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LECTURES ON THE COHOMOLOGY OF FINITE GROUP SCHEMES

by

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Abstract. — We provide an introduction to the cohomology of finite group schemes, a class of objects which includes finite groups and p-restricted Lie algebras. Various qualitative results, known earlier for finite groups by work of Quillen and others, are extended to this general context. Various computational techniques which arise from classical homological algebra are recalled. We then proceed to discuss the essential role of strict polynomial functors in the proof of the fundamental theorem which asserts that the cohomology of a finite group scheme is finitely generated.

***Résumé* (Cohomologie des schémas en groupes finis)**. — Ce texte est une introduction à la cohomologie des schémas en groupes finis. Cette classe d'objets contient les groupes finis et les algèbres de Lie restreintes. Plusieurs résultats qualitatifs, établis pour les groupes finis par Quillen et d'autres, leurs sont généralisés. On rappelle les méthodes de calcul de l'algèbre homologique, puis on explique l'intervention déterminante des foncteurs polynomiaux stricts dans la démonstration qui établit que la cohomologie d'un schéma en groupes fini est de type fini.

0. Introduction

The goal of these lectures (which were presented in a preliminary form at the Nantes meeting) is to provide an introduction to some of the techniques and computations of cohomology of finite group schemes over a field k of characteristic $p > 0$ which have been developed since the publication of J. Jantzen's book [**13**] and to explain the important role played by the cohomology of (strict polynomial) functors. The focal point of these lectures is a theorem of E. Friedlander and A. Suslin asserting that the cohomology of finite group schemes is finitely generated (see Theorem 4.7 below). The somewhat innovative proof of this theorem has led to numerous further results; in these lectures we have restricted attention to those results bearing on the qualitative description of the cohomology algebra of a finite group scheme.

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The reader can obtain a quick guide to these edited lectures by glancing at the table of contents. In the first lecture, we introduce the concepts and terminology which underline our subject. In particular, we recall the definition of the Frobenius kernels of an algebraic group and the Frobenius twists of a module. The second lecture summarizes some of the techniques which one can find for example in [**13**] which are used to compute cohomology. The relationship of this subject with the theme of the Nantes meeting, cohomology in categories of functors, is explained in the third lecture. Strict polynomial functors are introduced and their relationship with polynomial representations is explained. The fourth lecture is dedicated to an outline of the proof of finite generation of the cohomology of finite group schemes. Here, computations of cohomology in the category of strict polynomial functors plays a central role in the construction of certain universal classes; these computations follow closely the computations of V. Franjou, J. Lannes, and L. Schwartz [**8**] of ordinary functor cohomology. Finally, in Lecture 5 we describe how the techniques introduced to prove finite generation lead to a qualitative description of the cohomology algebra $\mathrm{H}^*(G, k)$ of a finite group scheme. This follows work of D. Quillen [**15**] who determined the maximal ideal spectrum of the cohomology of a finite group.

1. Affine group schemes

Let k be a field of characteristic $p > 0$, fixed throughout this paper. We begin our discussion by defining an affine group scheme (implicitly assumed to be over k) and considering a few interesting examples.

Definition 1.1. — An affine group scheme is a representable functor

$$G : (\text{fin.gen.comm.}k\text{-}alg) \longrightarrow (grps)$$

We denote by $k[G]$ the representing finitely generated commutative k-algebra (the *coordinate algebra*) of G. To give such a representable functor is equivalent to giving a finitely generated commutative Hopf algebra (over k).

Example 1.2. — $G = \mathbb{G}_a$, the additive group. This is the functor which takes a commutative k-algebra A to the underlying abelian group (which we might denote A^+). The coordinate algebra of $\mathbb{G}_a$ is $k[\mathbb{G}_a] = k[t]$, with coproduct $\Delta(t) = t \otimes 1 + 1 \otimes t$.

Example 1.3. — $G = \mathrm{GL}_n$, the general linear group, sends a commutative k-algebra A to the group of $n \times n$ invertible matrices $\{a_{i,j}\}$ with coefficients in A. The coordinate algebra of GL_n is given by

$$k[\mathrm{GL}_n] = k[x_{i,j}, t]_{1 \leqslant i,j \leqslant n} \big/ \det(x_{i,j})t - 1$$

with coproduct

$$\Delta(x_{i,j}) = \Sigma x_{i,k} \otimes x_{k,j}.$$

Example 1.4. — Let π be a (discrete) group. We view π as an affine group scheme by letting π also denote "the constant functor with value π." In other words, this functor sends a commutative k-algebra A to the group $\pi^{|\pi_0(A)|}$, where $\pi_0(A)$ is the set of indecomposable non-trivial idempotents in A and $|\pi_0(A)|$ denotes the cardinality of $\pi_0(A)$.

Example 1.5. — For any positive integer r, we consider the "r-th Frobenius kernel" of GL_n which is denoted $\mathrm{GL}_{n(r)}$. This is the functor which sends a commutative k-algeba A to the group of $n \times n$ invertible matrices $(a_{i,j})$ with coefficients in A which satisfy the property that $a_{i,j}^{p^r} = \delta_{i.j}$ (*i.e.*, equal to 1 if $i = j$ and 0 otherwise). The coordinate algebra $k[\mathrm{GL}_{n(r)}]$ is the quotient of $k[\mathrm{GL}_n]$ by the (Hopf) ideal generated by $x_{i,j}^{p^r} - \delta_{i,j}$. More explicitly, we can write $k[\mathrm{GL}_{n(r)}] = k[x_{i,j}]/(x_{i.j}^{p^r} - \delta_{i,j})$.

Similarly, the r-th Frobenius kernel of $\mathbb{G}_a$ sends A to the group of elements of A whose p^r-th power is 0. The coordinate algebra of $\mathbb{G}_{a(r)}$ is given by $k[\mathbb{G}_{a(r)}] = k[t]/t^{p^r}$, whereas the dual algebra is given by $k\mathbb{G}_{a(r)} = k[X_1, \dots, X_r]/(X_i^p)$ where one can view the dual generator X_i as the operator $\dfrac{1}{p^{i-1}!}\dfrac{d^{p^{i-1}}}{dt^{p^{i-1}}}$ on $k[t]$.

Example 1.6. — Let g be a finite dimensional p-restricted Lie algebra of k and let $V(g)$ denote its restricted enveloping algebra, the quotient of the universal enveloping algebra $U(g)$ of g by the ideal generated by $\{X^p - X^{[p]}, X \in g\}$ (where $(-)^{[p]} : g \to g$ is the p-th power operation of g), Then the k-linear dual of $V(g)$, which we denote by $V(g)^{\#}$, is a finite dimensional commutative Hopf algebra over k and thus corresponds to an affine group scheme over k.

Remark 1.7. — An affine group scheme G is said to be *finite* if $k[G]$ is finite dimensional. For example, if G corresponds to a finite group π as in Example 1.4 or if G is a group scheme as in Example 1.5 or G is associated to a finite dimensional p-restricted Lie algebra as in Example 1.6, then G is a finite group scheme. The linear dual is called the *group algebra* of G, denoted kG, consistent with the usual terminology of the group algebra of a discrete group π. In Example 1.6, the group algebra kG of the group scheme G associated to the p-restricted Lie algebra g is $V(g)$, the restricted enveloping algebra of g.

One usually refers to an affine group scheme G whose coordinate algebra is integral (*i.e.*, reduced and irreducible) as an (affine) algebraic group. For example, both $\mathbb{G}_a$ of Example 1.2 and GL_n of Example 1.3 are algebraic groups.

Remark 1.8. — A finite group scheme G is said to be *infinitesimal* if the coordinate algebra $k[G]$ is local. An infinitesimal group G is said to be of height $\leqslant r$ if G admits a closed embedding $G \hookrightarrow \mathrm{GL}_{n(r)}$ (*i.e.*, if $a^{p^r} = 0$ for every element a in the augmentation ideal of $k[G]$). For any infinitesimal group scheme G of height 1 we have an isomorphism of algebras:

$$kG \simeq V(LieG).$$

Conversely, if g is a finite dimensional p-restricted Lie algebra, then $V(g)^{\#}$ is the coordinate algebra of an infinitesimal group scheme G of height 1. This establishes an equivalence of categories between finite dimensional p-restricted Lie algebras and infinitesimal group schemes of height 1.

We next introduce the concept of a G-module for an affine group scheme (sometimes called a rational G-module).

Definition 1.9. — Let G be an affine group scheme over k. Then a G-module M is a k-vector space provided with an A-linear group action

$$G(A) \times (M \otimes A) \longrightarrow M \otimes A \tag{1.10}$$

for all finitely generated commutative k-algebras A, functorial with respect to A. (Here, and below, the tensor product is over k.)

Equivalently, such a G-module M is a k-vector space provided with the structure of a comodule for $k[G]$; namely, a k-linear map

$$\Delta_M : M \longrightarrow M \otimes k[G]. \tag{1.11}$$

To verify this equivalence, observe that the pairing (1.10) in the special case $A = k[G]$ is written

$$\mathrm{Hom}_{k\text{-alg}}(k[G], k[G]) \times (M \otimes k[G]) \longrightarrow M \otimes k[G].$$

This determines a comodule structure of the form (1.11) by restricting to $\mathrm{Id}_{k[G]} \in \mathrm{Hom}_{k\text{-alg}}(k[G], k[G])$. Conversely, given a comodule structure Δ_M, we get a pairing of the form (1.10) as the following composition

$$\begin{aligned}\mathrm{Hom}_{k\text{-alg}}(k[G], A) \times (M \otimes A) \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k[G], A) \times (M \otimes k[G] \otimes A)\\ \longrightarrow M \otimes A \otimes A \longrightarrow M \otimes A\end{aligned}$$

where the first map is given by Δ_M, the second by the natural pairing, and the third by the ring structure on A.

If M is a G-module, then the G-invariant submodule of M is the G-submodule with trivial G-action given by

$$M^G = \{m \in M \mid g \cdot (m \otimes 1) = m \otimes 1, \forall\, A, g \in G(A)\}$$

which is readily seen to be equal

$$M^G = \{m \in M \mid \Delta_M(m) = m \otimes 1\}.$$

If M, N are two G-modules, then the tensor product (over k) $M \otimes N$ has a natural structure of a G-module given by embedding G diagonally in $G \times G$; this is written succinctly in terms of Δ_M, Δ_N as the following composition involving the product structure of the ring $k[G]$:

$$\cdot \circ (\Delta_M \otimes \Delta_N) : M \otimes N \longrightarrow (M \otimes k[G]) \otimes (N \otimes k[G]) \longrightarrow M \otimes N \otimes k[G].$$

If the G-module M is finite dimensional (as a k vector space), we may give another useful formulation of the concept of a G-module. Namely, suppose that M is n-dimensional and identify the affine group scheme of k-automorphisms of M with GL_n. Then to give M the structure of a G-module is equivalent to giving a homomorphism $\rho_M : G \to \mathrm{GL}_n$ of affine group schemes.

An important example of a G-module is the coordinate algebra itself. We readily check that the coproduct on $k[G]$, $\Delta : k[G] \to k[G] \otimes k[G]$, corresponds to the right regular representation of G on the functions of G: $(g \in G, f(-) \in k[G]) \mapsto f(- \cdot g) \in k[G]$.

Suppose that $H \subset G$ is a closed subgroup scheme of the affine group scheme G (*i.e.*, $k[G] \to k[H]$ is surjective). Then for any H-module N, we consider the H-fixed points of $k[G] \otimes N$, where H acts on $k[G]$ via the right regular representation. We use the notation

$$\mathrm{Ind}_{\mathrm{H}}^{G} N = (k[G] \otimes N)^H$$

to denote the G-module with G action given by the left regular representation of G on $k[G]$.

One very useful aspect of this induction functor is given by the following theorem which is often called *Frobenius reciprocity.*

***Theorem 1.12* (*cf.* [13, 3.4]).** — *If $H \subset G$ is a closed subgroup of the affine group scheme G, then $\mathrm{Ind}_{\mathrm{H}}^{G}(-)$ is right adjoint to the restriction functor. In other words, for every H-module N and every G-module M, there is a natural isomorphism*

$$\mathrm{Hom}_H(M, N) \simeq \mathrm{Hom}_G(M, \mathrm{Ind}_{\mathrm{H}}^{G} N).$$

In particular, if N is an injective H-module, then $\mathrm{Ind}_{\mathrm{H}}^{G} N$ is an injective G-module. For example, $k[G] = \mathrm{Ind}_{e}^{G} k$ is an injective G-module.

Observe that sending $m \in M$ to $m \otimes \varepsilon \in M \otimes k[G]$ determines a homomorphism $M \to M \otimes k[G]$ of G-modules, where $\varepsilon : G \to k$ is evaluation at the identity (*i.e.*, the co-unit of the Hopf algebra $k[G]$). A direct calculation shows that the map $M \otimes k[G] \to M_{tr} \otimes k[G]$ defined by $m \otimes f \mapsto (1 \otimes f)\Delta_M(m)$ is an isomorphism of G-modules, where M_{tr} is a trivial G-module isomorphic to M as a k-vector space. Since $k[G]$ is an injective G-module, this verifies that any G-module can be embedded into an injective module.

Consequently, the category of G-modules is an abelian category with enough injectives, so that we may use standard homological algebra to define

$$\mathrm{Ext}_G^i(M, N) = R^i \,\mathrm{Hom}_G(M, -)(N)$$