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# TROPICAL CYCLE CLASSES FOR NON-ARCHIMEDEAN SPACES AND WEIGHT DECOMPOSITION OF DE RHAM COHOMOLOGY SHEAVES

BY Yifeng LIU

Abstract. – This article has three major goals. First, we define tropical cycle class maps for smooth varieties over non-Archimedean fields, valued in the Dolbeault cohomology defined in terms of real forms introduced by Chambert-Loir and Ducros. Second, we construct a functorial decomposition of de Rham cohomology sheaves, called weight decomposition, for smooth analytic spaces over certain non-Archimedean fields of characteristic zero, which generalizes a construction of Berkovich and solves a question raised by himself. Third, we reveal a connection between the tropical theory and the algebraic de Rham theory. As an application, we show that algebraic cycles that are trivial in the algebraic de Rham cohomology are trivial as currents for Dolbeault cohomology as well.

Résumé. – Cet article a trois objectifs majeurs. Premièrement, nous définissons des applications de classes de cycle tropicales pour des variétés lisses sur des corps non archimédiens, à valeurs dans la cohomologie de Dolbeault définie en termes de formes réelles introduites par Chambert-Loir et Ducros. Deuxièmement, nous construisons une décomposition fonctorielle des faisceaux de cohomologie de de Rham, appelée décomposition par le poids, pour des espaces analytiques lisses sur certains corps non archimédiens de caractéristique zéro, qui généralise une construction de Berkovich et résout une question posée par lui-même. Troisièmement, nous révélons une connexion entre la théorie tropicale et la théorie de de Rham algébrique. Comme application, nous montrons que les cycles algébriques qui sont triviaux dans la cohomologie de de Rham algébrique sont également triviaux en tant que courants pour la cohomologie de Dolbeault.

## 1. Introduction

This article has three major goals. First, we define tropical cycle class maps for smooth varieties over non-Archimedean fields, valued in the Dolbeault cohomology defined in terms of real forms introduced by Chambert-Loir and Ducros [9]. Second, we construct a functorial decomposition of de Rham cohomology sheaves, called weight decomposition, for smooth analytic spaces over non-Archimedean fields embeddable into $\mathbf{C}_{\mathbf{F}}$ (see below), which generalizes a construction of Berkovich and solves a question raised by himself in [6]. Third, we reveal a connection between the tropical theory and the algebraic de Rham theory. As an


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application, we show that algebraic cycles that are trivial in the algebraic de Rham cohomology are trivial as currents for Dolbeault cohomology as well.

In this article, by a non-Archimedean field, we mean a complete topological field with respect to a nontrivial non-Archimedean valuation of rank one. We fix a finite field $\mathbf{F}$ throughout the article. Denote by $\mathbf{Z}_{\mathbf{F}}$ the ring of Witt vectors in $\mathbf{F}$ and $\mathbf{Q}_{\mathbf{F}}$ the field of fractions of $\mathbf{Z}_{\mathbf{F}}$. Then $\mathbf{Q}_{\mathbf{F}}$ is naturally a non-Archimedean field, which is locally compact. Moreover, we fix a complete algebraic closure $\mathbf{C}_{\mathbf{F}}$ of $\mathbf{Q}_{\mathbf{F}}$, which is also a non-Archimedean field.

### 1.1. Tropical cycle class map

Let $K$ be a non-Archimedean field. In [9], Chambert-Loir and Ducros define, for every $K$-analytic (Berkovich) space [(1)] $X$, a bicomplex $(\mathscr{A}_X^{\bullet,\bullet}, \mathrm{d}', \mathrm{d}'')$ of sheaves of real vector spaces on $X$ concentrated in the first quadrant [(2)]. If $X$ is paracompact, then we define the *Dolbeault cohomology* (Definition 3.1 and Remark 3.2) of $X$ to be

$$\mathrm{H}^{p,q}(X) := \frac{\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,q}(X) \to \mathscr{A}_X^{p,q+1}(X))}{\mathrm{im}(\mathrm{d}''\colon \mathscr{A}_X^{p,q-1}(X) \to \mathscr{A}_X^{p,q}(X))}.$$

Moreover, we have an integration map

$$\int_X \colon \mathscr{A}_X^{n,n}(X)_c \to \mathbf{R}$$

for $n = \dim(X)$, where $\mathscr{A}_X^{n,n}(X)_c$ is the space of $(n,n)$-forms on $X$ whose support is compact and disjoint from the boundary of $X$.

By [19] and [9], we know that for every $p \geq 0$, the complex $(\mathscr{A}_X^{p,\bullet}, \mathrm{d}'')$ is a fine resolution of the sheaf $\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$. In Section 3, we will construct a canonical $\mathbf{Q}$-subsheaf $\mathscr{T}_X^p$ of $\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$ such that the induced map

$$\mathscr{T}_X^p \otimes_{\mathbf{Q}} \mathbf{R} \to \ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$$

is an isomorphism. In particular, we have a canonical isomorphism

$$\mathrm{H}^q(X, \mathscr{T}_X^p) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathrm{H}^{p,q}(X)$$

for every $p, q \geq 0$.

Recall that in the complex world, for a smooth complex algebraic variety $\mathscr{X}$, we have a cycle class map from $\mathrm{CH}^p(\mathscr{X})$ to the classical Dolbeault cohomology $\mathrm{H}_{\overline{\partial}}^{p,p}(\mathscr{X}^{\mathrm{an}})$ of the associated complex manifold $\mathscr{X}^{\mathrm{an}}$. Over a non-Archimedean field $K$ (see below), then we may associate a separated scheme $\mathscr{X}$ of finite type over $K$ to a $K$-analytic space $\mathscr{X}^{\mathrm{an}}$ [3, Section 2.6]. We put

$$\mathrm{H}_{\mathrm{trop}}^{p,q}(\mathscr{X}) := \mathrm{H}^q(\mathscr{X}^{\mathrm{an}}, \mathscr{T}_{\mathscr{X}^{\mathrm{an}}}^p).$$

The following theorem is an analogue of the above cycle class map in the non-Archimedean setup.

(1) In this article, we assume that all $K$-analytic spaces are good, Hausdorff and strictly $K$-analytic. See Section 1.5.
(2) See [9, Remarque 1.2.12] for an analogy with the complex case.

THEOREM 1.1 (Definition 3.8, Theorem 3.9, Corollary 3.12). – *Let $K$ be a non-Archimedean field and $\mathscr{X}$ a separated smooth scheme of finite type over $K$ of dimension $n$. Then there is a* tropical cycle class map

$$\mathrm{cl}_{\mathrm{trop}}\colon \mathrm{CH}^p(\mathscr{X}) \to \mathrm{H}^{p,p}_{\mathrm{trop}}(\mathscr{X}),$$

*functorial in $\mathscr{X}$ and $K$, such that for every algebraic cycle $\mathscr{Z}$ of $\mathscr{X}$ of codimension $p$,*

$$\int_{\mathscr{X}^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(\mathscr{Z}) \wedge \omega = \int_{\mathscr{Z}^{\mathrm{an}}} \omega \tag{1.1}$$

*for every $\mathrm{d}''$-closed form $\omega \in \mathscr{A}^{n-p,n-p}_{\mathscr{X}^{\mathrm{an}}}(\mathscr{X}^{\mathrm{an}})$ with compact support.*

*In particular, if $\mathscr{X}$ is proper and $\mathscr{Z}$ is of dimension* 0*, then*

$$\int_{\mathscr{X}^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(\mathscr{Z}) = \deg \mathscr{Z}.$$

The above theorem has the following corollary.

COROLLARY 1.2 (Corollary 3.13). – *Let $K$ be a non-Archimedean field and $\mathscr{X}$ a proper smooth scheme over $K$. Let $\mathrm{NS}^p(\mathscr{X})$ be the quotient of $\mathrm{CH}^p(\mathscr{X})$ modulo numerical equivalence. Then we have*

$$\dim \mathrm{H}^{p,p}_{\mathrm{trop}}(\mathscr{X}) \geq \dim \mathrm{NS}^p(\mathscr{X}) \otimes \mathbf{Q}$$

*for every $p \geq 0$.*

REMARK 1.3. – Let the situation be as in Theorem 1.1.

1. The tropical cycle class respects products on both sides. More precisely, for $\mathscr{Z}_i \in \mathrm{CH}^{p_i}(\mathscr{X})$ with $i = 1, 2$, we have
$$\mathrm{cl}_{\mathrm{trop}}(\mathscr{Z}_1 \cdot \mathscr{Z}_2) = \mathrm{cl}_{\mathrm{trop}}(\mathscr{Z}_1) \wedge \mathrm{cl}_{\mathrm{trop}}(\mathscr{Z}_2),$$
where we have used the natural pairing
$$\wedge\colon \mathrm{H}^{p_1,q_1}_{\mathrm{trop}}(\mathscr{X}) \times \mathrm{H}^{p_2,q_2}_{\mathrm{trop}}(\mathscr{X}) \to \mathrm{H}^{p_1+p_2,q_1+q_2}_{\mathrm{trop}}(\mathscr{X}).$$
Such compatibility is used in deducing Corollary 3.13.
2. We may regard the Formula (1.1) as a tropical version of the Cauchy formula in multi-variable complex analysis.
3. Based on this theorem, we will give a counterexample of the Künneth decomposition for the cohomology theory $\mathrm{H}^{\bullet,\bullet}_{\mathrm{trop}}$ in Example 3.14.

### 1.2. Weight decomposition

Suppose that $K$ is of characteristic zero. We have the following complex of $\mathfrak{c}_X$-modules in either analytic or étale topology:

$$\Omega^\bullet_X\colon \mathscr{O}_X = \Omega^0_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \Omega^2_X \xrightarrow{\mathrm{d}} \cdots,$$

known as the *de Rham complex*, where $\mathfrak{c}_X := \ker(\mathrm{d}\colon \mathscr{O}_X \to \Omega^1_X)$ is the sheaf of constants. It is *not* exact from the term $\Omega^1_X$ if $\dim(X) \geq 1$. The cohomology sheaves of the de Rham complex $\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X$ are called *de Rham cohomology sheaves*.