FROM THE YAMABE PROBLEM TO THE EQUIVARIANT YAMABE PROBLEM

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Abstract. The formulation and solution of the equivariant Yamabe problem are presented in this study. As a result, every compact Riemannian manifold distinct from the sphere posseses a conformal metric of constant scalar curvature which is also invariant under the action of the whole conformal group. This answers an old question of Lichnerowicz.

Résumé. Une étude du problème de Yamabe équivariant est présentée. En particulier, nous montrons que toute variété riemannienne compacte distincte de la sphère possède une métrique conforme à courbure scalaire constante dont le groupe d'isométries est le groupe conforme tout entier. Ceci répond à une question posée par Lichnerowicz.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let (X, g) be a compact Riemannian manifold of dimension $n \geq 3$. The classical Yamabe problem can be stated as follows: "prove that there exists a metric conformal to g with constant scalar curvature". As is well known, it is equivalent to proving the existence of a positive solution $u \in C^\infty(X)$ of the equation

$$(E) \quad \Delta u + \frac{n-2}{4(n-1)} \mathrm{Scal}(g) u = C u^{(n+2)/(n-2)}$$

where $\Delta u = -g^{ij}(\partial_{ij} u - \Gamma_{ij}^k \partial_k u)$ in a local chart, and where $\mathrm{Scal}(g)$ is the scalar curvature of g.

Let J denote the functional defined on $W^{1,2}(X)/\{0\}$ by

$$J(u) = \frac{\int_X |\nabla u|^2 \, dv(g) + \frac{n-2}{4(n-1)} \int_X \mathrm{Scal}(g)\, u^2 \, dv(g)}{\left(\int_X |u|^{2n/(n-2)} \, dv(g)\right)^{(n-2)/n}} .$$

The positive critical points of J are smooth solutions of (E). We denote by ω_n the volume of the standard unit sphere S^n and $\mu(S^n) = \frac{1}{4} n(n-2)\, \omega_n^{2/n}$.

A positive answer to the problem was given by Yamabe [Y] in 1960, but his demonstration was incomplete as Trudinger [T] pointed out in 1968. Nevertheless :

(i) Trudinger [T] proved in 1968 that, if $\mathrm{Inf} J \leq 0$, then $\mathrm{Inf} J = \mathrm{Min} J$ and there exists a unique positive solution to (E) ;

(ii) T. Aubin [A1] proved in 1976 that, if $\mathrm{Inf} J < \mu(S^n)$, then again $\mathrm{Inf} J = \mathrm{Min} J$ and there exists a positive solution to (E). (When $\mathrm{Inf} J > 0$, many solutions may exist. See for instance [HV3] and [S2]). In addition, he proved that we always have $\mathrm{Inf} J < \mu(S^n)$ if (X, g) is a non locally conformally flat manifold of dimension $n \geq 6$.

(iii) Schoen [S1] proved in 1984 that $\mathrm{Inf} J < \mu(S^n)$ if $(X, [g]) \neq (S^n, [\mathrm{st.}])$ and $n = 3, 4, 5$ or (X, g) locally conformally flat. (Here, st. denotes the standard metric of S^n). As a consequence, the classical Yamabe problem is completely solved.

Let us now turn our attention to the equivariant Yamabe problem. Since we know that every compact Riemannian manifold has a conformal metric of constant scalar curvature, we will try to get some more precise geometric informations. As a matter of fact, we will ask to have a conformal metric with constant scalar curvature and prescribed isometry group. This new problem was first brought to our attention by Bérard-Bergery (UCLA, 1990). The precise statement of the problem is the following. "Given (X, g) a compact Riemannian manifold of dimension $n \geq 3$ and G a compact subgroup of the conformal group $C(X, g)$ of g, prove that there exists a conformal G-invariant metric to g which is of constant scalar curvature". We solved the problem in [HV2], namely

Theorem 1 (Hebey-Vaugon [HV2]). — *Let (X, g) be a compact Riemannian manifold and G a compact subgroup of $C(X, g)$. Then, there always exists a conformal G-invariant metric g' to g which is of constant scalar curvature. In addition, g' can be chosen such that it realizes the infimum of $\mathrm{Vol}(\widetilde{g})^{(n-2)/n} \int_X \mathrm{Scal}(\widetilde{g}) dv(\widetilde{g})$ over the G-invariant metrics conformal to g.*

In fact, we just have to prove the second point of the theorem, which can be restated as follows. Given (X, g) a compact Riemannian manifold and G a compact subgroup of $I(X, g)$, there exists $u \in C^\infty(X)$, $u > 0$ and G-invariant, which realizes $\mathrm{Inf} J(u)$ where the infimum is taken over the G-invariant functions of $W^{1,2}(X)/\{0\}$. Let us denote by $\mathrm{Inf}_G J(u)$ this infimum. A generalization of Aubin's result is needed here. Let $[g]$ be the conformal class of g and $O_G(x)$ be the G-orbit of $x \in X$. This generalization can be stated as follows.

Theorem 2 (Hebey-Vaugon [HV2]). — *If $\mathrm{Inf}_G J(u) < \mu(S^n)\ (\mathrm{Inf}_{x \in X} \sharp O_G(x))^{2/n}\ (*)$, then the infimum $\mathrm{Inf}_G J(u)$ is achieved and $[g]$ carries a G-invariant metric of constant scalar curvature. In addition, the non strict inequality always holds.*

($\sharp O_G(x) \in \mathbb{N}^* \cup \{\infty\}$ denotes the cardinal number of $O_G(x)$). As a consequence, the proof of Theorem 1 is straightforward if all the orbits of G are infinite. If not, the proof proceeds by choosing appropriate test functions.

This improvement of the classical Yamabe problem allows us to cover a conjecture of Lichnerowicz. This conjecture can be stated as follows: "$I_o(X, g) = C_o(X, g)$ as soon as $\mathrm{Scal}(g)$ is constant and $(X, [g]) \neq (S^n, [\mathrm{st.}])$" (where $I_o(X, g)$ and $C_o(X, g)$ are the connected components of the identity in the isometry group $I(X, g)$ of g and in the conformal group $C(X, g)$ of g). This statement is true when $\mathrm{Scal}(g)$ is nonpositive (since the metric of constant scalar curvature is unique), but can be false when $\mathrm{Scal}(g)$ is positive. One sees this by considering $S^1(T) \times S^{n-1}$ as $I_o(S^1(T)) \times I_o(S^{n-1})$ acts transitively on the product, which for T large possesses many conformal metrics of constant scalar curvature (see [HV3] and [S2]). In fact, the conjecture should be restated as follows: " Given (X, g) a compact Riemannian manifold, $(X, [g]) \neq (S^n, [\mathrm{st.}])$, there exists at least one g' in $[g]$ which has constant scalar curvature and which satisfies $I(X, g') = C(X, g)$". This is the best result possible and was proved in Hebey-Vaugon [HV2]. Using the work of Lelong-Ferrand [LF] (see also Schoen [S2]), this result can be seen as a corollary of Theorem 1. (Lelong-Ferrand proved that for any compact Riemannian manifold (X, g) distinct from the sphere, there exists $g' \in [g]$ such that $I(X, g') = C(X, g)$.)

Theorem 3 (Hebey-Vaugon [HV2]). — *Every compact Riemannian manifold (X, g), distinct from the sphere, possesses a conformal metric of constant scalar curvature which has $C(X, g)$ as isometry group.*

In the following, $R(g)$ denotes the Riemann curvature tensor of g, $\mathrm{Weyl}(g)$ denotes the Weyl tensor of g and $\mathrm{Ric}(g)$ denotes the Ricci tensor of g.

2. SOME WORDS ABOUT THE CLASSICAL YAMABE PROBLEM

We give here a new solution of the classical Yamabe problem which unifies the works of Aubin [A1] and Schoen [S1]. For completeness, we mention that other proofs have also been presented in [Ba], [BB], [LP], [S2] and [S3].

Proposition 4 (Hebey-Vaugon [HV1]). — *When $(X, [g]) \neq (S^n, [\mathrm{st.}])$, the test functions*

$$\begin{cases} u_{\varepsilon,x} = (\varepsilon + r^2)^{1-n/2} & \text{if } \; r \leq \delta, \; \delta > 0 \\ u_{\varepsilon,x} = (\varepsilon + \delta^2)^{1-n/2} & \text{if } \; r \geq \delta \end{cases}$$

give the strict inequality $\mathrm{Inf} J < \mu(S^n)$. *(Here, r is the distance from x fixed in X, δ and ε are small).Therefore,* $\mathrm{Inf} J(u)$ *is achieved and every compact Riemannian manifold carries, in its conformal class, a metric of constant scalar curvature.*