EXTREMAL ISOSYSTOLIC METRICS FOR COMPACT SURFACES

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Abstract. Given a closed, orientable surface M of genus ≥ 2, one seeks an *extremal isosystolic metric* on M: this is a Riemannian metric that induces on M the smallest possible area, subject to the constraint that the corresponding *systole*, or shortest length of any non-contractible closed curve, is a fixed, positive number. The geometric problem is rendered into an analytic one by reducing it to solving a nonlinear, partial differential equation with free boundaries. Examples are shown, to illustrate some possible candidates for solutions of the problem in special cases.

Résumé. Sur une surface M compacte orientable de genre ≥ 2, on cherche une métrique isosystolique extrémale : c'est une métrique riemannienne d'aire la plus petite possible sous la contrainte que la systole, i.e. la courbe fermée lisse non contractible de longueur minimale, soit un nombre positif fixé. Le problème géométrique est transformé en un problème analytique en le réduisant à la résolution d'une équation aux dérivées partielles non-linéaire à frontière libre. Des exemples sont donnés pour illustrer des candidats possibles à être solution du problème dans des cas particuliers.

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1. INTRODUCTION

Given a compact Riemannian or Finslerian manifold (M, g), where g denotes the Riemannian (respectively, Finsler) metric, a base point $x_0 \in M$, and an element γ in the fundamental group $\pi_1(M, x_0)$, the *local systole* $Sys_\gamma(M, x_0, g)$ is defined to be the minimum length of any loop path through x_0 in the homotopy class γ. Denote by $\overline{\gamma}$ the conjugacy class of γ in $\pi_1(M, x_0)$; then the free local systole of (M, g) at $\overline{\gamma}$ is defined to be the minimum length of any closed path representing the free homotopy class $\overline{\gamma}$, and is denoted by $Sys_{\overline{\gamma}}(M, g) = Inf_{x_0 \in M}(Sys_\gamma(M, x_0, g))$. The systole (with no added qualifier) $Sys(M, g)$ is understood to be the least value of $Sys_{\overline{\gamma}}(M)$ as $\overline{\gamma}$ ranges over all non-trivial free homotopy classes.

In the terminology of M. Gromov [6], an n-dimensional, differentiable manifold M is called *essential*, if, for all Riemannian (respectively, Finsler) metrics g in M, the isosystolic ratio [1] $Vol(M, g)/(Sys(M, g))^n$ has a positive lower bound depending only on the topology of M. Gromov's compactness theorem asserts that, if M is essential, then for any positive constant c the function space of all metrics g in M, normalized by a positive factor so that $Sys(M, g) = 1$ and satisfying the volume inequality $Vol(M, g) \leq c$, is compact in the Fréchet-Hausdorff topology. In particular, all closed, 2-dimensional surfaces except for the 2-sphere are essential. With these facts in mind, it is natural to raise the question of estimating the minimum isosystolic ratio for any closed surface, orientable or not, in terms of its genus. Many variants of this question have been studied, some of them formulated to include more general spaces, such as manifolds with boundary, others dealing with restricted classes of metrics, such as Riemannian metrics with non-positive, or constant, negative curvature, or metrics in a given, conformal class, to name a few. While some statements in this paper apply to surfaces with boundary, we shall limit our consideration almost exclusively to Riemannian metrics in closed, orientable surfaces, leaving other cases for another occasion. The only types of closed surfaces for which one knows an explicit, extremal

[1] In Gromov's definition the isosystolic ratio is expressed by $Sys(M,g)/(Vol(M,g))^{1/n}$.

isosystolic metric, i.e. a Riemannian metric minimizing the isosystolic ratio, are the projective plane (P.M. Pu, [7]), the torus (C. Loewner, unpublished, cf. M. Berger, [3,4]) and the Klein Bottle (C. Bavard, [1,2]). For each of the other types of surfaces (i.e. for surfaces with negative Euler characteristic) there is a very wide gap between the best available estimates of upper and of lower bounds for the extremal isosystolic ratio. The main purpose of the paper is to reduce the problem of extremal isosystolic metric to a variational problem that may be studied by the methods of classical calculus of variations. At the end of this paper we shall exhibit for the record two explicit examples of metrics in an orientable surface of genus 3: both metrics attain locally minimum values of the isosystolic ratio, relative to small deformations of the metric in its function space, the second metric having an isosystolic ratio about 1.5% lower than the first ; it is believed that the value achieved by the second metric ($(7\sqrt{3})/8 \approx 1,51554$) is very close to, if not actually equal to the absolute minimum value for surfaces of genus 3. The two examples consist of piecewise flat metrics in the surface, each one constructed in terms of a corresponding, explicit, well known triangulation, with a large group of symmetries.

No similar construction has been found to yield an extremal isosystolic metric in surfaces of any genus $g = 2$, or ≥ 4, suggesting that the genera of surfaces whose extremal isosystolic metrics are piecewise flat may be quite sparse: it is this particular observation that has motivated the present study ; its ultimate goal is that of studying the general local properties of extremal isosystolic metrics, especially when they are not piecewise flat. Unfortunately the partial differential equations obtained have not yielded methods to construct any non-trivial, explicit solutions. However it is shown in Sections 6 and 7 that, merely by using the maximum principle, one can obtain some fairly close *a priori* estimates of the minimum isosystolic ratio in two examples, that illustrate also a useful generalization of the isosystolic problem. The first example consists of seeking a Riemannian metric in a 2-disk, admitting the group of symmetries of a regular hexagon, that minimizes the area subject to the condition that the least distance between each of the three pairs of opposite "sides" equals 2; the second example deals with the extremal isosystolic metrics in a torus with one open disc deleted: in this case the "systole" consists of two independent, positive, real numbers, representing, respectively, the "boundary systole" and the least length of any closed path representing a non-trivial homology class of cycles. Both of these examples illustrate some of the singularities that extremal isosystolic metrics may exhibit in general.

2. STRUCTURE OF k-REGULAR DOMAINS

Let M be a closed, orientable surface of genus $g \geq 2$ and consider the complete function space G of singular, generalized Riemannian (respectively, Finsler) metrics g on M, such that:

(i) g is bounded, locally, from above and below, by smooth Riemannian metrics ;

(ii) the g-length functional on the space of rectifiable arcs (the latter with the Fréchet topology) is lower semicontinuous.

This class of metrics is invariant under homeomorphisms of M of Lipschitz class; its definition ensures the compactness of any set of paths of bounded length, in any compact domain. In particular, the g-distance $d(x, y)$ between any two points $x, y \in M$ is achieved by a compact (non-empty) set of shortest paths. The function space G has the topology of uniform Lipschitz convergence of $d(x, y)$ in each compact subset of M: this topology ensures both the equivalence of the area functional $Vol(D) = Vol_g(D)$ [2] with the Lebesgue measure of any Borel set $D \subset M$ and its continuity with respect to the metric $g \in G$. Given any element $\overline{\gamma}$ in the set $\overline{\pi}_1^*(M)$ of non-trivial, homotopy classes of free, closed paths in M, the (free) local systole $Sys_{\overline{\gamma}}(M, g)$ is achieved by a compact family of oriented, closed paths of length $Sys_{\overline{\gamma}}(M, g)$, representing the class $\overline{\gamma}$: such closed paths will be referred to as <u>systole-long</u> paths ; for any given, positive real number A, the set $\Gamma_A \subset \overline{\pi}_1^*(M)$ consisting of all classes $\overline{\gamma}$ such that $Sys_{\overline{\gamma}}(M, g) \leq A$ is a finite set. The metrics in the class G may be discontinuous: for example, they may include isolated "short-cut" (or "fast-track") curves ; however it is a complete function space, to which an

[2] In the case of a Finsler metric g, the volume element form $dVol_g$ in terms of local parameters (u,v) is defined to be $\pi^{-1}\sigma(u,v)|du\wedge dv|$, where $\sigma(u,v)$ denotes the area of the unit g^*-disc in the cotangent bundle of M, with respect to the dual Finsler form g^* of g.