GEOMETRY OF TOTAL CURVATURE

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Abstract. This is a survey article on geometry of total curvature of complete open 2-dimensional Riemannian manifolds, which was first studied by Cohn-Vossen ([Col, Co2]) and on which after that much progress was made. The article consists of three topics : the ideal boundary, the mass of rays, and the behaviour of distant maximal geodesics.

Résumé. Cet article présente une synthèse sur la géométrie de la courbure totale des surfaces riemanniennes ouvertes, qui fut d'abord étudiée par Cohn-Vossen ([Co1, Co2]), et à propos de laquelle de grands progrès ont été faits ensuite. L'article couvre trois sujets : le bord idéal, la masse des rayons, et le comportement des géodésiques maximales à l'infini.

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INTRODUCTION

The total curvature of a closed Riemannian 2-manifold is determined only by the topology of the manifold. On the other hand, that of a complete open Riemannian 2-manifold is not a topological invariant but depends on the metric. The geometric meaning of the total curvature is an interesting subject. In this article, we survey some of our own results concerning the relations between the total curvature $c(M)$ of M and various geometric properties of M when M is a finitely connected, complete, open and oriented Riemannian 2-manifold.

Gromov [BGS] first defined the ideal boundary and its Tits metric for an n-dimensional Hadamard manifold as the set of equivalence classes of rays with respect to the asymptotic relation and investigated its geometric properties. This turns out to be useful in studying nonpositively curved n-manifolds. Here, the nonpositiveness of the sectional curvature implies that the asymptotic relation, which is originally due to Busemann [Bu], becomes an equivalence relation. However this is not true in general. The emphasis of the present article is that the ideal boundary together with the Tits metric can be constructed for M by a new equivalence relation between rays by using the total curvature. In particular, our construction is a natural generalization of that of Gromov, because both coincide on every Hadamard 2-manifold. It is natural to ask the influence of our Tits metric on the ideal boundary upon the geometric properties of M. The Tits metric defined here can be precisely described in terms of the total curvature of M, which plays an essential role throughout this article.

In Chapter 1, we construct the ideal boundary of M and its generalized Tits metric. For the Euclidean plane, the Tits distance between two points represented by two rays emanating from a common point is just the angle between the initial vectors of these rays. In the general case, we have various geometric properties on the analogy with the Euclidean case. All these properties are connected with the asymptotic behaviour. We apply these to the study of the detailed behaviour of

Busemann functions.

In Chapter 2, we investigate on the mass of rays in M. We view this as the Lebesgue measure $\mathfrak{M}(A_p)$ of the set A_p of all unit vectors which are initial vectors of rays emanating from a point p in M. A pioneering work of Maeda ([Md1], [Md2]) states that the infimum of $\mathfrak{M}(A_p)$ for all $p \in M$ is equal to $2\pi - c(M)$ provided M is a nonnegatively curved Riemannian plane (i.e., a complete nonnegatively curved manifold homeomorphic to $\mathbb{R}^2$). We investigate the asymptotic behaviour of the measure $\mathfrak{M}(A_p)$ for a general M with total curvature as p tends to infinity and the mean of $\mathfrak{M}(A_p)$ with respect to the volume of M.

In Chapter 3, we study the behaviour of maximal geodesics close enough to infinity (i.e., outside a large compact set) in a complete 2-manifold homeomorphic to $\mathbb{R}^2$ with total curvature less than 2π. Such manifolds will be called strict Riemannian planes. Any such maximal geodesic becomes proper as a map of $\mathbb{R}$ into M and has almost the same shape as that of a maximal geodesic in a flat cone. Moreover, we give an estimate for its rotation number and show that it is close to $\pi/(2\pi - c(M))$. Here, we have extended the notion of the rotation number of a closed curve due to Whitney [Wh] to that of a proper curve.

Basic concepts

The total curvature $c(M)$ of an oriented Riemannian 2-manifold M is defined to be the possibly improper integral $\int_M G\,dM$ of the Gaussian curvature G of M with respect to the volume element dM of M. We define the total positive curvature $c_+(M)$ and the total negative curvature $c_-(M)$ by $c_\pm(M) := \int_M G_\pm\,dM$, where $G_-+(p) := \max\{G(p), 0\}$ and $G_--(p) := \max\{-G(p), 0\}$ for $p \in M$. Then, the total curvature $c(M)$ exists if and only if at least one of $c_+(M)$ or $c_-(M)$ is finite. A well-known theorem due to Cohn-Vossen [Co1] states that if M is finitely connected and admits total curvature, then $c(M) \leq 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. When M is infinitely connected and admits total curvature, Huber's theorem [Hu] (cf. [Ba1]) states that $c(M) = -\infty$. Therefore, the total curvature exists if and only if the total positive curvature is finite.

Throughout this article, assume that M is a finitely connected, complete, open and oriented Riemannian 2-manifold admitting total curvature and that all geodesics of M are normal. The finite connectivity of M implies that there exists a homeomorphism $\varphi : M \to N - E$, where N is a closed and oriented 2-manifold and E is a finite subset of N. We call each point in E an *endpoint* of M. For instance, if M is a *Riemannian plane* (i.e., a complete Riemannian 2-manifold homeomorphic to $\mathbb{R}^2$), then N is homeomorphic to S^2 and E consists of a single point in N. A subset U of M is called a *neighbourhood of an endpoint* $e \in E$ if $\varphi(U) \cup \{e\}$ is a neighbourhood of e in N. For each endpoint e of M, we denote by $\mathfrak{U}(e)$ the set of all neighbourhoods of e which are diffeomorphic to closed half-cylinders with smooth boundary. Following Busemann [Bu], we call an element of $\mathfrak{U}(e)$ a *tube* of M.

For any region D of M with piecewise smooth boundary ∂D parameterized positively relative to D, we define the *total geodesic curvature* $\kappa(D)$ by the sum of the integrals of the geodesic curvature of ∂D together with the exterior angles of D at all vertices. Here, we allow $\kappa(D)$ to be infinite. When $\partial D = \phi$ (i.e., $D = M$), we set $\kappa(D) := 0$. The Gauss-Bonnet theorem states that if a region D has piecewise smooth boundary and is compact and finitely connected, then

$$\kappa(D) + c(D) = 2\pi\chi(D) \ .$$

For any region D of M admitting $\kappa(D) + c(D)$ (i.e., so that $\kappa(D)$ and $c(D)$ exist and if both $\kappa(D)$ and $c(D)$ are infinite, they have the same sign), we define

$$\kappa_\infty(D) := 2\pi\chi(D) - \kappa(D) - c(D) \ .$$

A slight generalization of Cohn-Vossen's theorem (cf. [Co2], [Sy5]) states that

$$\kappa_\infty(D) \geq \pi\chi(\partial D) \ ,$$

where $\chi(\partial D)$ is the Euler characteristic of ∂D, namely the number of connected components of ∂D which is homeomorphic to $\mathbb{R}$.

Geometrically, $\kappa_\infty(D)$ may be thought of as the total geodesic curvature of the boundary at infinity of D. This is seen as follows. Let $\{D_j\}$ be a monotone increasing sequence of compact regions with piecewise smooth boundary such that $\cup D_j = D$ and that the inclusion map from each D_j into D is a strong deformation retraction. Since $\chi(D_j) = \chi(D)$ for all j and $\lim_{j\to\infty} c(D_j) = c(D)$, the Gauss-Bonnet theorem implies that

$$\kappa_\infty(D) = 2\pi\chi(D) - \kappa(D) - \lim_{j\to\infty} c(D_j) = \lim_{j\to\infty} \kappa(D_j) - \kappa(D) \ .$$