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**CONIC SINGULARITIES METRICS WITH PRESCRIBED
SCALAR CURVATURE: A PRIORI ESTIMATES
FOR NORMAL CROSSING DIVISORS**

BY LI LONG, WANG JIAN & ZHENG KAI

ABSTRACT. — The purpose of this paper is to prove a priori estimates for constant scalar curvature Kähler metrics with conic singularities along normal crossing divisors. The zeroth order estimates are proved by a reformulated version of Alexandrov's maximum principle. The higher order estimates follow from Chen-Cheng's framework, equipped with new techniques to handle the singularities. Finally, we extend these estimates to the twisted equations.

RÉSUMÉ (*Métriques singulières coniques avec courbure scalaire prescrite: estimées à priori pour les diviseurs à croisements normaux*). — Le but de ce papier est de montrer les estimées à priori pour la métrique Kählerienne à courbure scalaire constante avec les singularités coniques lelong les diviseurs à croisements normaux. La estimée d'ordre zéro est démontrée par une version reformulée du principe maximal d'Alexandrov. Les estimées d'ordre supérieure suivent des travaux de Chen-Cheng équipés avec les nouvelles techniques pour traiter les singularités. Finalement, on étend les estimées aux les équations tordues.

Dedicated to Prof. Jean-Pierre Demailly

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1. Introduction

Recently Chen and Cheng [6, 7, 8] established the a priori estimates for the constant scalar curvature Kähler (cscK) metrics equation, which are fundamental to the Yau-Tian-Donaldson conjecture on the existence of the cscK metrics. Their estimates led to the resolution of the properness conjecture and Donaldson's geodesic stability conjecture.

Our goal is to prove a singular version of the Yau-Tian-Donaldson conjecture, and this first paper aims at generalising Chen-Cheng's a priori estimates to *log-smooth klt pairs*. That is to say, our metrics develop cone like singularities along normal crossing divisors. In the subsequent papers, we will discuss the existence problem for cscK metrics on *singular klt pairs*.

Let (X, D) be a log smooth klt pair, where $D := \sum_{k=1}^d (1 - \beta_k) D_k$ is an \mathbb{R} -divisor on the compact Kähler manifold X . Here the index $\beta := \{\beta_k\}_{k=1}^d$ is a collection of angles $0 < \beta_k < 1$. For some $0 < \alpha < \min_k \{\frac{1}{\beta_k} - 1, 1\}$, we consider the conic Hölder space $\mathcal{C}^{2,\alpha,\beta}$ first introduced by Donaldson [12].

Suppose $(\varphi, F) \in \mathcal{C}^{2,\alpha,\beta}$ is a pair satisfying the conic cscK equation (Definition 2.1). Denote $H_\beta(\varphi)$ by the entropy of a conic Kähler potential φ , with respect to the background metric ω_β (equation (5)):

$$H_\beta(\varphi) := \int_X \log \frac{\omega_\varphi^n}{\omega_\beta^n} \omega_\varphi^n.$$

The following estimates can then be proven.

THEOREM 1.1. — *Let (φ, F) be a $\mathcal{C}^{2,\alpha,\beta}$ -conic cscK pair on (X, D) . Suppose that its entropy $H_\beta(\varphi)$ is bounded by a uniform constant C . Then there exists another uniform constant \tilde{C} , such that the following holds:*

(i) *the C^0 -estimate*

$$\|\varphi\|_0 < \tilde{C};$$

(ii) *the non-degeneracy estimate*

$$-\tilde{C} < F < \tilde{C};$$

(iii) *the gradient F -estimate and the C^2 -estimate*

$$\max_X |\nabla_\varphi F|_\varphi + \max_X \text{tr}_{\omega_\beta} \omega_\varphi < \tilde{C}.$$

The C^0 estimate is proved in Theorem 4.5 and Corollary 4.6, and the non-degeneracy estimate is proved in Lemma 4.8. Comparing with Chen-Cheng's work [6], the new difficulty is that Alexandrov's maximum principle (AMP) fails in the conic case. More precisely, the constant appearing in Chen-Cheng's estimate depends on the diameter of the coordinate balls, on which we applied this maximum principle. However, the diameter has to become smaller and

smaller when the ball is approaching the divisors, and we then lose the control of the constant.

In order to overcome this difficulty, we developed a new version of AMP, the *generalised Alexandrov’s maximum principle* (GAMP) in Theorem 3.5. The key observation is that this maximum principle still works for a function u if the upper contact set Γ_u^+ of this function is disjoint from the singular locus of u . Therefore, we can utilise this new maximum principle in the estimates, by adding an extra “extremely” pseudo-convex auxiliary function near the divisor.

The integral method on a compact manifold (iteration without assuming a uniform Sobolev constant on varying metrics) from Chen-He [9] is important in Chen-Cheng’s work. Following their basic framework, the gradient F -estimate and the C^2 -estimate in the conic setting are also proved via the following $W^{2,p}$ type estimate.

THEOREM 1.2 (Theorem 5.1). — *Let (φ, F) be a $\mathcal{C}^{2,\alpha,\beta}$ -conic cscK pair on (X, D) . For any $1 < p < +\infty$, there exists a uniform constant C' such that*

$$\int_X (\text{tr}_{\omega_\beta} \omega_\varphi)^p \omega_\beta^n < C'.$$

Here this constant C' only depends on p , $\|\varphi\|_0$, $\|F\|_0$ and the conic background metrics ω_β, ω_D on (X, D) .

Unlike the smooth case, our $W^{2,p}$ and C^2 -estimates have to switch back and forth between the two different conic background metrics, ω_β and the Donaldson metric ω_D [12]. The advantage of the Donaldson metric ω_D is that the growth rate of its bisectional curvature can be explicitly computed, and hence quite clear near the divisors. Moreover, notice that L^p -norm $\|n + \Delta\varphi\|_{L^p(\omega_\beta^n)}$ is uniformly equivalent to $\|n + \Delta\varphi\|_{L^p(\omega_D^n)}$, since ω_D and ω_β are quasi-isometric to each other.

For later purposes, we assumed that the cscK pair (φ, F) lies in the conic Hölder space $\mathcal{C}^{2,\alpha,\beta}$. In practice, all these a priori estimates work through perfectly if we only assume $(\varphi, F) \in \mathcal{C}_\beta^{1,\bar{1}}$ (Definition 4.1) at the very beginning.

In order to investigate the existence of the conic cscK metrics, we are further led to study the following continuity path on $Y := X \setminus \text{Supp}(D)$:

$$t(R_\varphi - \underline{R}_\beta) = (1 - t)(\text{tr}_\varphi \tau_\beta - \underline{\tau}_\beta),$$

for $t \in [0, 1]$, where τ is a closed $(1, 1)$ -form varying in a fixed Kähler class. More precisely, we assume that $\tau := \tau_0 + dd^c f \geq 0$ for some fixed smooth $(1, 1)$ -form τ_0 on Y with $|\tau_0|_{\omega_\beta}$ uniformly bounded, and the function f satisfies

$$\sup_X f = 0; \quad \int_X e^{-p_0 f} \omega_\beta^n < +\infty, \quad \text{for some } p_0 > 1.$$

With these constraints, a triple $(\varphi, F, f) \in \mathcal{C}^{2,\alpha,\beta}$ is a solution to the twisted conic-cscK equation if they satisfy equations (109) and (110). Then we extend our estimates to the following.

THEOREM 1.3. — *Let $(\varphi, F, f) \in \mathcal{C}^{2,\alpha,\beta}$ be a triple of the twisted equations. Suppose the entropy $H_\beta(\varphi)$ is bounded by a uniform constant C . Then there exists another uniform constant C'' , such that the following holds:*

(iv) *the C^0 -estimate*

$$\|\varphi\|_0 < C'';$$

(v) *the non-degeneracy estimate*

$$-C'' < F < C'';$$

(vi) *there exists a constant k_n , only depending on the dimension n , such that if $p_0 > k_n$, then we have*

$$|\nabla_\varphi(F + f)|_\varphi < C''.$$

Since the upper bound of the $(1, 1)$ form τ is out of control in the twisted case, we no longer expect the C^2 -estimate directly. However, the C^2 -estimate can be actually deduced from the gradient F -estimate, by a conic version of Chen-He's integral estimate.

Similarly, the gradient F -estimate is proved via the following $W^{2,p}$ -estimate.

THEOREM 1.4 (Theorem 7.4). — *Let $(\varphi, F, f) \in \mathcal{C}^{2,\alpha,\beta}$ be a triple of the twisted equations. For any $p \geq 1$ there exists a constant \hat{C} satisfying*

$$\int_X e^{-(p-1)f} (\mathrm{tr}_{\omega_\beta} \omega_\varphi)^p \omega_\beta^n \leq \hat{C},$$

where the constant \hat{C} depends only on p , p_0 (uniform if p_0 is bounded away from 1), $\|\varphi\|_0$, $\|F + f\|_0$, and the background metrics ω_β and ω_D .

More generally, our zero order estimates (the C^0 and non-degeneracy estimates) can also be used on singular klt pairs. In fact, after being pulled back to a log-resolution, the metric has conic singularities along normal crossing divisors, but it is possibly degenerate along some exceptional divisors. Therefore, we can apply our techniques on the resolution, and then the estimates follow from GAMP again.

Furthermore, the higher order estimates on singular klt pairs, like the $W^{2,p}$ -estimate and C^2 -estimate, can also be realised on a compact domain away from the divisor. These topics will be discussed in a sequel paper.

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2. Preliminary

Let (X, ω) be an n -dimensional compact complex Kähler manifold. Suppose $D := \sum_{k=1}^d (1 - \beta_k) D_k$ is an \mathbb{R} -divisor on X with simple normal crossing support, and the angle $\beta_k \in (0, 1)$ for all k . We call such a pair (X, D) a *log smooth klt pair*.

Near a point p on the support of D , there exists a holomorphic coordinate system $\{z_i\}$ such that the $\text{Supp}(D)$ is defined by the equation $\{z_1 \cdots z_d = 0\}$. Then a model conic metric ω_{cone} with cone angle β_k along D_k can be written as

$$(1) \quad \omega_{cone} := \sum_{k=1}^d \frac{\sqrt{-1} dz_k \wedge d\bar{z}_k}{|z_k|^{2-2\beta_k}} + \sum_{k=d+1}^n \sqrt{-1} dz_k \wedge d\bar{z}_k.$$

A positive current $\omega_\varphi := \omega + dd^c \varphi$ is said to be a conic Kähler metric with cone angle β_k along D_k , if it is smooth on $X \setminus (\bigcup D_k)$ and quasi-isometric to the model metric ω_{cone} near each point $p \in \text{Supp}(D)$, i.e. it satisfies

$$C^{-1} \omega_{cone} \leq \omega_\varphi \leq C \omega_{cone},$$

for some constant $C > 0$.

When the divisor D is a smooth hypersurface, Donaldson [12] introduced the conic Hölder spaces for potentials $\varphi \in \mathcal{C}^{0, \alpha, \beta}$ or $\varphi \in \mathcal{C}^{2, \alpha, \beta}$, for $\alpha \in (0, 1)$ with $\alpha\beta < 1 - \beta$. Moreover, he proved a version of the Schauder estimate [12, 4] for the conic Laplacian operator.

2.1. Conic Kähler-Einstein metrics. — Let $(L_k, \phi_k), 1 \leq k \leq d$ be a set of hermitian line bundles with non-trivial sections $s_k \in H^0(X, L_k)$. Assume that the divisors $D_k := \{s_k = 0\}$ are smooth and intersect transversally. For simplicity, we write the norm of the sections as $|s_k|^2 := |s_k|^2 e^{-\phi_k}$. Then a simple example