A hitchhiker's guide to Khovanov homology

Paul Turner

Panoramas et Synthèses

Numéro 48

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique

Panoramas & Synthèses
48, 2016, p. 91–129

A HITCHHIKER'S GUIDE TO KHOVANOV HOMOLOGY

by

Paul Turner

I dedicate these notes to the memory of Ruty Ben-Zion

Abstract. – These notes from the 2014 summer school Quantum Topology at the CIRM in Luminy attempt to provide a rough guide to a selection of developments in Khovanov homology over the last fifteen years.

***Résumé* (Guide pratique de l'homologie de Khovanov).** – Cet article reprend les notes du cours sur l'homologie de Khovanov donné en 2014 à l'école thématique « Topologie quantique » du CIRM à Luminy : une vue d'ensemble et des repères pour s'orienter dans les recherches effectuées dans ce domaine au cours des 15 dernières années.

Foreword

There are already too many introductory articles on Khovanov homology and another is not really needed. On the other hand by now—15 years after the invention of the subject—it is quite easy to get lost after having taken those first few steps. What could be useful is a rough guide to some of the developments over that time and the summer school *Quantum Topology* at the CIRM in Luminy has provided the ideal opportunity for thinking about what such a guide should look like. It is quite a risky undertaking because it is all too easy to offend by omission, misrepresentation or other. I have not attempted a complete literature survey and inevitably these notes reflect my personal view, jaundiced as it may often be. My apologies in advance for any offense unwittingly caused.

I would like to express my warm thanks to David Cimasoni, Christine Lescop, Lukas Lewark, Alex Shumakovitch, Liam Watson and Ben Webster.

2010 Mathematics Subject Classification. – 57M25, 57M27.
Key words and phrases. – Knot theory, Khovanov homology.

1. A beginning

There are a number of introductions to Khovanov homology. A good place to start is Dror Bar-Natan's exposition of Khovanov's work

– *On Khovanov's categorification of the Jones polynomial* (Bar-Natan, [**4**]) ,

followed by Alex Shumakovitch's introduction

– *Khovanov homology theories and their applications* (Shumakovitch, [**83**]),

not forgetting the original paper by Mikhail Khovanov

– *A categorification of the Jones polynomial* (Khovanov, [**34**]).

Another possible starting point is

– *Five lectures on Khovanov homology* (Turner, [**88**]).

1.1. There is a link homology theory called Khovanov homology. – What are the minimal requirements of something deserving of the name *link homology theory*? We should expect a functor

$$H\colon \mathbf{Links} \to \mathbf{A}$$

where **Links** is some category of links in which isotopies are morphisms and **A** another category, probably abelian, where we have in mind the category of finite dimensional vector spaces, $\mathbf{Vect}_R$, or of modules, $\mathbf{Mod}_R$, over a fixed ring R. This functor should satisfy a number of properties.

- *Invariance.* If $L_1 \to L_2$ is an isotopy then the induced map $H(L_1) \to H(L_2)$ should be an isomorphism.
- *Disjoint unions.* Given two disjoint links L_1 and L_2 we want the union expressed in terms of the parts
$$H(L_1 \sqcup L_2) \cong H(L_1) \,\square\, H(L_2)$$
where $\square$ is some monoidal operation in **A** such as $\oplus$ or $\otimes$.
- *Normalization.* The value of $H(\text{unknot})$ should be specified. (Possibly also the value of the empty knot).
- *Computational tool.* We want something like a long exact sequence which relates homology of a given link with associated "simpler" ones—something like the Meyer-Vietoris sequence in ordinary homology.

If these are our expectations then Khovanov homology is bound to please. Let us take **Links** to be the category whose objects are oriented links in S^3 and whose morphisms are link cobordisms, that is to say compact oriented surfaces-with-boundary in $S^3 \times I$ defined up to isotopy. All manifolds are assumed to be smooth.

Theorem 1.1 **(Existence of Khovanov homology)**. – *There exists a (covariant) functor*

$$\mathrm{Kh}\colon \mathbf{Links} \to \mathbf{Vect}_{\mathbb{F}_2}$$

satisfying

1. *If* $\Sigma\colon L_1 \to L_2$ *is an isotopy then* $\mathrm{Kh}(\Sigma)\colon \mathrm{Kh}(L_1) \to \mathrm{Kh}(L_2)$ *is an isomorphism,*

2. $\mathrm{Kh}(L_1 \sqcup L_2) \cong \mathrm{Kh}(L_1) \otimes \mathrm{Kh}(L_2)$,
3. $\mathrm{Kh}(\mathit{unknot}) = \mathbb{F}_2 \oplus \mathbb{F}_2$ *and* $\mathrm{Kh}(\varnothing) = \mathbb{F}_2$,
4. *If L is presented by a link diagram a small piece of which is then there is an exact triangle*

$$\begin{tikzcd} \mathrm{Kh}(\;) \arrow[rr] & & \mathrm{Kh}(\;) \arrow[dl] \\ & \mathrm{Kh}(\;). \arrow[ul] & \end{tikzcd}$$

In fact a little more is needed to guarantee something non-trivial and in addition to the above we demand that Kh carries a bigrading

$$\mathrm{Kh}^{*,*}(L) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Kh}^{i,j}(L)$$

and with respect to this

- a link cobordism $\Sigma\colon L_1 \to L_2$ induces a map $\mathrm{Kh}(\Sigma)$ of bidegree $(0, \chi(\Sigma))$ where $\chi(\Sigma)$ is the Euler characteristic of the surface,
- the generators of the unknot have bidegree $(0,1)$ and $(0,-1)$ (and for the empty knot bidegree $(0,0)$),
- the exact triangle unravels as follows:

  Case I: For each j there is a long exact sequence

$$\xrightarrow{\delta} \mathrm{Kh}^{i,j+1}(\;) \longrightarrow \mathrm{Kh}^{i,j}(\;) \longrightarrow \mathrm{Kh}^{i-\omega,j-1-3\omega}(\;) \xrightarrow{\delta} \mathrm{Kh}^{i+1,j+1}(\;) \longrightarrow$$

  where ω is the number of negative crossings in the chosen orientation of minus the number of negative crossings in .

  Case II: For each j there is a long exact sequence

$$\longrightarrow \mathrm{Kh}^{i-1,j-1}(\;) \xrightarrow{\delta} \mathrm{Kh}^{i-1-c,j-2-3c}(\;) \longrightarrow \mathrm{Kh}^{i,j}(\;) \longrightarrow \mathrm{Kh}^{i,j-1}(\;) \xrightarrow{\delta}$$

  where c is the number of negative crossings in the chosen orientation of minus the number of negative crossings in .

To prove the theorem one must construct such a functor, but first let's see a few consequences relying only on existence and standard results.

Proposition 1.2. – *If a link L has an odd number of components then* $\mathrm{Kh}^{*,even}(L)$ *is trivial. If it has an even number of components then* $\mathrm{Kh}^{*,odd}(L)$ *is trivial.*

Proof. – The proof is by induction on the number of crossing and uses the following elementary result.

Lemma 1.3. – *In the discussion of the long exact sequences above (i) if the strands featured at the crossing are from the same component then ω is odd and c is even, and (ii) if they are from different components then ω is even and c odd.*

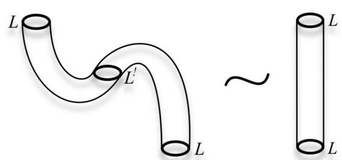
For the inductive step we use this and, depending on the case, one of the long exact sequence shown above, observing that in each case two of the three groups shown are trivial. □

Proposition 1.4. – *If $L^!$ denotes the mirror image of the link L then $\mathrm{Kh}^{i,j}(L^!) \cong \mathrm{Kh}^{-i,-j}(L)$.*

Proof. – There is a link cobordism $\Sigma\colon L^! \sqcup L \to \varnothing$ with $\chi(\Sigma) = 0$ obtained by bending the identity cobordism (a cylinder) $L \to L$. Since Kh is a functor there is an induced map of bidegree $(0, \chi(\Sigma)) = (0, 0)$

$$\Sigma_*\colon \mathrm{Kh}^{*,*}(L^!) \otimes \mathrm{Kh}^{*,*}(L) \to \mathrm{Kh}^{*,*}(\varnothing) = \mathbb{F}_2.$$

By a standard "cylinder straightening isotopy" argument



the bilinear form is non-degenerate, and the result follows recalling that we are in a bigraded setting so

$$(\mathrm{Kh}^{*,*}(L^!) \otimes \mathrm{Kh}^{*,*}(L))^{0,0} = \bigoplus_{i,j} \mathrm{Kh}^{i,j}(L^!) \otimes \mathrm{Kh}^{-i,-j}(L).$$ □

Exercise 1. – Theorem 1.1 includes the statement that $\mathrm{Kh}(\text{unknot}) = \mathbb{F}_2 \oplus \mathbb{F}_2$. In fact we could assume the weaker statement: the homology of the unknot is concentrated in degree zero. Use this along with the diagram , the long exact sequence, the property on disjoint unions and the invariance of Khovanov homology to show that $\dim(\mathrm{Kh}(\text{unknot})) = 2$.

Proposition 1.5. – *For any oriented link L,*

$$\frac{1}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}} \sum_{i,j} (-1)^{i+j+1} t^{\frac{j}{2}} \dim(\mathrm{Kh}^{i,j}(L))$$

is the Jones polynomial of L.