A CONVERGENCE THEOREM IN THE GEOMETRY OF ALEXANDROV SPACES

Takao YAMAGUCHI

Department of Mathematics
Kyushu University
Fukuoka 812 (Japan)

Abstract. The fibration theorems in Riemannian geometry play an important role in the theory of convergence of Riemannian manifolds. In the present paper, we extend them to the Lipschitz submersion theorem for Alexandrov spaces, and discuss some applications.

Résumé. Les théorèmes de fibration de la géométrie riemannienne jouent un rôle important dans la théorie de la convergence des variétés riemanniennes. Dans cet article, on les étend au cadre lipschitzien des espaces d'Alexandrov, et on donne quelques applications.

M.S.C. Subject Classification Index (1991) : 53C.

TABLE OF CONTENTS

0. INTRODUCTION

An Alexandrov space is a metric space with length structure and with a notion of curvature. In the present paper we study Alexandrov spaces whose curvatures are bounded below. Such a space occurs for instance as the Hausdorff limit of a sequence of Riemannian manifolds with curvature bounded below. Understanding such a limit space is significant in the study of structure of Riemannian manifolds themselves also, and it is a common sense nowadays that there is interplay between Riemannian geometry and the geometry of Alexandrov spaces through Hausdorff convergence.

Recently Burago, Gromov and Perelman [BGP] have made important progress in understanding the geometry of Alexandrov spaces whose curvatures are bounded below. Especially, they proved that the Hausdorff dimension of such a space X is an integer if it is finite and that X contains an open dense set which is a Lipschitz manifold. A recent result in the revised version [BGP2] and also Otsu and Shioya [OS] has extended the later result by showing that such a regular set actually has full measure. Since the notion of Alexandrov space is a generalization of Riemannian manifold, it seems natural to consider the problem : what extent can one extend results in Riemannian geometry to Alexandrov spaces ?

The notion of Hausdorff distance introduced by Gromov [GLP] has brought a number of fruitful results in Riemannian geometry. For instance, the convergence theorems and their extension, the fibration theorems, or other related methods have played important roles in the study of global structure of Riemannian manifolds. The main motivation of this paper is to extend the fibration theorem ([Y]) to Alexandrov spaces. In the Riemannian case we assumed that the limit space is a Riemannian manifold. Here, we employ an Alexandrov space as the limit whose singularities are quite nice in the following sense.

Let X be an n-dimensional complete Alexandrov space with curvature bounded below. In [BGP], it was proved that the space of directions Σ_p at any point $p \in X$

is an $(n-1)$-dimensional Alexandrov space with curvature ≥ 1, and that if Σ_p is Hausdorff close to the unit $(n-1)$-sphere S^{n-1}, then a neighborhood of p is bi-Lipschitz homeomorphic to an open set in $\mathbf{R}^n$. This fact is also characterized by the existence of (n,δ)-strainer. (For details, see Section 1). For $\delta > 0$, we now define the *δ-strain radius* at $p \in X$ as the supremum of $r > 0$ such that there exists an (n,δ)-strainer at p with length r, and the *δ-strain radius of X* by

$$\delta\text{-str. rad }(X) = \inf_{p\in X} \delta\text{-strain radius at } p\,.$$

For instance, X has a positive δ-strain radius if X is compact and if Σ_p is Hausdorff close to S^{n-1} for each $p \in X$.

For every two points x, y in X, a minimal geodesic joining x to y is denoted by xy, and the distance between them by $|xy|$. The angle between minimal geodesics xy and xz is denoted by $\angle yxz$. Under this notaton, we say that a surjective map $f : M \to X$ between Alexandrov spaces is an *ϵ-almost Lipschitz submersion* if

(0.1.1) — it is an ϵ-Hausdorff approximation.

(0.1.2) — For every $p, q \in M$ if θ is the infimum of $\angle qpx$ when x runs over $f^{-1}(f(p))$, then

$$\left| \frac{|f(p)f(q)|}{|pq|} - \sin\theta \right| < \epsilon\,.$$

Remark that the notion of ϵ-almost Lipschitz submersion is a generalization of ϵ-almost Riemannian submersion. Our main result in this paper is as follows.

Theorem 0.2. — *For a given positive integer n and $\mu_0 > 0$, there exist positive numbers $\delta = \delta_n$ and $\epsilon = \epsilon_n(\mu_0)$ satisfying the following. Let X be an n-dimensional complete Alexandrov space with curvature ≥ -1 and with δ-str.rad$(X) > \mu_0$. Then, if the Hausdorff distance between X and a complete Alexandrov space M with curvature ≥ -1 is less than ϵ, then there exists a $\tau(\delta,\epsilon)$-almost Lipschitz submersion $f : M \to X$. Here, $\tau(\delta,\sigma)$ denotes a positive constant depending on n, μ_0 and δ, ϵ and satisfying* $\lim_{\delta,\epsilon\to 0} \tau(\delta,\epsilon) = 0$.

Because of the lack of differentiability in X, it is unclear at present if the map f is actually a locally trivial fiber bundle. The author conjectures that this is true. In fact, in the case when both X and M have natural differentiable structures of class C^1, we can take a locally trivial fibre bundle as the map f. (See Remark 4.20).

Remark 0.3. — Under the same assumption as in Theorem 0.2, for any $x \in X$ let Δ_x denote the diameter of $f^{-1}(x)$. Then, there exists a compact nonnegatively curved Alexandrov space N such that the Hausdorff distance between N and $f^{-1}(x)$ having the metric multiplied by $1/\Delta_x$ is less than $\tau(\delta, \epsilon)$ for every $x \in X$. (See the proof of Theorem 5.1 in §5.)

In Theorem 0.2, if $\dim M = \dim X$ it turns out that

Corollary 0.4. — *Under the same assumptions as in Theorem 0.2, if* $\dim M = n$, *then the map f is $\tau(\delta, \sigma)$-almost isometric in the sense that for every $x, y \in M$*

$$\left| \frac{|f(x)f(y)|}{|xy|} - 1 \right| < \tau(\delta, \sigma) .$$

Remark 0.5. — In [BGP2], Burago, Gromov and Perelman have proved Corollary 0.4 independently. And Wilhelm [W] has obtained Theorem 0.2 under stronger assumptions. He assumed a positive lower bound on the injectivity radius of X and that M is an almost Riamannian space. His constant ϵ in the result depends on the particular choice of X. It should also be noted that Perelman [Pr1] has obtained a version of Corollary 0.4 in the general situation. He proved that any compact Alexandrov space X with curvature ≥ -1 has a small neighborhood with respect to the Hausdorff distance such that every Alexandrov space of the same dimension as X with curvature ≥ -1 which lies in the neighborhood is homeomorphic to X.

By using Corollary 0.4, one can prove the volume convergence.

Corollary 0.6. ([Pr2]) — *Suppose that a sequence (M_i) of n-dimensional compact Alexandrov spaces with curvature ≥ -1 converges to an n-dimensional one, say M, with respect to the Hausdorff distance. Then, the Hausdorff n-measure of M_i converges to that of M.*

As in the Riemannian case, Theorem 0.2 has a number of applications. The results in Riemannian geometry which essentially follow from the splitting theorem ([T],[CG],[GP1],[Y]) and the fibration theorem are still valid for Alexandrov spaces.