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WINDOWS, CORES AND SKINNING MAPS

BY JEFFREY F. BROCK, KENNETH W. BROMBERG, RICHARD D. CANARY AND YAIR N. MINSKY

ABSTRACT. – We give a generalization of Thurston's Bounded Image Theorem for skinning maps, which applies to pared 3-manifolds with incompressible boundary that are not necessarily acylindrical. Along the way we study properties of divergent sequences in the deformation space of such a manifold, establishing the existence of compact cores satisfying a certain notion of uniform geometry.

RÉSUMÉ. – Nous donnons une généralisation du théorème de l'image bornée de Thurston aux variétés de dimension 3 apprêtées à bord incompressible qui ne sont pas nécessairement acylindriques. En cours de route nous étudions des propriétés de suites divergeant dans l'espace des déformations d'une telle variété, établissant l'existence de coeurs compacts qui satisfont une certaine notion de géométrie uniforme.

1. Introduction

Critical to Thurston's Geometrization Theorem for Haken 3-manifolds was a fixed-point problem, phrased for a self-mapping of the deformation space of a hyperbolic 3-manifold with boundary. This *skinning map* implicitly describes how to enhance a topological gluing of a 3-manifold along its boundary with geometric information; a fixed point for the skinning map realizes a geometric solution to the gluing problem, resulting in a hyperbolic structure on the gluing.

Beyond its utility in geometrization, the map itself reveals more quantitatively the relationship between topological and geometric features of a hyperbolic 3-manifold. Indeed, Thurston's Bounded Image Theorem (see Thurston [44], Morgan [38], Kent [28]) which provides the desired fixed-point, guarantees that for an acylindrical 3-manifold, the image of the skinning map is a bounded subset of Teichmüller space. In this paper, we investigate

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the skinning map in the more general case when the 3-manifold is only assumed to have incompressible boundary. Here, the image of the skinning map need not be bounded, but the restriction onto any essential subsurface of the boundary that is homotopic off of the characteristic submanifold is bounded. One may think of this as a strong form of Thurston's Relative Compactness Theorem. Along the way, we refine our understanding of the interior geometry of a hyperbolic 3-manifold, establishing existence of a uniformly controlled family of compact cores for each deformation space of such manifolds.

Let M be a compact oriented hyperbolizable 3-manifold whose boundary is incompressible and contains no tori and let $CC_0(M)$ denote the set of convex cocompact hyperbolic structures on the interior of M. A convex cocompact hyperbolic structure N_ρ on the interior of M gives rise to a holonomy representation $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ and induces a well-defined conformal structure on its boundary ∂M. Bers [8] shows that one obtains an identification of $CC_0(M)$ with $\mathcal{T}(\partial M)$, the Teichmüller space of conformal structures on ∂M. The skinning map

$$\sigma_M : CC_0(M) \to \mathcal{T}(\overline{\partial M})$$

records the asymptotic geometry of the "inward-pointing" end of the cover associated to each boundary component. If M has connected boundary S and N_ρ is in $CC_0(M)$, then the cover N_S of N_ρ associated to $\pi_1(S)$ is quasifuchsian, i.e., a point in $CC_0(S \times [0,1])$, so may be identified to a point $(X, Y) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$. Then, $\sigma_M(\rho) = Y$ and X is the point in $\mathcal{T}(S)$ associated to ρ by the Bers parametrization. (The skinning map will be defined more carefully, and in greater generality, in Section 2.1.)

A compact, oriented, hyperbolizable 3-manifold M is said to be *acylindrical* if it contains no essential annuli, or, equivalently, if $\pi_1(M)$ does not admit a non-trivial splitting over a cyclic subgroup. In this setting, Thurston's Bounded Image Theorem has the following form.

THURSTON'S BOUNDED IMAGE THEOREM. – *If M is a compact, oriented, acylindrical, hyperbolizable 3-manifold with no torus boundary components, then the skinning map $\sigma_M : CC_0(M) \to \mathcal{T}(\overline{\partial M})$ has bounded image.*

The skinning map has been studied extensively when M is acylindrical. This study has focused on obtaining bounds on the diameter of the skinning map in terms of the topology of M and the geometry of its unique hyperbolic metric with totally geodesic boundary, see Kent [28] and Kent-Minsky [29]. In the case that M is not required to be acylindrical, it is known that the skinning map is non-constant (Dumas-Kent [22]) and finite-to-one (Dumas [21]), while Gaster [24] demonstrated that it need not be injective.

In order to state our generalization of Thurston's Bounded Image Theorem to the setting where M is only assumed to have incompressible boundary, we recall that the *characteristic submanifold* $\Sigma(M)$ is a minimal collection of solid and thickened tori and interval bundles in M whose frontier is a collection of essential annuli such that every (embedded) essential annulus in M is isotopic into $\Sigma(M)$ (see Johannson [26] or Jaco-Shalen [25]). Thurston [46] defines the *window* of M to be the union of the interval bundles in $\Sigma(M)$, together with a regular neighborhood of each component of the frontier of $\Sigma(M)$ which is not homotopic into an interval bundle. Let $\partial_{nw} M$ denote the intersection of ∂M with the complement of the window. Components of $\partial_{nw} M$ are either components of the intersection of ∂M with the (relatively) acylindrical pieces of the decomposition, or annuli in the boundaries of the solid or thickened tori pieces.

Our main theorem asserts that any curve in $\partial_{nw} M$, equivalently any curve in ∂M which may be homotoped off the characteristic submanifold, has uniformly bounded length in the hyperbolic structures which arise in the skinning image.

THEOREM 1.1. – *Let M be a compact, oriented, hyperbolizable 3-manifold whose boundary is incompressible. For each curve α in $\partial_{nw} M$, its length $\ell_Y(\alpha)$ is bounded as Y varies over the image of σ_M.*

If W is an essential subsurface of ∂M, then σ_M induces a map

$$\sigma_M^W : CC_0(M) \to \mathcal{F}(W),$$

where $\mathcal{F}(W)$ is the Fricke space of all hyperbolic structures on the interior of W. Notice that these hyperbolic structures are allowed to have either finite or infinite area. Our main theorem immediately translates to the fact that σ_M^W has bounded image:

COROLLARY 1.2. – *Let M be a compact, oriented, hyperbolizable 3-manifold whose boundary is incompressible. If W is a component of $\partial_{nw} M$, then the image of σ_M^W is bounded in $\mathcal{F}(W)$.*

Note that these statements allow ∂M to have torus components, and moreover the theorem applies more generally, when we consider the space $AH_0(M)$ of all hyperbolic structures on the interior of M. Given an element of $AH_0(M)$ with holonomy representation $\rho : \pi_1(M) \to PSL(2, \mathbb{C})$ and quotient manifold N_ρ, we obtain an end invariant $\sigma_M(\rho)$ on the non-toroidal part of the boundary, which records the asymptotic geometry of the inward-pointing end of the covers of N_ρ associated to those boundary components of M. This ending invariant consists of a multicurve, known as the parabolic locus, and either a finite area hyperbolic structure or a minimal filling geodesic lamination on each component of the complement. If a curve α on ∂M is homotopic into a component of the complement of the parabolic locus which has a hyperbolic structure, then $\ell_{\sigma_M(\rho)}(\alpha)$ denotes the length of the geodesic representative of α in this hyperbolic structure; if α is homotopic into the parabolic locus then $\ell_{\sigma_M(\rho)}(\alpha) = 0$; and otherwise $\ell_{\sigma_M(\rho)}(\alpha) = +\infty$.

With these definitions, Theorem 1.1 continues to hold. In addition, σ_M^W can still be defined on $AH_0(M)$ when W is a component of $\partial_{nw} M$, and Corollary 1.2 holds as well. Notice that Corollary 1.2 contains Thurston's original Bounded Image Theorem as a special case. Both results also have natural generalizations to the pared setting, which we will state and prove in Section 5.

Along the way to proving Theorem 1.1, we will develop some tools and structural results on the geometry of hyperbolic 3-manifolds that may be of independent interest. One is a new tool for studying the geometry of surface groups, and the other is a uniformity statement for compact cores of manifolds in a deformation space $AH_0(M)$.