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A LEFSCHETZ THEOREM FOR OVERCONVERGENT ISOCRYSTALS WITH FROBENIUS STRUCTURE

BY TOMOYUKI ABE AND HÉLÈNE ESNAULT

ABSTRACT. – We show a Lefschetz theorem for irreducible overconvergent F -isocrystals on smooth varieties defined over a finite field. We derive several consequences from it.

RÉSUMÉ. – Nous montrons un théorème de Lefschetz pour les F -isocristaux surconvergeants sur des variétés lisses définies sur un corps fini. Nous en tirons plusieurs conséquences.

Introduction

Let X_0 be a normal geometrically connected scheme of finite type defined over a finite field \mathbb{F}_q , let \mathcal{F}_0 be an irreducible lisse Weil $\overline{\mathbb{Q}}_\ell$ -sheaf with finite determinant (thus in fact \mathcal{F}_0 is an étale sheaf as well), where $\ell \neq p = \text{char}(\mathbb{F}_q)$. In Weil II [8, Conj. 1.2.10], Deligne conjectured the following.

- (i) The sheaf \mathcal{F}_0 is of weight 0.
- (ii) There is a number field $E \subset \overline{\mathbb{Q}}_\ell$ such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, the characteristic polynomial $f_x(\mathcal{F}_0, t) := \det(1 - tF_x \mid \mathcal{F}_{0,\bar{x}})$ lies in $E[t]$, where F_x is the geometric Frobenius of x .
- (iii) For any $\ell' \neq p$ and any embedding $\sigma: E \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$, for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, any root of $\sigma f_x(\mathcal{F}_0, t)$ is an ℓ' -adic unit.
- (iv) For any σ as in (iii), there is an irreducible $\overline{\mathbb{Q}}_{\ell'}$ -lisse sheaf $\mathcal{F}_{0,\sigma}$, called the σ -companion, such that $\sigma f_x(\mathcal{F}_0, t) = f_x(\mathcal{F}_{0,\sigma}, t)$.
- (v) There is a crystalline version of (iv).

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Deligne's conjectures (i)–(iv) have been proved by Lafforgue [18, Thm. VII.6] when X_0 is a smooth curve, as a corollary of the Langlands correspondence, which is proven showing that automorphic forms are in some sense motivic.

When X_0 has dimension at least 2, the automorphic side on which one could rely to prove Deligne's conjectures is not available: there is no theory of automorphic forms in higher dimension. The problem then becomes how to reduce, by geometry, the statements to dimension 1. For (i) and (iii), one proves a Lefschetz theorem (see [12, Thm. 2.15], [10, 1.5–1.9], [13, B1]):

0.1. THEOREM. – *On X_0 smooth, for any closed point x_0 , there exists a smooth curve C_0 and a morphism $C_0 \rightarrow X_0$ such that $x_0 \rightarrow X_0$ lifts to $x_0 \rightarrow C_0$, and such that the restriction of \mathcal{F}_0 to C_0 remains irreducible.*

Using Theorem 0.1, Deligne proved (ii) ([10, Thm. 3.1]), and Drinfeld, using (ii), proved (iv) in ([12, Thm. 1.1]), assuming in addition X_0 to be smooth. In particular Drinfeld proved in [12, Thm. 2.5] the following key theorem.

0.2. THEOREM. – *If X_0 is smooth, given a number field $E \subset \overline{\mathbb{Q}_\ell}$, and a place λ of E dividing ℓ , a collection of polynomials $f_x(t) \in E[t]$ indexed by any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, such that the following two conditions are satisfied:*

- (i) *for any smooth curve C_0 with a morphism $C_0 \rightarrow X_0$ and any $n > 0$ and $x \in C_0(\mathbb{F}_{q^n})$, there exists a lisse étale $\overline{\mathbb{Q}_\ell}$ -sheaf $\mathcal{F}_0^{C_0}$ on C_0 with monodromy in $\mathrm{GL}(r, E_\lambda)$ such that $f_x(t) = f_x(\mathcal{F}_0^{C_0}, t)$, where E_λ is the completion of E with respect to the place λ ;*
- (ii) *there exists a finite étale cover $X'_0 \rightarrow X_0$ such that $\mathcal{F}_0^{C_0}$ is tame on all C_0 factoring through $X'_0 \rightarrow X_0$.*

Then there exists a lisse $\overline{\mathbb{Q}_\ell}$ -sheaf \mathcal{F}_0 on X_0 with monodromy in $\mathrm{GL}(r, E_\lambda)$, such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, $f_x(t) = f_x(\mathcal{F}_0, t) \in E[t]$.

Further, to realize the assumptions of Theorem 0.2 in order to show the existence of $\mathcal{F}_{0,\sigma}$, Drinfeld uses Theorem 0.1 in [12, 4.1]. He constructs step by step the residual representations with monodromy in $\mathrm{GL}(r, \mathcal{O}_{E_\lambda}/\mathfrak{m}^n)$ for n growing, where \mathcal{O}_{E_λ} is the ring of integers of E_λ and \mathfrak{m} is its maximal ideal.

The formulation of (v) has been made explicit by Crew [7, 4.13]. The conjecture is that the crystalline category analogous to the category of Weil $\overline{\mathbb{Q}_\ell}$ -sheaves is the category of overconvergent F -isocrystals (see Section 1.1 for the definitions). In order to emphasize the analogy between ℓ and p , one slightly reformulates the definition of companions. One replaces σ in (iii) by an isomorphism $\sigma: \overline{\mathbb{Q}_\ell} \rightarrow \overline{\mathbb{Q}_{\ell'}}$ (see [13, Thm. 4.4]), and keeps (iv) as it is. Here ℓ, ℓ' are any two prime numbers. For $\ell' = p, \ell \neq p$, and \mathcal{F} an irreducible lisse $\overline{\mathbb{Q}_\ell}$ -sheaf, one requests the existence of an overconvergent F -isocrystal M_0 on X_0 with eigenpolynomial $f_x(M_0, t)$ such that $f_x(M_0, t) = \sigma f_x(\mathcal{F}, t) \in \sigma(E)[t]$ for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, where $f_x(\mathcal{F}, t)$ is the characteristic polynomial of the geometric Frobenius at x on \mathcal{F} (see Section 1.4 for the definitions). The isocrystal M_0 is called a σ -companion to \mathcal{F} . Given an irreducible overconvergent F -isocrystal M_0 with finite determinant on X_0 , and σ as above, a lisse ℓ -adic Weil sheaf \mathcal{F} on X_0 is a σ^{-1} -companion if $\sigma^{-1} f_x(M_0, t) = f_x(\mathcal{F}, t) \in E[t]$

at $x \in X_0(\mathbb{F}_{q^n})$ (see Definition 1.4). Similarly we can assume $p = \ell = \ell'$. This way we can talk on ℓ -adic or p -adic companions of either an M_0 or an \mathcal{F} . The companion correspondence should preserve the notions of irreducibility, finiteness of the determinant, the eigenpolynomials at closed points of X_0 , and the ramification.

The conjecture in the strong form has been proven by the first author when X_0 is a smooth curve ([1, Intro. Thm.]). The aim of this article is to prove the following analog of Theorem 0.1 on X smooth.

0.3. THEOREM (Theorem 3.10). – *Let X_0 be a smooth geometrically connected scheme over \mathbb{F}_q . Let M_0 be an irreducible overconvergent F -isocrystal with finite determinant. Then for every closed point $x_0 \rightarrow X_0$, there exists a smooth irreducible curve C_0 defined over k , together with a morphism $C_0 \rightarrow X_0$ and a factorization $x_0 \rightarrow C_0 \rightarrow X_0$, such that the pull-back of M_0 to C_0 is irreducible.*

Theorem 0.3, together with [10, Rmk. 3.10], footnote 2, and [1, Thm. 4.2.2] enable one to conclude that there is a number field $E \subset \overline{\mathbb{Q}}_p$ such that for any $n > 0$ and $x \in X_0(\mathbb{F}_{q^n})$, $f_x(M_0, t) \in E[t]$ (see Lemma 4.1). This yields the p -adic analog of (i) over a smooth variety X_0 . (See Section 4.6 when X_0 is normal). Then Theorem 0.2 implies the existence of ℓ -adic companions to a given irreducible overconvergent F -isocrystal M_0 with finite determinant (see Theorem 4.2). We point out that the existence of ℓ -adic companions has already been proven by Kedlaya in [17, Thm. 5.3] in a different way, using weights (see [17, §4, Intro.]), however not their irreducibility. The Lefschetz Theorem 3.10 implies that the companion correspondence preserves irreducibility.

Theorem 0.3 has other consequences (see Section 4), aside of the existence already mentioned of ℓ -adic companions. Deligne's finiteness theorem [13, Thm. 1.1] transposes to the crystalline side (see Corollary 4.3): on X_0 smooth, there are finitely many isomorphism classes of irreducible overconvergent F -isocrystals in bounded rank and bounded ramification, up to twist by a character of the finite field. One can also kill the ramification of an F -overconvergent isocrystal by a finite étale cover in Kedlaya's semistability reduction theorem (Remark 4.4).

We now explain the method of proof of Theorem 0.3. We replace M_0 by the full Tannakian subcategory $\langle M \rangle$ of the category of overconvergent F -isocrystals spanned by M over the algebraic closure $\overline{\mathbb{F}}_q$ (we drop the lower indices $_0$ to indicate this, see Section 1.1 for the definitions). We slightly improve the theorem ([11, Prop. 2.21 (a), Rmk. 2.29]) describing the surjectivity of an homomorphism of Tannaka groups in categorical terms in Lemma 1.6: the restriction functor $\langle M \rangle \rightarrow \langle M|_C \rangle$ to a curve $C \rightarrow X$ is an equivalence when it is fully faithful and any F -overconvergent isocrystal of rank 1 on C is torsion. Class field theory for F -overconvergent isocrystals ([2, Lem. 6.1]⁽¹⁾) implies the torsion property. As for full faithfulness, the problem is of cohomological nature, one has to compute that the restriction homomorphism $H^0(X, N) \rightarrow H^0(C, N|_C)$ is an isomorphism for all objects N in $\langle M \rangle$. In the tame case, this is performed in Section 2 using the techniques developed in [3]. As a corollary, ℓ -adic companions exist in the tame case (see Proposition 2.8). In the wild case, Kedlaya's

⁽¹⁾ See Remark 4.6 for a correction of a mistake in this lemma.