RICCI CURVATURE MODULO HOMOTOPY

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Abstract. This article is a report summarizing recent progress in the geometry of negative Ricci and scalar curvature. It describes the range of general existence results of such metrics on manifolds of dimension ≥ 3. Moreover it explains flexibility and approximation theorems for these curvature conditions leading to unexpected effects. For instance, we find that "modulo homotopy" (in a specified sense) these curvatures do not have any of the typical geometric impacts.

Résumé. Cet article est un résumé des progrès récents dans la géométrie des variétés riemanniennes à courbure de Ricci ou scalaire négative. Il décrit le domaine de validité des résultats généraux d'existence pour de telles métriques sur les variétés de dimension ≥ 3. De plus, il explique les théorèmes de flexibilité et d'approximation pour ces conditions de courbure, ce qui conduit à des résultats inattendus. Par exemple, nous montrons que "modulo homotopie" (dans un sens précis), ces conditions de courbure n'impliquent aucune des conditions géométriques usuelles.

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INTRODUCTION

This paper reports on recent progress in understanding negative Ricci and scalar curvature. We mainly intended to write a guide summarizing and tabulating the main results. We also alluded to some technical (or rather philosophical) background while this is just enough to give some orientation.

As will become clear, $\mathrm{Ric} < 0$-metrics can be met quite frequently in geometry, in a way unexpected before.

One of the insights is concerned with the contrast between positive and negative curvatures. In the case of sectional curvature the implied topological conditions exclude each others, while Ricci and scalar curvature behave quite differently. Here, one may think of a certain maximal amount of positive curvature which could be carried by a given manifold. Now, starting from any metric one can deform it into more and more strongly negatively curved ones. In other words, on each manifold there is an (individual) upper but definitely no lower bound for the spectrum of such an "amount" of Ricci or scalar curvature.

Beside other features there is an amazing resemblance to some existence theories in completely different contexts, for instance, Smale-Hirsch immersion theory. Namely, one may say that these geometric problems can be understood "modulo homotopy" from the algebraic structure of the differential relation which formalizes the geometric condition (e.g. $\mathrm{Ric} < 0$ as partial differential inequality of second order). We will discuss these things in more details in a later chapter.

Now, in order to start our $\mathrm{Ric} < 0$-story, we may notice that it was not even known whether each manifold could admit a $\mathrm{Ric} < 0$-metric. As this paper intends to lead beyond this first order question we start with a short sketch of how to prove that each closed manifold M^n of dimension $n \geq 3$ admits a metric with $\mathrm{Ric} < 0$.

First of all, we mention that it is an easier matter to get a $\mathrm{Ric} < 0$-metric on open manifolds, and thus it does not hurt to use this here. Secondly, we start only in dimension $n \geq 4$. The case $n = 3$, omitted here, can be handled similarly, but needs an extra argument.

Now, if $B \subset M^n$ is a ball, then B contains a closed submanifold N^{n-2} admitting a metric with $\mathrm{Ric} < 0$ and whose normal bundle is trivial. This is easily done in case $n = 4$ using the embedding of a hyperbolic surface in $\mathbb{R}^3 \subset \mathbb{R}^4$.

In higher dimensions we can use induction : S^{n-2}, $n \geq 5$, admits a metric with $\mathrm{Ric} < 0$ and we take the usual embedding $S^{n-2} \hookrightarrow \mathbb{R}^{n-1} \subset \mathbb{R}^n$. (Of course these metrics are not the induced metrics coming from the embedding.)

As mentioned above, we have a metric with $\mathrm{Ric} < 0$ on the open manifold $M \setminus N$, and, in addition, we can get a warped product metric on a tubular neighborhood U of N such that $U \setminus N$ may be identified with $]0, r[\times S^1 \times N$ equipped with $g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g_N$ for some strongly increasing $f \in C^\infty(\mathbb{R}, \mathbb{R}^{>0})$. The manifold $(]0, r[\times S^1, g_{\mathbb{R}} + f^2 \cdot g_{S^1})$ looks like the spreading open end of the pseudosphere, and we would be done if it was possible to "close" this with a metric with Gaussian curvature $K < 0$. But this is impossible by the Gauß-Bonnet theorem.

On the other hand, we can use the additional factor (N, g_N). We can take a singular metric $g_{\mathrm{sing.}}$ with $K < 0$ on the disk D such that the metric near the boundary looks like $(]0, r[\times S^1, g_{\mathbb{R}} + f^2\, g_{S^1})$ with $\{0\} \times S^1 = \partial D(!)$. Now, we can use $\mathrm{Ric}(g_N) < 0$ to smooth the singularities of $g_{\mathrm{sing.}}$ getting a warped product metric with $\mathrm{Ric} < 0$ on $D \times N$ and glue it to $M \setminus U$. Thus, we have closed M again and it is equipped with a metric with $\mathrm{Ric} < 0$. Details and extensions are described in [L4].

We hope that including this rough existence argument already in the introduction motivates the search for refinements (in various directions) as treated in this paper. In the course of describing such results we will meet some important features of how $\mathrm{Ric} < 0$-metrics are "assembled" in general.

I. COLLECTION OF RESULTS

One of the main features of Ric $<$ 0-geometry is that many problems can be condensed into a local one and that, on the other hand, the local solution can be globalized.

In this chapter we start to describe the results available using this method of attack. It turns out that this particular interplay yields insights into the behaviour of Ric $<$ 0-metrics in a natural way.

I.1. General Existence Theorems.

I.1.1. Theorem. — *Each manifold $M^n, n \geq 3$, admits a complete metric g_M with*

$$-a(n) < r(g_M) < -b(n)\,,$$

for some constants $a(n) > b(n) > 0$ depending only on the dimension n.

We also have another result motivated partly by the existence of complete, finite area metrics with $K < -1$ on open surfaces, partly by S.T. Yau's theorem that each complete non-compact manifold with Ric > 0 has infinite volume.

I.1.2. Theorem. — *Each manifold M^n, $n \geq 3$, admits a complete metric g'_M with $r(g'_M) < -1$ and $\mathrm{Vol}(M^n, g'_M) < +\infty$.*

I.1.1 - I.1.2 are proved in [L2].