CLASSICAL, EXCEPTIONAL, AND EXOTIC HOLONOMIES : A STATUS REPORT

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Abstract. I report on the status of the problem of determining the groups that can occur as the irreducible holonomy of a torsion-free affine connection on some manifold.

Résumé. Il s'agit d'un rapport sur le problème de la détermination des groupes qui peuvent être les groupes d'holonomie de connexions affines sans torsion.

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TABLE OF CONTENTS

INTRODUCTION

0.1. Overview. The goal of this report is to present, in a unified way, what is known about the problem of prescribed holonomy of torsion-free affine connections smooth manifolds.

In §1, I give the fundamental definitions and develop the algebra needed to formulate Berger's criteria which a subgroup of $\mathrm{GL}(T_xM)$ must satisfy if it is to be the holonomy of a torsion-free affine connection on M which is not locally symmetric. I also develop the closely related notion of a torsion-free H-structure. The fundamental strategy is to 'classify' the torsion-free connections with a given holonomy H by first 'classifying' the torsion-free H-structures and then examining the problem of determining for any given torsion-free H-structure, its space of compatible torsion-free connections. In nearly all cases, there is a unique compatible torsion-free connection, but there are important exceptions that are closely related to the second-order homogeneous spaces.

I formulate the classification problem for general torsion-free H-structures as a problem treatable by the methods of Cartan-Kähler theory. Finally, I conclude this section with an appendix containing definitions of the various Spencer constructions that will be needed and a discussion of the history of the classification of the irreducible second-order homogeneous spaces. This classification turns out to be important in the classification of the affine torsion-free holonomies in §3.

In §2, I review Berger's list of the possible irreducible holonomies for pseudo-Riemannian metrics which are not locally symmetric. In the course of the review, I analyze each of the possibilities and determine the degree of generality of each one. Among the notable results are, first, that the group $\mathrm{SO}(n,\mathbb{H})$, which appeared on Berger's original list turns out not to be possible as the holonomy of a torsion-free connection, and, second, that there are two extra cases left off the usual lists (see §2.7-8). These can be viewed as alternate real forms of a group whose compact form is $\mathrm{Sp}(p)\cdot\mathrm{Sp}(1)$, the holonomy group of the so-called 'quaternionic-Kähler' metrics.

In §3, I turn to Berger's list of the possible irreducible holonomies for affine connections which are not locally symmetric and do not preserve any non-zero quadratic form. This list turns out to be quite interesting and the examples display a wide variety of phenomena. Actually, one has to remember that Berger's original list was only meant to cover all but a finite number of the possibilities, leaving open the possibility of a finite number of 'exotic' examples. Moreover, in Berger's original list, there was no attempt to deal with the different possibilities for the holonomy of the central part of the group; Berger's classification deals mainly with the classification of the semi-simple part of the irreducible holonomies. It turns out that the center of the group plays a very important role and gives rise to a wealth of examples that had heretofore not been anticipated.

Finally, in §4, I discuss what is known about the exotic examples so far (see Table 4). Perhaps the most interesting of these examples, aside from the examples in dimension 4 first discussed in [Br2], are the ones associated to the 'exceptional' second-order homogeneous spaces of dimension 16 and 27. For example, a consequence of this is that $E_6^{\mathbb{C}} \subset \mathrm{SL}(27,\mathbb{C})$ can occur as the holonomy of a torsion-free (but not locally symmetric) connection on a complex manifold of dimension 27! Unfortunately, as of this writing, the full classification of the possible exotic examples is far from complete.

0.2. Notation. In this report, I have adopted a slightly non-standard nomenclature for the various groups that are to be discussed. This subsection will serve to fix this notation, which is closely related to that used in [KoNa].

I will need to work with vector spaces over $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$. Conjugation has its standard meaning in $\mathbb{C}$ and $\mathbb{H}$; in each case, the fixed subalgebra is $\mathbb{R}$. The symbol $\mathbb{F}$ will be used to denote any one of these division algebras. The elements of the standard n-space $\mathbb{F}^n$ are to be thought of as columns of elements of $\mathbb{F}$ of height n. It is convenient to take all vector spaces over $\mathbb{H}$ to be *right* vector spaces.

For any vector space V over F, the group of invertible $\mathbb{F}$-linear endomorphisms of V will be denoted $\mathrm{GL}(V,\mathbb{F})$ or just $\mathrm{GL}(V)$ when there is no danger of confusion. The algebra of n-by-n matrices with entries in $\mathbb{F}$ will be denoted by $M_n(\mathbb{F})$. This algebra acts on the left of $\mathbb{F}^n$ by the obvious matrix multiplication, representing the algebra $\mathrm{End}_{\mathbb{F}}(\mathbb{F}^n)$. As usual, let $\mathrm{GL}(n,\mathbb{F}) \subset M_n(\mathbb{F})$ denote the Lie group consisting of the invertible matrices in $M_n(\mathbb{F})$, i.e., $\mathrm{GL}(n,\mathbb{F}) = \mathrm{GL}(\mathbb{F}^n)$. When $\mathbb{F} = \mathbb{R}$, the group $\mathrm{GL}(V)$ has two components and it is occasionally useful to use the notation $\mathrm{GL}^+(V)$ for the identity component. For any $A \in M_n(\mathbb{F})$, define $A^* \in M_n(\mathbb{F})$ to be the conjugate transpose of A, so that $(AB)^* = B^*A^*$ for all $A, B \in M_n(\mathbb{F})$.

For a vector space V over $\mathbb{R}$ or $\mathbb{C}$, the notation $\mathrm{SL}(V)$ has its standard meaning. There is no good notion of a quaternionic determinant; however, the obvious identification $\mathbb{H}^n \simeq \mathbb{R}^{4n}$ induces an embedding $\mathrm{GL}(n,\mathbb{H}) \hookrightarrow \mathrm{GL}(4n,\mathbb{R})$ and the subgroup $\mathrm{SL}(n,\mathbb{H}) \subset \mathrm{GL}(n,\mathbb{H})$ is then defined by $\mathrm{SL}(n,\mathbb{H}) = \mathrm{GL}(n,\mathbb{H}) \cap \mathrm{SL}(4n,\mathbb{R})$. Note that $\mathrm{SL}(n,\mathbb{H})$ has codimension 1 (not 4) in $\mathrm{GL}(n,\mathbb{H})$. In Chevalley's nomenclature, $\mathrm{SL}(n,\mathbb{H})$, which is a real form of $\mathrm{SL}(2n,\mathbb{C})$, is denoted $\mathrm{SU}^*(2n)$. My notation for the other real forms of $\mathrm{SL}(n,\mathbb{C})$ are the standard ones: $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SU}(p,q) = \{\, A \in \mathrm{SL}(n,\mathbb{C}) \,|\, A^*\, I_{p,q}\, A = I_{p,q} \,\}$. For simplicity, $\mathrm{SU}(n)$ denotes $\mathrm{SU}(n,0)$.

When $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ and Q is a non-degenerate quadratic form on a vector space V over $\mathbb{F}$, the slightly non-standard usage $\mathrm{SO}(V,Q)$ (respectively, $\mathrm{CO}(V,Q)$) will refer to the identity component of the subgroup of $\mathrm{GL}(V)$ that fixes Q (respectively, that fixes Q up to a multiple). The notations $\mathrm{SO}(p,q)$ ($= \mathrm{SO}(p)$ when $q = 0$) and $\mathrm{CO}(p,q)$ ($= \mathrm{CO}(p)$ when $q = 0$) denote the identity components of the standard subgroups of $\mathrm{GL}(p{+}q,\mathbb{R})$, while $\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{CO}(n,\mathbb{C})$ denote the standard subgroups of $\mathrm{GL}(n,\mathbb{C})$. Finally, $\mathrm{SO}(n,\mathbb{H})$ stands for the subgroup consisting of those $A \in \mathrm{GL}(n,\mathbb{H})$ that satisfy $A^*\, iI_n\, A = iI_n$. In Chevalley's nomenclature, $\mathrm{SO}(n,\mathbb{H})$, which is a real form of $\mathrm{SO}(2n,\mathbb{C})$, is denoted $\mathrm{SO}^*(2n)$.

Finally, when $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ and Ω is a non-degenerate skew-symmetric bilinear form on a vector space V over $\mathbb{F}$, the notation $\mathrm{Sp}(V,\Omega)$ (respectively, $\mathrm{CSp}(V,\Omega)$) will stand for the subgroup of $\mathrm{GL}(V)$ that fixes Ω (respectively, that fixes Ω up to a multiple.) The notations $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{CSp}(n,\mathbb{R})$ denote the standard subgroups of $\mathrm{GL}(2n,\mathbb{R})$ while $\mathrm{Sp}(n,\mathbb{C})$ and $\mathrm{CSp}(n,\mathbb{C})$ denote the standard subgroups of $\mathrm{GL}(2n,\mathbb{C})$. (In Chevalley's notation, $\mathrm{Sp}(n,\mathbb{R})$ is denoted by $\mathrm{Sp}^*(n)$.) As for the other real forms of $\mathrm{Sp}(n,\mathbb{C})$, I use the usual $\mathrm{Sp}(p,q)$ to denote the subgroup of $\mathrm{GL}(p{+}q,\mathbb{H})$ consisting of those matrices $A \in M_{p+q}(\mathbb{H})$ that satisfy $A^*\, I_{p,q}\, A = I_{p,q}$, with $\mathrm{Sp}(n,0)$ abbreviated to $\mathrm{Sp}(n)$.

Now define the following subspaces

$$S_n(\mathbb{R}) = \{\ A \in M_n(\mathbb{R}) \bigm| A = {}^tA\ \}$$