-dimensional. The best known relation between the colored Jones polynomial and the fundamental group is the Melvin-Morton conjecture (now a theorem, see Subsection 4.6), which relates the colored Jones polynomial to the Alexander polynomial. The famous volume conjecture would connect the colored Jones polynomial to the hyperbolic structure of the knot complement. The Alexander polynomial is an *abelian* invariant of the knot complement, since it can be defined using abelian representations of the knot group. A finer invariant, the two variable A-polynomial, is defined using non-abelian representations of the knot group and its peripheral system. The A-polynomial has been important in geometric topology. The AJ conjecture would relate the colored Jones polynomial to the A-polynomial.

The goal of this note is to give a friendly introduction to the colored Jones polynomial, to explain the AJ conjecture, and to sketch a proof of the AJ conjecture for a class of knots which includes infinitely many two-bridge knots and all pretzel knots $(-2, 3, 6n \pm 1)$.

In Section 2 we define the Jones polynomial through the Kauffman bracket and give a proof (due to Kauffman, Murasugi, and Thistlethwaite) of the Tait conjecture on the crossing number of alternating links. In Section 3 we give an overview of quantum link invariants coming from quantum groups associated to simple Lie algebras. Section 4 is devoted to properties of the colored Jones polynomial, the Melvin-Morton conjecture, and the growth of the colored Jones polynomial. In Section 5 we show that for every knot, the color Jones function satisfies a recurrence relation, and we define the recurrence polynomial. In Section 6 we explain the Kauffman bracket skein module and its relation to character varieties. Section 7 is devoted to the AJ conjecture.

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2. The Jones polynomial

In this section we give the definition of the Jones polynomial via the Kauffman bracket, establish its basic properties, and sketch a proof (due to Kauffman, Murasugi, and Thistlethwaite) of the Tait conjecture on the crossing number of alternating links. The estimate of the degree bounds found in the proof of the Tait conjecture will be used in later sections. All the results in this section are now classic, and can be found for examples in textbooks [**39, 50**].

2.1. Knots and links in $\mathbb{R}^3 \subset S^3$. – Fix the standard 3-dimensional space $\mathbb{R}^3$. *An oriented link* L is a compact 1-dimensional oriented smooth submanifold of $\mathbb{R}^3 \subset S^3$. A link of 1 component is called a *knot*. By convention, the empty set is also considered a link.

A *framed oriented link* L is a link equipped with a smooth normal vector field V, which is a function $V : L \to \mathbb{R}^3$, such that $V(x)$ is not in the tangent space $T_x L$ for every $x \in L$.

Two (framed) oriented links are *equivalent* if one can be smoothly deformed into another in the class of (framed) oriented links.

A (framed) oriented link is *ordered* if there is an order on the set of its components.

Usually we don't distinguish between a link and its equivalence class. Un-oriented links, un-oriented framed links and their equivalence classes are defined similarly.

A link invariant is a map

$$I : \{\text{equivalence classes of links}\} \to S,$$

where S is a set.

Example 2.1. – For unoriented unframed links, the link group $\pi_1(L) := \pi_1(\mathbb{R}^3 \setminus L)$ is a link invariant.

2.2. Link diagram, blackboard framing. – One often studies an (oriented or unoriented) link L by studying one of its diagrams on $\mathbb{R}^2$, which is a projection D of L onto $\mathbb{R}^2$ (in general position), together with the "over/under" information at each crossing point. An (oriented) link diagram D of a link L determines the equivalence class of the (oriented) link L. Link diagrams are considered up to isotopy of the plane $\mathbb{R}^2$.

A link diagram comes with the *blackboard framing*, in which the framing vectors are in the plane $\mathbb{R}^2$. We say that a link diagram D is a *blackboard diagram* of a framed link L if the framed link determined by D together with its blackboard framing is equivalent to L.

It is known that two unoriented link diagrams define the same equivalence class of unoriented unframed links if and only if they are related by a sequence of Reidemeister moves RI, RII, and RIII (and isotopies of the plane). The Reidemeister moves are listed in Figure 1 and 2. For framed unoriented link diagrams one replaces RI by RI_f. For oriented links one allows all possible orientations of the strands in the figures. For details see e.g., **[7, 50]**.

Thus, the map associating an unoriented unframed link diagram to its link class descends to an isomorphism of sets

$$\{\text{link diagrams}\}/(\text{RI,RII,RIII}) \xrightarrow{\cong} \{\text{equiv. classes of links}\}.$$

If I is an invariant of unoriented link diagrams which is invariant under Reidemeister moves, then I descends to an invariant of unoriented unframed links.

The mirror image of a (framed, oriented) link L, denoted by $L!$, is the image of L under a reflection in a plane in $\mathbb{R}^3$. It is easy to see that the equivalence class of $L!$

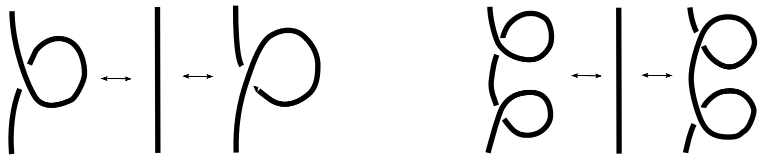

FIGURE 1. Reidemeister move RI on the left and RI_f on the right.

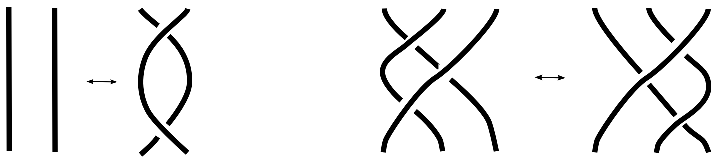

FIGURE 2. Reidemeister move RII on the left and RIII on the right.

depends only on the equivalence class of the original link L. If L has a (blackboard) framing D, then $L!$ has as a link diagram the *mirror image* of D, which is the result of switching all the crossings of D from over to under and vice versa.

2.3. Sign of a crossing, linking number, writhe. – Up to isotopies of the plane $\mathbb{R}^2$ there are two types of crossings of oriented link diagrams, see Figure 3. The crossing on

FIGURE 3. A positive crossing and a negative crossing

the left is called a positive crossing, while the one on the right is called a negative crossing.

For a 2-component oriented link diagram $D = D_1 \cup D_2$, define

$$\mathrm{lk}(D) = \frac{1}{2} \sum_x \varepsilon(x),$$

where the sum is over all the crossings between D_1 and D_2, and $\varepsilon(x)$ is the sign of x.

Exercise 2.2. – (a) Show that $\mathrm{lk}(D)$ does not change under oriented Reidemeister moves and hence defines an invariant of 2-component oriented links, known as the *linking number*.

(b) Suppose $L = L_1 \cup L_2$ be a 2-component oriented link. Define the Gauss map

$$\gamma : L_1 \times L_2 \to S^2 = \{z \in \mathbb{R}^3 \mid ||z|| = 1\}, \quad \gamma(x, y) = \frac{x - y}{||x - y||}.$$