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ABSOLUTE SETS AND THE DECOMPOSITION THEOREM

BY NERO BUDUR AND BOTONG WANG

ABSTRACT. – We show that any natural (derived) functor on constructible sheaves on smooth complex algebraic varieties can be used to construct a special kind of constructible sets of local systems, called absolute sets. We conjecture that the absolute sets satisfy a "special varieties package," similar to the André-Oort conjecture. The conjecture gives a simple proof of the Decomposition Theorem for all semi-simple perverse sheaves, assuming it for the geometric ones. We prove the conjecture in the rank-one case: the closed absolute sets are finite unions of torsion-translated affine tori. This extends a structure result of the authors for cohomology jump loci to any other natural jump loci. For example, to jump loci of intersection cohomology and Leray filtrations. We also show that the Leray spectral sequence for the open embedding in a good compactification degenerates for all rank one local systems at the usual page, not just for unitary local systems.

RÉSUMÉ. – Nous montrons que tout foncteur naturel sur des faisceaux constructibles sur des variétés algébriques complexes lisses peut être utilisé pour construire un type spécial d'ensembles constructibles de systèmes locaux, appelés ensembles absolus. Nous conjecturons que les ensembles absolus satisfont des propriétés spéciales similaires à la conjecture d'André-Oort. La conjecture donne une preuve simple du théorème de décomposition pour tous les faisceaux pervers semi-simples, en le supposant vrai pour ceux d'origine géométrique. Nous démontrons la conjecture dans le cas de rang 1: les ensembles absolus fermés sont des unions finies de tores affines translatés par torsion, ce qui étend un résultat des auteurs pour les lieux de saut de cohomologie à tout autre lieu de saut naturel. Nous montrons aussi que la suite spectrale de Leray pour l'immersion ouverte dans une bonne compactification dégénère à la page habituelle pour tous les systèmes locaux de rang 1.

1. Introduction

1.1. Original motivation

Let X be an algebraic variety over $\mathbb{C}$. Let $\mathcal{M}_B(X, 1)$ be the moduli space of rank-one local systems on X. This variety, also called the character variety, is easy to describe, the set of $\mathbb{C}$-points being

$$\mathcal{M}_B(X, 1)(\mathbb{C}) = \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*),$$

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that is, finitely many copies of an affine torus $(\mathbb{C}^*)^n$. Let now

$$\phi : \mathcal{M}_B(X,1)(\mathbb{C}) \to \mathbb{Z}$$

be a function defined as the composition of functors

$$\mathcal{M}_B(X,1)(\mathbb{C}) \to \mathbf{D}^b_c(X_1,\mathbb{C}) \to \mathbf{D}^b_c(X_2,\mathbb{C}) \to \cdots \to \mathbf{D}^b_c(X_r,\mathbb{C}) \xrightarrow{\dim_{\mathbb{C}} H^i(\_)} \mathbb{Z},$$

where X_i are smooth complex algebraic varieties, $X = X_1$, X_r is a point, $\mathbf{D}^b_c(X_i,\mathbb{C})$ is the derived category of bounded complexes of $\mathbb{C}$-modules with constructible cohomology, H^i is cohomology, such that the first arrow attaches to a local system the complex of sheaves concentrated in degree zero given by that local system, and the other arrows are some natural derived functors. The question is: *is ϕ a classical constructible function ?*

We show that the answer is yes, and that more is true, ϕ is an absolute function, see Theorem 9.1.1. This means that the irreducible components of the inverse images of ϕ are constructible sets of a very restricted type from both a geometric and arithmetic point of view, see Theorem 1.3.1 below. If in addition the varieties are projective, the derived categories are replaced by moduli of local systems, and the functors are restricted to certain types, this has been shown by Simpson [56].

1.2. General framework for constructibility.

Classical constructible sets and functions are produced in algebraic geometry via morphisms of schemes. First we develop a framework to produce constructible functions from natural functors between categories, without the need to resort to the existence of a morphism of moduli spaces that models the functor.

The idea is that we should still view a collection of categories $\{\mathbf{D}^b_c(X,R)\}_R$, where R are rings of coefficients, as some kind of space, with the functors between such collections as some kind of morphisms, preserving a uniform notion of constructibility that we formulate in Definition 2.7.1. Uniform, that is, recovering the classical notion of constructibility for collections $\{\mathcal{M}(R)\}_R$ where $\mathcal{M}$ is some variety, moduli space of objects of some subcategory.

We do this by introducing unispaces and their morphisms. A unispace is nothing new, it is just a functor from finite type regular $\mathbb{C}$-algebras to sets. The morphisms of unispaces are defined so that they preserve the general constructible functions of Definition 2.7.1. Morphisms of unispaces provide a way to produce classical constructible functions. Such morphisms can arise from functors on categories which do not necessarily admit scheme-theoretic moduli spaces. In short, we provide a general recipe for producing classical constructible functions.

More precisely, a morphism of $\mathbb{C}$-schemes, or algebraic $\mathbb{C}$-stacks, of finite type gives a morphism of the associated unispaces. A basic unispace will be the constructible-functions unispace $\underline{\mathbb{Z}}^{\mathrm{cstr}}$. The morphisms of unispaces $\mathscr{X} \to \underline{\mathbb{Z}}^{\mathrm{cstr}}$ generalize the notion of constructible functions on a scheme or algebraic stack of finite type over $\mathbb{C}$. A morphism of unispaces $X \to \mathscr{X}$ between a scheme or algebraic stack X of finite type over $\mathbb{C}$ and a unispace $\mathscr{X}$, composed with a unispace morphism $\mathscr{X} \to \underline{\mathbb{Z}}^{\mathrm{cstr}}$, gives a classical constructible function on X. A typical such morphism $X \to \mathscr{X}$ arises when $\mathscr{X}$ is a moduli functor and X is representing it. Most of the moduli functors are not represented by a scheme or an algebraic stack, although they are unispaces. Hence one can view our theory of constructible functions for unispaces as a notion of constructibility which frees the moduli functors from being represented in a specific category.

All this is done in Section 2. In Section 3 we introduce the general notion of constructibility over a subfield of $\mathbb{C}$ from the point of view of unispaces.

The main application of our general framework for constructibility is to answer the question from the original motivation above. More precisely, we show that any natural functor encountered on local systems, constructible sheaves, perverse sheaves, or bounded complexes with constructible cohomology, on $\mathbb{C}$ algebraic varieties lifts to a morphism of unispaces, see Theorem 5.14.1. Since we do not know how to define "any" in the previous sentence, in practice we prove this claim for a huge list of natural functors. By composition, taking fiber products, or inverse images of appropriate subcategories, or by looking at morphisms between such functors, we can generate many, if not "any," other natural functors.

We also address the field of definition of such morphisms of unispaces, most of the time this being $\mathbb{Q}$ rather than $\mathbb{C}$. A particular case of this observation is the well-known fact that the moduli spaces of local systems are $\mathbb{Q}$-schemes.

Besides morphisms of unispaces, one also has plain natural transformations of unispaces. These can be thought as continuous maps, since they produce classical semi-continuous functions rather than constructible functions. Many of the natural (derived) functors are actually natural transformations of unispaces and hence produce closed sets, rather than just constructible sets, see Theorem 5.14.3. A particular case of this observation is the well-known fact that the cohomology jump loci are closed subschemes of the moduli of local systems. However, other functors such as intersection cohomology do not satisfy this property. In particular, intersection cohomology jump loci of local systems are constructible, sometimes not closed, sets.

1.3. Absolute sets

When we restrict to derived or perverse functors on smooth $\mathbb{C}$ algebraic varieties, the constructible sets they produce come with extra structure due to the Riemann-Hilbert correspondence between regular holonomic algebraic $\mathscr{D}$-modules and perverse sheaves, see Theorem 6.4.3. We call these sets absolute. For local systems, the absolute sets have been introduced in [56], with the difference that here we do not use the Dolbeault picture at all. Another difference is that our definition of absolute constructible sets of local systems circumvents the existence of de Rham moduli spaces, yet it implies constructibility in the de Rham moduli, Proposition 7.4.5.

The major question is: *just how special these absolute sets are*? For this question to remain concrete, we address it mainly for the moduli of local systems $\mathscr{M}_B(X)$ on a smooth $\mathbb{C}$-algebraic variety. Recall that the $\mathbb{C}$-points of this space are the semi-simple local systems. Building on Simpson's conjectures [57, 58], we conjecture that the absolute sets of local systems satisfy a "special varieties package," which can be viewed as an analog for semi-simple local systems of the Manin-Mumford, Mordell-Lang, and André-Oort conjectures [36, 50, 21, 41, 34, 63, 62], see Conjecture 10.4.1.

We prove the "special varieties package" conjecture for absolute sets of rank one local systems. More precisely, we show: