RAMIFICATIONS OF THE CLASSICAL SPHERE THEOREM

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Abstract. The paper describes old and new developments, within as well as outside of Riemannian geometry, originating from the classical sphere theorem.

Résumé. Cet article décrit des développements anciens et récents, en géométrie riemannienne et ailleurs, provenant du classique théorème de caractérisation des sphères.

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INTRODUCTION

Although Comparison Geometry can be traced back to the previous century, it did not really take root as a discipline until the 1930's through the work of Morse, [M1,2], Schoenberg [S], Myers [My] and Synge [Sy]. The real breakthrough came in the 1950's with the pioneering work of Rauch [R] and the foundational work of Alexandrov and Toponogov [T]. Since then, the simple idea of comparing the geometry of an arbitrary Riemannian manifold with the geometries of constant curvature spaces has witnessed a tremendous evolution.

Sphere Theorems have often played a pivotal role in this evolution. In fact many of the powerful ideas and techniques known today were first conceived in connection with investigations around potential sphere theorems (cf. also [Sh]). Their significance is also measured by their implications for the local structure of general Riemannian manifolds and other related, but more singular spaces.

Our aim here is to trace out paths of developments, still under construction, originating from the classical sphere theorem [R,K2] and the associated rigidity theorem by Berger [B1]. In doing so, it is our hope to reveal that there is an abundance of challenging open problems in this area whose solutions will yet again involve the conception of new ideas and tools.

1. DEVELOPMENTS FROM WITHIN

In this section we describe evolutions associated with constructions on a fixed Riemannian manifold.

It all began with Rauch's comparison theorem for the length of Jacobi fields [R] and subsequently with the global Alexandrov-Toponogov triangle comparison theorem [T]. And it culminated in the now classical theorem.

Theorem 1.1 (Rauch-Berger-Klingenberg). — *Let M be a closed simply connected Riemannian manifold whose sectional curvature satisfies $1 \leq \sec M \leq 4$. Then, either*

(i) M is a twisted sphere, or

(ii) M is isometric to a rank one symmetric space.

Under the assumptions stated in the theorem, one of the key ingredients is the injectivity radius estimate, $\operatorname{inj} M \geq \frac{\pi}{2}$. In the original approach, this was achieved via Morse theory of geodesics [K1,2], [CG] and [KS] (cf. also [E]). Before moving on to the natural generalization suggested by this estimate, let us point out that so far, no positively curved exotic spheres are known!

Quite recently, it was shown by M. Weiss that some exotic spheres do not admit 1/4-pinched metrics [W]. His method is based on the observation that a 1/4-pinched sphere M has maximal so called *Morse perfection*, i.e., there is a $\dim M$-dimensional ($\mathbb{Z}_2$ -equivariant) spherical family of Morse functions on M. On the other hand, sophisticated methods from algebraic K-theory reveal that some exotic spheres have smaller Morse perfection. It is interesting to note that this is also related to the so called *Gromoll-filtrations* of homotopy spheres, an idea which arose in the first proof that there are no exotic δ_n-pinched n-dimensional spheres when δ_n is sufficiently close to 1 [G]. Another completely different method to prove the same result was conceived independently by Shikata [S2]. He constructed a distance between differentiable structures [S1], an idea which has since been expanded tremendously (cf. [Gr]). The best estimate for $\delta_n = \delta$ is due to Suyama [Su]. His method combines the earlier methods for achieving a dimension independent constant, the first due to Shiohama [SS] and the second to Ruh [R1,2]. Here, Ruh's method of approximating an almost flat connection with a flat connection has evolved quite far and has had many subsequent applications (cf. [R3]).

Another natural question related to the classical sphere theorem is: what happens if M is not simply connected? So far, all known (strictly) 1/4-pinched manifolds are diffeomorphic to space forms. Moreover, at least these are the only manifolds which admit a δ-pinched riemannian metric, with δ sufficiently close to 1 [GKR1,2], [IR]. The general nonlinear "Riemannian center of mass" was developed in connection with the first proof of this result [GK].

Recall that the *radius* $\operatorname{rad} M$ and *diameter* $\operatorname{diam} M$ are given by $\operatorname{rad} M = \min_p \max_q \operatorname{dist}(p,q) \leq \max_{p,q} \operatorname{dist}(p,q) = \operatorname{diam} M$. Since $\operatorname{inj} M \geq \frac{\pi}{2}$ for 1-connected Riemannian manifolds M with $1 \leq \sec M \leq 4$, we also have $\operatorname{diam} M \geq \operatorname{rad} M \geq \frac{\pi}{2}$ for such manifolds. In particular,

$$\begin{aligned}\{M \mid 1 \leq \sec M \leq 4, \pi_1(M) = \{1\}\} &\subset \{M \mid 1 \leq \sec M, \operatorname{rad} M \geq \frac{\pi}{2}\} \\ &\subset \{M \mid 1 \leq \sec M, \operatorname{diam} M \geq \frac{\pi}{2}\} .\end{aligned}$$

For the largest of these classes we have the following diameter sphere Theorem [GS] a homotopy version of which was first proved in [B2] and its associated rigidity theorem [GG1,2].

Theorem 1.2. (Gromoll-Grove-Shiohama) — *Let M be a closed Riemannian manifold with $\sec M \geq 1$ and $\operatorname{diam} M \geq \frac{\pi}{2}$. Then, either*

(i) M is a twisted sphere, with the possible exception that $H^*(M) \simeq H^*(CaP^2)$,

(ii) M is isometric to one of

(a) a rank 1 *symmetric space,*

(b) $\mathbb{C}P^{\text{odd}}/\mathbb{Z}_2$,

(c) $S^n/\Gamma, \Gamma \subset O(n+1)$ *acts reducibly on* $\mathbb{R}^{n+1}$.

The principal new tool discovered in the proof of this sphere theorem was a "critical point theory" for nonsmooth distance functions. This signaled the beginning of intense investigations of manifolds with a lower curvature bound only (for surveys, cf. [C], [Gro]).

Aside from trying to deal with the exceptional case of the Cayley plane in the above result, the most obvious questions related to the theorem are

Problem 1.3.

(i) Are there any exotic spheres M with $\sec M \geq 1$ and $\operatorname{diam} M \geq \frac{\pi}{2}$?

(ii) Are there "new" manifolds M with $\sec M \geq 1$ and $\operatorname{diam} M \geq \frac{\pi}{2} - \epsilon$, which are not on the list of the above theorem ?

At this moment it appears to be too ambitious to answer these questions at the level of generality at which they were posed (cf. the discussion in the next section).