ON THE MOTION OF A CURVE TOWARDS ELASTICA

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Abstract. We consider a non-linear 4-th order parabolic equation derived from bending energy of wires in the 3-dimensional Euclidean space. We show that a solution exists for all time, and converges to an elastica when t goes to ∞.

Résumé. On considère une équation parabolique du 4^{e} ordre non linéaire provenant de l'énergie de flexion d'un câble dans l'espace euclidien de dimension 3. On montre qu'une solution existe pour tout temps, et converge lorsque t tend vers l'infini vers un "elastica".

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INTRODUCTION

Consider a springy circle wire in the Euclidean space $\mathbb{R}^3$. We characterize such a wire as a closed curve γ with fixed line element and fixed length. We treat curves $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with $|\gamma'| \equiv 1$. We denote by x the parameter of the curve, and denote by $'$, $''$ or $^{(n)}$ the derivatives with respect to x.

For such a curve, its elastic energy is given by

$$E(\gamma) := \oint |\gamma''|^2 \, dx \ .$$

Solutions of the corresponding Euler-Lagrange equation are called *elastic curves*. We discuss the corresponding parabolic equation in this paper. We will see that the equation becomes

$$\begin{cases} \partial_t \gamma = -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ . \end{cases}$$

Theorem. — *For any* C^∞ *initial data* $\gamma_0(x)$ *with* $|\gamma_0'| = 1$*, the above equation has a unique solution* $\gamma(x,t)$ *for all time, and the solution converges to an elastica when* $t \to \infty$*.*

We refer to Langer and Singer [13] for the classification of closed elasticae in the Euclidean space. They also discuss Palais-Smale's condition C and the gradient flow in [14]. However, their flow is completely different from ours. Our equation represents the physical motion of springy wire under high viscosity, while their flow has no physical meaning.

This paper is organized as follows. First, we prepare some basic facts. Section 1 : The equation (introduce the above equation), Section 2 : Notations, Section 3 : Basic inequalities, Section 4 : Estimations for ODE ($-v'' + av = b$). After this preparation, the proof of Theorem goes as usual. Section 5 : Linearized equation, Section 6 : Short time existence (by open–closed method), Section 7 : Long time existence, Section 8 : Convergence (using real analyticity of the Euclidean space).

1. THE EQUATION

To derive an equation of motion governed by energy, we perturb the curve $\gamma = \gamma(x)$ with a time parameter $t : \gamma = \gamma(x,t)$. Then, the elastic energy changes as

$$\frac{d}{dt}|_{t=0}E(\gamma) = 2\oint(\gamma'', \partial_t|_{t=0}\gamma'')\,dx = 2\oint(\gamma^{(4)}, \partial_t|_{t=0}\gamma)\,dx\ ,$$

where $\gamma(x,0) = \gamma(x)$. Therefore, $-\gamma^{(4)}$ would be the most efficient direction to minimize the elastic energy. However, this direction does not preserve the condition $|\gamma'| \equiv 1$. To force to preserve the condition we have to add certain terms. Let V be the space of all directions satisfying the condition in the sense of first derivative, i.e., $V = \{\eta \mid (\gamma', \eta') = 0\}$.

We can check that a direction is L^2 orthogonal to V if and only if it has a form $(w\gamma')'$ for some function $w(x)$. Therefore, the "true" direction has a form $-\gamma^{(4)} + (w\gamma')'$, where the function w has to satisfy the condition $\big((-\gamma^{(4)} + (w\gamma')')', \gamma'\big) = 0$. Namely, we consider the equation

$$\text{(1.1)}\qquad \begin{cases} \partial_t\gamma = -\gamma^{(4)} + (w\gamma')'\ , \\ (-\gamma^{(5)} + (w\gamma')'', \gamma') = 0\ , \\ |\gamma'| = 1\ . \end{cases}$$

Note that both γ and w are unknown functions on $S^1 \times \mathbb{R}_+$.

The second equality of (1.1) is reduced as follows. By the third condition, we see

$$\begin{aligned} (\gamma'', \gamma') &= 0\ , \\ (\gamma^{(3)}, \gamma') &= -|\gamma''|^2\ , \\ (\gamma^{(4)}, \gamma') &= -\frac{3}{2}(|\gamma''|^2)'\ , \\ (\gamma^{(5)}, \gamma') &= -2(|\gamma''|^2)'' + |\gamma^{(3)}|^2\ . \end{aligned}$$

Hence,

$$((w\gamma')'', \gamma') = w'' - |\gamma''|^2 w \ ,$$

and the second equality in (1.1) becomes

$$-w'' + |\gamma''|^2 w = 2(|\gamma''|^2)'' - |\gamma^{(3)}|^2 \ .$$

If we put $v = w + 2|\gamma''|^2$, then we get

$$-v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ .$$

We conclude that equation (1.1) is equivalent to the equation

$$\text{EP} \qquad \begin{cases} \partial_t \gamma = -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ . \end{cases}$$

The equation of elastic curves is

$$\text{EE} \qquad \begin{cases} -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' = 0 \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ , \\ |\gamma'| = 1 \ . \end{cases}$$

The first equality gives

$$0 = (\gamma', -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')') = \frac{3}{2}(|\gamma''|^2)' + (v - 2|\gamma''|^2)' \ .$$

Hence, the equation of elastic curves reduces to the equation

$$-\gamma^{(4)} - ((\frac{3}{2}|\gamma''|^2 + c)\gamma')' = 0 \ ,$$

where c is an arbitrary number.

2. NOTATIONS

Throughout this paper, we use variables x on $S^1 = \mathbb{R}/\mathbb{Z}$ and t on $\mathbb{R}_+ = [0, \infty)$. Symbols $*'$ and $*^{(n)}$ denote the derivation with respect to the variable x, even for a