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STABLE K-THEORY IS BIFUNCTOR HOMOLOGY (AFTER A. SCORICHENKO)

by

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Abstract. — For many rings R, the homology with coefficients of the infinite general linear group $\mathrm{GL}(R)$ is the tensor product of its homology with trivial coefficients with another term, which has been identified as the stable K-theory of the ring. Scorichenko's theorem states that stable K-theory is functor homology.

***Résumé* (La K-théorie stable est l'homologie des foncteurs (d'après A. Scorichenko))**
Pour beaucoup d'anneaux R, l'homologie du groupe linéaire infini avec coefficients s'obtient en effectuant le produit tensoriel de son homologie avec coefficients triviaux par un autre terme, qui n'est autre que la K-théorie stable de l'anneau. Le théorème de Scorichenko exprime la K-théorie stable comme homologie des foncteurs.

0. Introduction

The purpose of this chapter is to present A. Scorichenko's work for his dissertation at Northwestern.

***Theorem 0.1* ([20]).** — *For a ring R, let $\mathbb{P}(R)$ be the category of finitely generated projective left R-modules, and let $D : \mathbb{P}(R)^{\mathrm{op}} \times \mathbb{P}(R) \to \mathrm{Ab}$ be a bifunctor. If D has finite degree with respect to both variables, then there is an isomorphism between Waldhausen's stable K-theory and the homology of $\mathbb{P}$:*

$$K_*^{\mathrm{st}}(R, D) \longrightarrow \mathrm{H}_*(\mathbb{P}(R), D).$$

This proves a conjecture stated in [**4**]. The conjecture first appeared in [**15**] for biadditive bifunctors, a case proved in [**6**] (see also [**19**] for the outline of another approach). In the case of finite fields, the conjecture was proved for general bifunctor coefficients in [**3**] and in [**8**, Appendix]. This special case can be reformulated in terms of functor cohomology, whose computation is a main topic in this book. Indeed, the

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conjecture has been a motivation for developing computation tools in categories of functors. The homology $\mathrm{H}_*(\mathbb{P}(R), D)$ can be expressed purely in terms of homological algebra in categories of functors, which is well understood in many cases (see [**9, 10, 8**] or the article *Introduction to functor homology* in this volume). For example, when $D(X,Y) = \mathrm{Hom}_R(X, P \otimes_R Y)$, $\mathrm{H}_*(\mathbb{P}(R), D)$ is isomorphic to the topological Hochschild homology [**16**] and to the MacLane homology [**11**] of R with coefficients in the bimodule P. When $D(X,Y) = \mathrm{Hom}_R(A(X), B(Y))$ for polynomial functors A and B in $\mathcal{F}(R)$, and if R is a field, then $\mathrm{H}_*(\mathbb{P}(R), D)$ is dual to $\mathrm{Ext}^*_{\mathcal{F}(R)}(A, B)$ as studied in this book.

Stable K-theory is precisely related to homology of invertible matrices: Waldhausen explained [**21**, Section 6] that stable K-theory gives access to homology of the general linear group, with twisted coefficients, through the spectral sequence discussed in sections 1 and 5. One point of the theorem is that although stable K-theory is defined in terms of invertible matrices, it is naturally isomorphic to a more manageable theory, expressed in terms of all matrices. The isomorphism of Theorem 0.1 is induced by the inclusion of invertible matrices in all matrices. There are variations on this, as will be seen with Scorichenko's use of the category of epimorphisms.

1. Homology of general linear groups and stable K-theory

Let R be a ring and $\mathrm{GL}_n(R)$ be the group of invertible matrices over R. For a bimodule P over R, the R-bimodule of $n \times n$-matrices $\mathrm{gl}_n(P)$ is a $\mathrm{GL}_n(R)$-module for the conjugation action: $X * M := X^{-1}MX$. We embed $\mathrm{GL}_n(R)$ as a subgroup in $\mathrm{GL}_{n+1}(R)$ by: $X \mapsto \left(\begin{smallmatrix} X & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and define the direct limit $\mathrm{GL}(R) = \bigcup_n \mathrm{GL}_n(R)$. We embed $\mathrm{gl}_n(P)$ in $\mathrm{gl}_{n+1}(P)$ by: $M \mapsto \left(\begin{smallmatrix} M & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and define the direct limit $\mathrm{gl}(P) = \bigcup_n \mathrm{gl}_n(P)$. This yields the conjugation action of $\mathrm{GL}(R)$ on $\mathrm{gl}(P)$.

The homology groups with twisted coefficients $\mathrm{H}_*(\mathrm{GL}(R); \mathrm{gl}(P))$ appear as the E^2_{n1}-terms of the following change of rings spectral sequence. Let:

$$0 \longrightarrow P \longrightarrow S \longrightarrow R \longrightarrow 0$$

be a singular extension of rings. Thus S is a ring and P is a two-sided ideal of S such that $P^2 = 0$ and $R = S/P$. There is a short exact sequence of groups

$$0 \longrightarrow \mathrm{gl}(P) \longrightarrow \mathrm{GL}(S) \longrightarrow \mathrm{GL}(R) \longrightarrow 1,$$

where the inclusion is by the exponential map $x \mapsto 1 + x$. It yields a Hochschild-Serre spectral sequence

$$E^2_{pq} = \mathrm{H}_p(\mathrm{GL}(R), \mathrm{H}_q(\mathrm{gl}(P))) \Longrightarrow \mathrm{H}_{p+q}(\mathrm{GL}(S)).$$

Since $\mathrm{gl}(P)$ is an abelian group, its homology $\mathrm{H}_*(\mathrm{gl}(P))$ is known [**18**, Section 8]. Here is a way to put these groups in a more general framework.

Let $\mathbb{P}(R)$, or simply $\mathbb{P}$, be the category of finitely generated projective left R-modules. The category $\mathbb{P}$ is equivalent to a small category and therefore we can do homological algebra in $\mathrm{Func}(\mathbb{P}, \mathrm{Ab})$. For a bifunctor $D : \mathbb{P}^{\mathrm{op}} \times \mathbb{P} \to \mathrm{Ab}$ the abelian group $D(R^n, R^n)$ has a natural $\mathrm{GL}_n(R)$-module structure, with action on both variable. Define $p_n : R^{n+1} \to R^n$ and $i_n : R^n \to R^{n+1}$ by: $p_n(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$ and $i_n(x_1 \dots, x_n) = (x_1, \dots, x_n, 0)$. They yield an homomorphism

$$D(p_n, i_n) : D(R^n, R^n) \longrightarrow D(R^{n+1}, R^{n+1})$$

which is compatible with the inclusions $\mathrm{GL}_n(R) \subset \mathrm{GL}_{n+1}(R)$. At the limit, one gets a $\mathrm{GL}(R)$-module $D_\infty := \mathrm{colim}_n\, D(R^n, R^n)$. For example, when $D(X, Y) = \mathrm{Hom}_R(X, P \otimes_R Y)$ for a given bimodule P, then $D_\infty = \mathrm{gl}(P)$. Considering the bifunctor defined by $D(X, Y) = \mathrm{H}_q(\mathrm{Hom}_R(X, P \otimes_R Y))$ recovers $D_\infty = \mathrm{H}_q(\mathrm{gl}(P))$.

We are left with the general problem of understanding the groups $\mathrm{H}_*(\mathrm{GL}(R), D_\infty)$. This is achieved by comparing it with an appropriate notion of homology of a small category for the category $\mathbb{P}$ (see Section 2.5). The group $\mathrm{GL}_n(R)$ appears as the subcategory of $\mathbb{P}$ consisting of the automorphisms of R^n, and this inclusion induces an homomorphism

$$\psi_* : \mathrm{H}_*(\mathrm{GL}(R), D_\infty) \longrightarrow \mathrm{H}_*(\mathbb{P}(R), D).$$

Unfortunately the homomorphism ψ_* is very far from being an isomorphism in general. Indeed, if D is a constant bifunctor, then $\mathrm{H}_*(\mathbb{P}, D)$ vanishes in positive dimensions, because $\mathbb{P}$ has a zero object, while the homology of the general linear group is highly nontrivial in general. There is a trick due to Waldhausen [**21**, p. 387–388], which simplifies the situation. Define the stable K-theory $K_*^{\mathrm{st}}(R, D)$ of R with coefficients in D as the homology of the homotopy fiber of $B\,\mathrm{GL}(R) \to B\,\mathrm{GL}(R)^+$, with twisted coefficients in D_∞. In the resulting Serre spectral sequence

$$E^2_{pq} = \mathrm{H}_*(\mathrm{GL}(R), K_*^{\mathrm{st}}(R, D)) \Longrightarrow \mathrm{H}_*(\mathrm{GL}(R), D_\infty) \tag{1}$$

the action of $\mathrm{GL}(R)$ on $K_*^{\mathrm{st}}(R, D)$ is *trivial* (see [**12**]). The spectral sequence (1) degenerates at E^2 in many cases (see [**4**], or Section 5 in this paper). Moreover there is a natural transformation

$$\nu_* : K_*^{\mathrm{st}}(R, D) \longrightarrow \mathrm{H}_*(\mathbb{P}(R), D)$$

because $\mathrm{H}_*(\mathbb{P}, -)$ is a universal sequence of functors defined on $\mathrm{Func}(\mathbb{P}^{\mathrm{op}} \times \mathbb{P}, \mathrm{Ab})$ (see Lemma 2.1).

Scorichenko's theorem 0.1 states that ν_* is an isomorphism, if D has finite degree with respect to both variables. For the definition of functors of finite degree we refer the reader to Section 3. Symmetric, exterior or divided powers all have finite degree, as does indeed the bifunctor defined by $D(X, Y) = \mathrm{H}_q(\mathrm{Hom}_R(X, P \otimes_R Y))$, which is relevant to the above change of rings spectral sequence.

2. Preliminaries from homological algebra

2.1. Universal sequences of functors. — We assume the reader to be familiar with the basics of homological algebra and category theory, as in [**5**]. We recall the following axiomatic characterization of derived functors, to be used several times in this paper. Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories. A *connected sequence of functors* is a sequence of additive functors $(T_n : \mathbf{A} \to \mathbf{B})_{n\geqslant 0}$ together with homomorphisms

$$\partial_n : T_{n+1}(C) \longrightarrow T_n(A)$$

for each exact sequence in $\mathbf{A}$

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{s} C \longrightarrow 0$$

which are natural in respect of maps of short exact sequences. A connected sequence is *exact* if for each exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{s} C \to 0$ in $\mathbf{A}$, the long sequence in $\mathbf{B}$

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\partial} T_n(A) \xrightarrow{i_*} T_n(B) \xrightarrow{s_*} T_n(C) \longrightarrow \cdots \longrightarrow T_0(C) \longrightarrow 0$$

is exact. Assume $\mathbf{A}$ has enough projective objects. A *universal sequence of functors* is an exact connected sequence of functors such that $T_n(P) = 0$ for all positive n and all projective P. The following is a particular case of [**5**, Proposition III.5.2].

Proposition 2.1. — *Let $T : \mathbf{A} \to \mathbf{B}$ be an additive covariant functor. Its left derived functors $(L_nT : \mathbf{A} \to \mathbf{B})_{n\geqslant 0}$ form a universal sequence of functors. Conversely, if $(T_n : \mathbf{A} \to \mathbf{B})_{n\geqslant 0}$ is an exact connected sequence of functors, then there is a unique morphism of connected sequence of functors $(\xi_n : T_n \to L_n(T_0))_{n\geqslant 0}$ such that $\xi_0 : T_0 \to L_0T_0$ is the canonical isomorphism. Furthermore ξ_n is an isomorphism for all $n \geqslant 0$ provided $(T_n : \mathbf{A} \to \mathbf{B})_{n\geqslant 0}$ is a universal sequence of functors.*

2.2. A lemma on collapsing spectral sequences. — We now extend these notions to spectral sequences of functors. A *∂-spectral sequence* is for each $A \in \mathbf{A}$ a upper-half-plane spectral sequence $(E^r_{pq}(A), d^r)_{r\geqslant 2}$ in $\mathbf{B}$, which is natural in $A \in \mathbf{A}$, together with homomorphisms

$$\partial_r : E^r_{pq}(C) \longrightarrow E^r_{p,q-1}(A)$$

for each short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{s} C \to 0$ in $\mathbf{A}$, which are natural in respect of maps of short exact sequences, and such that:

(i) for each $r \geqslant 2$, ∂_{r+1} is the map induced in homology by ∂_r

(ii) the diagrams

$$\begin{array}{ccc} E^r_{pq}(C) & \xrightarrow{d^r} & E^r_{p-r,q+r-1}(C) \\ \partial\downarrow & & \downarrow\partial \\ E^r_{p,q-1}(A) & \xrightarrow{d^r} & E^r_{p-r,q+r-2}(A) \end{array}$$

commute for all integers p, q, and $r \geqslant 2$.

Lemma 2.2. — *Let* $\mathbf{A}$ *be an abelian category and let* $(E^r_{pq})_{r\geqslant 2}$ *be a* ∂*-spectral sequence. Assume that the following condition holds: For any* C *in* $\mathbf{A}$*, there is a short exact sequence* $0\to A\to B\to C\to 0$ *in* $\mathbf{A}$ *such that the maps* $\partial^2 : E^2_{pq}(C)\to E^2_{p,q-1}(A)$ *are monomorphisms. Then the spectral sequence* $(E^r_{pq}(C), d^r)_{r\geqslant 2}$ *stops at* E^2 *for any* C *in* $\mathbf{A}$.

Proof. — We need to show that $d^r = 0$ for each r. Let C be in $\mathbf{A}$, and let $0\to A\to B\to C\to 0$ be a short exact sequence as in the statement. Starting at the E^2-level, let us consider the commutative diagram:

$$\begin{array}{ccc} E^2_{pq}(C) & \xrightarrow{d^2} & E^2_{p-2,q+1}(C) \\ \partial\downarrow & & \downarrow\partial \\ E^2_{p,q-1}(A) & \xrightarrow{d^2} & E^2_{p-2,q}(A) \end{array}$$

By hypothesis, the right vertical map is mono. When $q = 0$, the left bottom term is 0: hence $d^2_{p0} = 0$. We then proceed by induction on q, applying the induction hypothesis to A to show that the bottom map is 0.

At the next stage, we have $E^3 = E^2$ and $\partial_3 = \partial_2$, by the first condition of a ∂-spectral sequence. Hence the conditions on the E^2-term carry over to the E^3-term, and we repeat the argument *ad lib.* □

2.3. Categories of functors. — For a small category $\mathcal{C}$ and a category $\mathbf{A}$ we let $\mathrm{Func}(\mathcal{C},\mathbf{A})$ be the category of all functors from $\mathcal{C}$ to $\mathbf{A}$ and natural transformations between them. The category $\mathrm{Func}(\mathcal{C},\mathbf{A})$ carries lots of the properties of $\mathbf{A}$. It has limits (resp. colimits) provided $\mathbf{A}$ has limits (resp. colimits). The limits and colimits in $\mathrm{Func}(\mathcal{C},\mathbf{A})$ are computed pointwise. In particular, if $\mathbf{A}$ is an abelian category, then $\mathrm{Func}(\mathcal{C},\mathbf{A})$ is also an abelian category: A sequence

$$0\longrightarrow F\longrightarrow G\longrightarrow H\longrightarrow 0$$

is an exact sequence in $\mathrm{Func}(\mathcal{C},\mathbf{A})$ if

$$0\longrightarrow F(X)\longrightarrow G(X)\longrightarrow H(X)\longrightarrow 0$$

is exact for all $X\in\mathcal{C}$.

We are especially interested in the case when $\mathbf{A}$ is the category R-Mod of left modules over a ring R. We restrict to this case for the rest of the section. To describe projective generators in the category $\mathrm{Func}(\mathcal{C},\mathbf{A})$, we recall the Yoneda lemma.

Lemma 2.3 (**[13]**). — *Let* X *be an object in* $\mathcal{C}$*. For any functor*

$$T:\mathcal{C}\longrightarrow \mathrm{Sets}$$

to the category of sets, there is a natural (in X*) bijection*

$$\mathrm{Hom}_{\mathrm{Func}(\mathcal{C},\mathrm{Sets})}(\mathrm{Hom}_{\mathcal{C}}(X,-),T)\ \cong\ T(X),$$