Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 148
Fascicule 1

2020

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 99-132

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 148, mars 2020

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Stéphane SEURET

Bull. Soc. Math. France
148 (1), 2020, p. 99-132

UPPER BOUNDS FOR COURANT-SHARP NEUMANN AND ROBIN EIGENVALUES

by Katie Gittins & Corentin Léna

Abstract. — We consider the eigenvalues of the Laplacian, with a Neumann or Robin boundary condition, on an open, bounded, connected set in $\mathbb{R}^n$ with a C^2 boundary. We obtain upper bounds for the eigenvalues that have a corresponding eigenfunction that achieves equality in Courant's Nodal Domain theorem. In the case where the set is also assumed to be convex, we obtain explicit upper bounds in terms of some of the geometric quantities of the set.

Résumé (*Majoration des valeurs propres Courant strictes de Neumann et Robin*). — Nous considérons les valeurs propres du laplacien sur un ouvert borné connexe de $\mathbb{R}^n$ à bord C^2, avec condition au bord de Neumann ou de Robin. Nous majorons celles qui ont une fonction propre dont le nombre de domaines nodaux atteint la borne de Courant (dites *Courant strictes*). Lorsque l'ouvert est convexe, nous présentons une majoration explicite en fonction de grandeurs géométriques.

1. Introduction

1.1. Statement of the problem. — Let Ω be an open, bounded, connected set in $\mathbb{R}^n$, $n \geq 2$, with a Lipschitz boundary $\partial\Omega$. Consider the Neumann Laplacian

Texte reçu le 6 novembre 2018, modifié le 8 mai 2019, accepté le 20 mai 2019.

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Mathematical subject classification (2010). — 35P15, 49R05, 35P05.

Key words and phrases. — Courant's nodal domain theorem, Neumann eigenvalues, Robin eigenvalues, Euclidean domains.

doi:10.24033/bsmf.2800

acting on $L^2(\Omega)$ and note that it has discrete spectrum since Ω is bounded. The Neumann eigenvalues of Ω can hence be written in a non-decreasing sequence, counted with multiplicity,

$$0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \cdots \leq \mu_k(\Omega) \leq \ldots,$$

where the only accumulation point is $+\infty$.

By Courant's Nodal Domain theorem, any eigenfunction corresponding to $\mu_k(\Omega)$ has at most k nodal domains. If u_k is an eigenfunction corresponding to $\mu_k(\Omega)$ with k nodal domains, then we call it a Courant-sharp eigenfunction. In this case, we also call $\mu_k(\Omega)$ a Courant-sharp eigenvalue of Ω.

The Courant-sharp property was first considered by Pleijel [15] in 1956 for the Dirichlet Laplacian. In particular, Pleijel proved that there are only finitely many Courant-sharp Dirichlet eigenvalues of a bounded, planar domain with a sufficiently regular boundary. See [6, 14] for generalisations of Pleijel's theorem to higher dimensions and other geometric settings. Following from Pleijel's result, for a given domain, natural questions are, how many such eigenvalues are there and how large are they?

The recent articles [5, 2] consider these questions and give upper bounds for the largest Courant-sharp Dirichlet eigenvalue and the number of such eigenvalues in terms of some of the geometric quantities of the underlying domain. In Theorem 1.3 of [5], the authors bound this number using the area, perimeter, maximal curvature and minimal cut-distance to the boundary, for a set in $\mathbb{R}^2$ which is sufficiently regular but not necessarily convex (the cut-distance will be defined in Section 3). In [2], such geometric upper bounds are obtained for an open set in $\mathbb{R}^n$ with finite Lebesgue measure. In the case where the domain is convex, the upper bound given in Example 1 of [2] could be expressed in terms of the isoperimetric ratio of the domain. From this, one can deduce that if the domain has a large number of Courant-sharp Dirichlet eigenvalues then its isoperimetric ratio is also large.

It was shown recently in [10] that if Ω is an open, bounded, connected set in $\mathbb{R}^n$ with a $C^{1,1}$ boundary, then the Neumann Laplacian acting on $L^2(\Omega)$ has finitely many Courant-sharp eigenvalues (we refer to [10] for a description of prior results). As mentioned in [5], the aforementioned questions are also interesting for the Courant-sharp eigenvalues of the Neumann Laplacian.

1.2. Goal of the article. — The aim of the present article is to obtain upper bounds for the Courant-sharp Neumann eigenvalues in the case where $\Omega \subset \mathbb{R}^n$ is open, bounded, and connected with a C^2 boundary. In the case where Ω is also convex, we obtain explicit upper bounds for the Courant-sharp Neumann eigenvalues of Ω, and for the number of such eigenvalues, in terms of some of the geometric quantities of Ω. These results correspond to some of those mentioned above for the Dirichlet case, with some additional hypotheses due to the difficulties in handling the Neumann boundary condition.

We follow the same strategy that was used in [10]. This involves distinguishing between the nodal domains of a Courant-sharp eigenfunction u for which the majority of the L^2 norm of u comes from the interior (bulk domains) and those for which the majority of the L^2 norm of u comes from near the boundary (boundary domains), and then obtaining upper bounds for the number of each type of nodal domain. In the first case, the argument used by Pleijel [15], which rests upon the Faber-Krahn inequality, can be used as the eigenfunction in a bulk domain almost satisfies a Dirichlet boundary condition. For the boundary domains, it is not possible to employ the same argument as Pleijel as these nodal domains have mixed Dirichlet-Neumann boundary conditions so the Faber-Krahn inequality cannot be employed. The strategy of [10] to deal with the boundary domains is to locally straighten the boundary of the domain Ω and then to reflect the nodal domain in order to obtain a new domain that almost satisfies a Dirichlet boundary condition. One then has to compare the L^2 norm of the gradient of an eigenfunction corresponding to a Courant-sharp eigenvalue on the boundary domain to the L^2 norm of the gradient of the reflected eigenfunction on the reflected domain. See Section 5.

We restrict our attention to Euclidean domains with a C^2 boundary. We can then make use of tubular coordinates in order to set up and describe the reflection procedure explicitly. This allows us to keep explicit control of the constants appearing in the aforementioned estimates in order to obtain estimates for the Courant-sharp Neumann eigenvalues.

In Proposition 7.1, we obtain an upper bound for the Courant-sharp Neumann eigenvalues of Ω in terms of some of its geometric quantities. More specifically, it depends on $|\Omega|$ the area of Ω, $\rho(\Omega)$ the square-root of the isoperimetric ratio, $t_+(\Omega)$ the smallest radius of curvature of the boundary, and the cut distance to the boundary (see Section 3 for precise definitions of the latter quantities).

A simpler presentation of this upper bound is possible in the case where Ω is convex, since one of the additional conditions in the general case is no longer required (see Section 8). In addition, we obtain an upper bound for the number of such eigenvalues by using the upper bound for the Neumann counting function which is proved in Appendix A. In particular, we have the following proposition.

Proposition 1.1. — *Let Ω be an open, bounded, convex set in $\mathbb{R}^2$ with a C^2 boundary. There exist constants $C > 0$ and $C' > 0$, that do not depend on Ω, such that for any Courant-sharp eigenvalue $\mu_k(\Omega)$,*

$$\mu_k(\Omega) \leq C \left(\frac{|\Omega|}{t_+(\Omega)^4} + \frac{\rho(\Omega)^8}{|\Omega|} \right) \tag{1}$$

and

$$k \leq C' \left(\frac{|\Omega|^2}{t_+(\Omega)^4} + \rho(\Omega)^8 \right). \tag{2}$$

We note that the left-hand side and right-hand side of Inequality (1) have the same homogeneity with respect to scaling, and that Inequality (2) is scaling invariant. In addition, in Section 8, we obtain an upper bound for the Courant-sharp Neumann eigenvalues which also depends upon the diameter of Ω.

By Proposition 1.1, we then observe that if Ω is a sufficiently regular convex set with a large number of Courant-sharp eigenvalues, it has a large isoperimetric ratio or a large curvature at some point of its boundary (or both). If we additionally assume that $\mu_k(\Omega)$ is large compared with $|\Omega|t_+(\Omega)^{-4}$, we can conclude that the isoperimetric ratio is large. We note that a large isoperimetric ratio is enough to generate a large number of Courant-sharp Neumann eigenvalues. Indeed, this is the case for a rectangle $(0,1)\times(0,L)$ with large L. By contrast, to the best of the authors' knowledge, it is not known whether a boundary point with large curvature alone can generate many Courant-sharp eigenvalues. It could be interesting to investigate this further.

By $-\Delta_\Omega^\beta$, we denote the Laplacian on Ω with the following Robin boundary condition

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \text{ on } \partial\Omega,$$

where $\frac{\partial u}{\partial \nu}$ is the exterior normal derivative and $\beta : \partial\Omega \to \mathbb{R}$ is a non-negative, Lipschitz continuous function. We denote the corresponding eigenvalues by $(\mu_k(\Omega,\beta))_{k\geq 1}$. It was shown in [10] that there are finitely many Courant-sharp eigenvalues of $-\Delta_\Omega^\beta$. By monotonicity of the Robin eigenvalues with respect to β, we obtain the same results for the Courant-sharp Robin eigenvalues (see Subsection 2.2).

In addition, we obtain analogous results to those mentioned above for any dimension $n \geq 3$, namely Propositions 9.2, 9.3 and 9.4 in Section 9.

1.3. Organisation of the article. — In Section 2, we show that in order to obtain upper bounds for the largest Courant-sharp eigenvalue μ, it is sufficient to obtain upper bounds for the number of nodal domains and the remainder of the Dirichlet counting function. Estimates for the latter are obtained in Section 6. To deal with the former, we first consider the 2-dimensional case and set up tubular coordinates in Section 3. Following [10], we then define cut-off functions in Section 4 that allow us to distinguish between bulk and boundary domains. In Subsection 5.1 we perform the straightening of the boundary procedure and obtain the desired estimates. We then use these estimates in Subsection 5.2 to obtain an explicit upper bound for the number of nodal domains. In Subsection 5.3 and Subsection 5.4, by taking the geometry of the domain into account, we improve the estimates from Subsection 5.1 in special cases. We then combine all of the preceding results in Section 7 to obtain an upper bound for the largest Courant-sharp eigenvalue. In Section 8, we obtain explicit upper bounds for the largest Courant-sharp eigenvalue and the number of Courant-sharp eigenvalues of an open and convex planar domain with a C^2 boundary that involve some of its geometric quantities. In particular, we prove Proposition 1.1. In Section 9, we obtain analogous results in arbitrary dimension $n \geq 3$. In Appendix A, we prove an upper bound for the Neumann counting function of a convex set, which is used in the two preceding sections to control the number of Courant-sharp eigenvalues.

2. Preliminaries

2.1. Strategy for Courant-sharp Neumann eigenvalues. — For $\mu > 0$, we define the Neumann counting function as follows:

$$N_\Omega^N(\mu) := \sharp\{k \in \mathbb{N}^* \ : \ \mu_k(\Omega) < \mu\}.$$

Let $(\lambda_k(\Omega))_{k\geq 1}$ denote the Dirichlet eigenvalues of the Laplacian on Ω. By the min-max characterisations of the Neumann and Dirichlet eigenvalues, we have, for $k \in \mathbb{N}^*$, that

$$\mu_k(\Omega) \leq \lambda_k(\Omega). \tag{3}$$

For $\mu > 0$, we define the Dirichlet counting function:

$$N_\Omega^D(\mu) := \sharp\{k \in \mathbb{N}^* \ : \ \lambda_k(\Omega) < \mu\},$$

and the corresponding *remainder* $R_\Omega^D(\mu)$ such that

$$N_\Omega^D(\mu) = \frac{\omega_n|\Omega|}{(2\pi)^n}\mu^{n/2} - R_\Omega^D(\mu), \tag{4}$$

where ω_n denotes the Lebesgue measure of a ball of radius 1 in $\mathbb{R}^n$, and the first term in the right-hand side of Equation (4) corresponds to Weyl's law. By (3), we have

$$N_\Omega^N(\mu) \geq N_\Omega^D(\mu),$$

and therefore

$$N_\Omega^N(\mu) \geq \frac{\omega_n|\Omega|}{(2\pi)^n}\mu^{n/2} - R_\Omega^D(\mu).$$

Consider an eigenpair (μ, u) for the Neumann Laplacian, and denote by $\nu(u)$ the number of its nodal domains. If u is a Courant-sharp eigenfunction associated with $\mu > 0$, $\mu = \mu_k(\Omega)$ with $\nu(u) = k$. On the other hand, Courant's Nodal Domain theorem implies that $\mu_{k-1}(\Omega) < \mu_k(\Omega)$, so that $N_\Omega^N(\mu) = k-1$. We therefore have

$$N_\Omega^N(\mu) - \nu(u) < 0. \tag{5}$$

Hence, in order to obtain upper bounds for μ, we require upper bounds for $\nu(u)$ and $R_\Omega^D(\mu)$. These will be obtained in Sections 5, 6 respectively.