SYSTOLES AND INTERSYSTOLIC INEQUALITIES

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Abstract. This articles surveys inequalities involving systoles in Riemannian geometry.

Résumé. Cet article présente l'ensemble des inégalités connues sur les systoles en géométrie riemannienne.

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1. PRELIMINARIES

The notion of the *k-dimensional systole* of a Riemannian manifold was introduced by Marcel Berger in 1972 following earlier work by Loewner (around 1949, unpublished), Pu (1952), Accola (1960) and Blatter (1961). Recall that according to Berger *the k-dimensional systole of a Riemannian manifold V is defined as the infimum of the k-dimensional volumes of the k-dimensional cycles (subvarieties) in V which are not homologous to zero in V.*

In fact, the idea of the 1-dimensional systole can be traced back to the classical geometry of numbers as one considers minima of quadratic forms on lattices in $\mathbb{R}^n$. The fundamental result here is an upper bound on such a minimum in terms of the discriminant of the form in question. This can be formulated in geometric language as follows.

1.A. Bound on the 1-systole of a flat torus. — *Let V be a flat Riemannian torus of dimension n. Then, the 1-systole of V can be bounded in terms of the volume of V by*

$$\text{systole} \le \text{const}_n \,(\text{Volume})^{\frac{1}{n}} \ ,$$

where $\text{const}_n = C\sqrt{n}$ *for some universal constant C (which is not far from one).*

Reformulation and proof. The torus V can be isometrically covered by $\mathbb{R}^n$ and so $V = \mathbb{R}^n/\Gamma$ for some lattice Γ, that is a discrete group of parallel translations of $\mathbb{R}^n$. (This group is *isomorphic* to $\mathbb{Z}^n$ but is *not*, in general, equal to the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ consisting of integral points in $\mathbb{R}^n$.) If a point $x \in \mathbb{R}^n$ is moved by some $\gamma \in \Gamma$ to $\gamma(x)$, then the segment $[x, \gamma(x)]$ joining x with $\gamma(x)$ in $\mathbb{R}^n$ projects to a closed curve S in $V = \mathbb{R}^n/\Gamma$ whose length equals $\text{dist}\big(x, \gamma(x)\big)$. Furthermore, if γ is a non-identity element in Γ (i.e., $x \neq \gamma(x)$), then S is non-homologous to zero in V. In fact, S is *non-homotopic* to zero by the elementary theory of covering spaces which implies "non-homologous to zero" since the group $\Gamma = \pi_1(V) = \mathbb{Z}^n$ is Abelian. Thus, the bound on the 1-systole of V is equivalent to the following estimate.

1.A.1. Displacement estimate. — *For the above lattice* Γ *acting on* $\mathbb{R}^n$ *by parallel translations, there exists a point* $x \in \mathbb{R}^n$ *and a non-identity element* $\gamma \in \Gamma$, *such that*

$$\operatorname{dist}\big(x, \gamma(x)\big) \leq \operatorname{const}_n \operatorname{Vol}(\mathbb{R}^n/\Gamma) \ .$$

Proof. Take a closed ball B of radius R in $\mathbb{R}^n$ such that the volume of B is greater than or equal to that of $V = \mathbb{R}^n/\Gamma$. Then, the projection $p : B \to V$ is *not* one-to-one and we have distinct points x and x' in B with $p(x) = p(x')$. This equality means that $x' = \gamma(x)$ for some $\gamma \in \Gamma$ (by the definition of the quotient space $\mathbb{R}^n/\Gamma$) and, since the diameter of B is $2R$, the distance between x and $\gamma(x) = x'$ is at most $2R$.

Now, we recall that the volume of $B = B(R)$ equals

$$\operatorname{Vol} B = \sigma_n R^n \ ,$$

where σ_n is given by the familiar formula involving the Γ-function (here, Γ has nothing to do with the lattice Γ),

$$\sigma_n = \pi^{n/2} / \Gamma\big(\frac{n}{2} + 1\big) \ .$$

Then, a pair of points x and x' with $p(x') = p(x)$ necessarily appears for

$$R = (\sigma_n)^{-\frac{1}{n}} (\operatorname{Vol} V)^{\frac{1}{n}} \ .$$

So we obtain the required displacement bound

$$\operatorname{dist}\big(x,\, \gamma(x)\big) \leq \operatorname{const}_n \operatorname{Vol} V$$

for

$$\operatorname{const}_n = 2\Big(\Gamma\big(\frac{n}{2} + 1\big)\Big)^{\frac{1}{n}} \Big/ \sqrt{\pi} \ ,$$

and the number const_n is bounded by $C\sqrt{n}$ according to Stirling's formula $\Gamma(n) \approx n^n$.

Remarks. (a) The above argument is classical, going back to Gauss (to Diophantus ?), Hermite and Minkowski. We dissected the proof in order to make visible the anatomy of our more general systolic inequalities discussed later on.

(b) Since Γ acts by parallel translations, the displacement $\operatorname{dist}\big(x, \gamma(x)\big)$ does not depend on x, and we may take the origin $0 \in \mathbb{R}^n$ for x. Then, our displacement estimate bounds the Euclidean norm on the lattice Γ embedded into $\mathbb{R}^n$ as the Γ-orbit of the origin by

$$\inf \|\gamma\|_{\mathbb{R}^n} \leq \operatorname{const}_n (\operatorname{Vol} \mathbb{R}^n/\Gamma)^{\frac{1}{n}} \ , \tag{$*$}$$

where inf is taken over $\gamma \in \Gamma - \{0\}$. (The squared Euclidean norm serves as the quadratic form referred to at the beginning of this discussion.)

The above $(*)$ is called the *Minkowski convex body theorem*. It remains valid (by the proof we gave) for an arbitrary Banach (Minkowski) norm on $\mathbb{R}^n$. In traditional language, *every convex centrally symmetric body B in $\mathbb{R}^n$ contains a non-zero point $\gamma \in \Gamma$, provided* $\operatorname{Vol} B \geq 2^n \operatorname{Vol}(\mathbb{R}^n/\Gamma)$.

(c) The value of const_2 and the extremal lattice $\Gamma \subset \mathbb{R}^2$ are known since Antiquity. Namely, $\operatorname{const}_2 = (2/\sqrt{3})^{\frac{1}{2}}$, and the extremal lattice has a regular hexagon as fundamental domain. (Such an hexagon of unit width has area $\sqrt{3}/2$.) *Thus, for every flat 2-torus one has*

$$\text{systole} \leq (2/\sqrt{3})^{\frac{1}{2}} (\text{Area})^{\frac{1}{2}} \ , \tag{$+$}$$

where equality holds if and only if the corresponding lattice $\Gamma \subset \mathbb{R}^2$ is hexagonal.

1.B. Loewner made an amazing discovery around 1949

Loewner torus theorem. — *Let V be the topological 2-torus with an arbitrary Riemannian metric. Then, the 1-systole of V satisfies the same inequality as in the flat case,*

$$\text{systole} \leq (2/\sqrt{3})^{\frac{1}{2}} (\text{Area})^{\frac{1}{2}} \ ,$$

and equality holds if and only if the metric on V is flat and the corresponding lattice is hexagonal.

Proof. The key argument is the following

The uniformization theorem for tori. — *For every V there exists a flat torus V_0* (which can be normalized by the condition $\operatorname{Area} V_0 = \operatorname{Area} V$) *admitting a conformal diffeomorphism $\varphi : V_0 \to V$.*