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THE FARGUES-FONTAINE CURVE AND DIAMONDS [d'après Fargues, Fontaine, and Scholze]

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1. INTRODUCTION

The goal of this text is to overview the Fargues-Fontaine curve, its role in p-adic Hodge theory, and its relation to Scholze's theory of perfectoid spaces and diamonds. On the other hand, we do not touch on the role of the curve in local class field theory [**11, 15**] or in the local Langlands correspondence [**14**].

1.0.1. *The literature.* — The definitive text on the foundations of the curve is the book by Fargues and Fontaine [**17**]. There exist several more introductory articles, in particular Colmez's extensive preface [**7**] to the book, Fargues' recent ICM text [**13**], and Fargues-Fontaine's Durham survey [**16**]. In view of these articles, which were very useful when preparing the current text and which we highly recommend to readers with a background in p-adic arithmetic geometry, we have attempted to present the theory here with the non-expert in mind. In particular, Sections 2–3 should be accessible to any reader with a knowledge of elementary algebraic geometry.

Concerning diamonds, Scholze's Berkeley lecture notes [**32**] contain the main concepts, while [**31**] is the source for the technical foundations, and his ICM text [**30**] gives an overview. Section 4 on diamonds is sparse on details but we have attempted to indicate some of the main ideas of the theory.

1.0.2. *What is the curve?* — Let us begin by recalling the old analogy between the integers $\mathbb{Z}$ and the ring $\mathbb{C}[z]$ of polynomials in one variable over the complex numbers. They are both principal ideal domains, even Euclidean domains, with Euclidean function given respectively by the usual absolute value $|\cdot|$ coming from $\mathbb{R}$ and by polynomial degree. Geometrically, $\mathbb{C}[z]$ (whose monic prime polynomials identify with the complex plane via $x \mapsto z - x$) is the set of functions on the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ whose only pole is at infinity, and the degree of a polynomial is precisely the order of this pole. Analogously, arithmetic geometry views $\mathbb{Z}$ as functions on the set of prime numbers, with an extra point at infinity being provided by the real numbers or equivalently by $|\cdot|$. Motivated by this analogy, it is not uncommon to develop analogues of geometric tools for the Riemann sphere (e.g., vector bundles, sheaves, cohomology,...) when doing arithmetic geometry over $\mathbb{Z}$. This approach, although fruitful, can only be taken so far, since the point at infinity for $\mathbb{Z}$ is no longer algebraic and so the compactification-at-infinity $\{\text{primes}\} \cup \{|\cdot|\}$ is no longer an algebro-geometric object.

The theory of Fargues and Fontaine takes this analogy much further if we focus on a given prime number p and replace arithmetic geometry over $\mathbb{Z}$ by arithmetic geometry over $\mathbb{Z}_p$ or $\mathbb{Q}_p$. The Euclidean domain $\mathbb{Z}$ or $\mathbb{C}[z]$ is now replaced by a certain $\mathbb{Q}_p$-algebra B_e (coming from p-adic Hodge theory), which is again (almost) a Euclidean domain with the Euclidean structure arising from a point at infinity. But, whereas in the case of $\mathbb{Z}$ the point at infinity was outside the world of algebraic geometry, we are now in a situation much closer to that of the Riemann sphere: there exists an actual curve (in a sense of algebraic geometry) X^{FF} whose functions regular away from a certain point at infinity are the ring B_e and whose geometric and cohomological properties (which are similar to those of $\mathbb{P}^1_{\mathbb{C}}$) encode significant information about arithmetic geometry over $\mathbb{Q}_p$. This is the *fundamental curve of p-adic Hodge theory*, or the *Fargues-Fontaine curve*.

1.0.3. *Overview*. — We will return to the above point-at-infinity perspective after Theorem 1.1 but first, given our goal of diamonds, we wish to introduce the Fargues-Fontaine curve as a space of untilts. Here "untilt" refers to the tilting–untilting correspondence of Scholze through which one passes between geometry over the characteristic zero field $\mathbb{Q}_p$ and over the characteristic p field $\mathbb{F}_p$ **[23, 29]**. For example, let $\mathbb{C}_p$ be the "p-adic complex numbers," i.e., the p-adic completion of the algebraic closure of $\mathbb{Q}_p$; then its *tilt* $\mathbb{C}_p^\flat$, whose definition we will recall in Section 2.1, is a field with similar superficial structure to $\mathbb{C}_p$ but it is an extension of $\mathbb{F}_p$ rather than of $\mathbb{Q}_p$. A fundamental motivating question for both the curve and for diamonds is the following:

> Putting $F = \mathbb{C}_p^\flat$, do there exist fields $C \supseteq \mathbb{Q}_p$ other than $\mathbb{C}_p$ such that $C^\flat = F$? Informally, do there exist other ways of passing back to characteristic zero from characteristic p ?

More precisely, since equality is clearly not the right notion, let $|Y_F|$ denote the set of *untilts* (C, ι), where C is a suitable extension of $\mathbb{Q}_p$ and $\iota : F \xrightarrow{\simeq} C^\flat$ is a specified isomorphism; such pairs are taken up to an obvious notion of equivalence. A coarser notion of equivalence is obtained by taking the Frobenius automorphism $\varphi : F \xrightarrow{\simeq} F$, $x \mapsto x^p$, into account, thereby leading to the set of untilts up to Frobenius equivalence $|Y_F|/\varphi^{\mathbb{Z}}$. The remarkable theorem of Fargues and Fontaine states that this set of untilts admits the structure of a "smooth, complete curve" (see Definition 2.6), now known as the *Fargues-Fontaine curve* X_F^{FF} *of* F.

THEOREM 1.1 (Fargues-Fontaine). — *The set* $|Y_F|/\varphi^{\mathbb{Z}}$ *is the underlying set of points of a complete curve* X_F^{FF}.

We can now make more precise Paragraph 1.0.2 about points at infinity; see Sections 2.2–2.3 for details. The original field $\mathbb{C}_p$ is itself an untilt of F, thereby giving us a preferred point $\infty \in X_F^{\mathrm{FF}}$. The ring of functions on X_F^{FF} which are regular away from ∞ turns out to equal the Frobenius-fixed subring $B_e := B_{\mathrm{crys}}^{\varphi=1}$ of the classical crystalline period ring of Fontaine **[21]**; meanwhile, the completed germs of meromorphic functions at ∞ equals his classical de Rham period ring B_{dR}. The classical (and subtle) so-called *fundamental exact sequence* of p-adic Hodge theory

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^{+} \longrightarrow 0$$

then translates into a simple cohomological vanishing statement about the curve X_F^{FF}. In this way the Fargues-Fontaine curve may be viewed as subtly gluing together B_e (which is almost a Euclidean domain) and B_{dR} (which is a complete discrete valuation field), in the same way as the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ glues together $\mathbb{C}[z]$ (= functions on $\mathbb{P}^1_{\mathbb{C}}$ regular away from infinity) and $\mathbb{C}((\frac{1}{z}))$ (= completed germs of meromorphic functions at infinity). Moreover, just as for $\mathbb{P}^1_{\mathbb{C}}$, the sum of the orders of zeros/poles of any meromorphic function on X_F^{FF} is zero, which is precisely what it means for X_F^{FF} to be "complete".

Another similarity between the Fargues-Fontaine curve and the Riemann sphere is their vector bundles. On the Riemann sphere, a theorem of Grothendieck states that any vector bundle is isomorphic to $\bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\lambda_i)$ for some unique sequence of integers $\lambda_1 \geq \cdots \geq \lambda_m$, where $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\lambda)$ is the usual twisted line bundle of degree λ. On the Fargues-Fontaine curve the situation is more complicated, as there exist non-decomposable "rational twists" $\mathcal{O}_{X_F^{\mathrm{FF}}}(\lambda)$, for $\lambda \in \mathbb{Q}$ (this is only a line bundle if $\lambda \in \mathbb{Z}$; in general its rank is given by the denominator of λ), but then Fargues and Fontaine establish the following analog of Grothendieck's theorem:

THEOREM 1.2 (Fargues-Fontaine). — *Let E be a vector bundle on X_F^{FF}. Then there exists a unique sequence of rational numbers $\lambda_1 \geq \cdots \geq \lambda_m$ such that E is isomorphic to $\bigoplus_{i=1}^m \mathcal{O}_{X_F^{\mathrm{FF}}}(\lambda_i)$.*

The proof of Theorem 1.2, which we discuss in Section 3.2 but which is beyond the scope of this survey, requires a range of deep techniques including p-divisible groups and p-adic period mappings. Conversely, it encodes enough information to have important applications to classical questions in p-adic Hodge theory. For example, we use it in Section 3.3 to explain a short proof of Fontaine's "weakly admissible implies admissible" conjecture about Galois representations from 1988 **[22]** (resolved first by Colmez-Fontaine in 2000 **[8]**). The key idea is that many linear algebraic objects of p-adic Hodge theory (modules with filtration, with Frobenius,...) may be used to build vector bundles on X_F^{FF}, which may then be analyzed through Theorem 1.2. An important technique in such analyses is the general rank-degree formalism of Harder and Narasimhan **[25]**, which applies to vector bundles on any curve; we review their theory in Section 3.1.

We now turn to Scholze's theory of diamonds. Recall that our motivating goal is to classify untilts of the characteristic p field F. In the world of diamonds, such an untilt corresponds to a "morphism" from $\mathbb{Q}_p$ to F: of course, algebraically there exist no homomorphisms between fields of different characteristic, but diamonds provide a theory of p-adic geometry in which everything is of characteristic p in some sense. Even more interestingly, the choice of two untilts of F (i.e., two points of $|Y_F|$) corresponds to a morphism from $\mathbb{Q}_p \otimes \mathbb{Q}_p$ to F, where $-\otimes-$ refers to an absolute tensor product for diamonds. (To avoid misleading the reader, we caution that there is no set-theoretic object $\mathbb{Q}_p \otimes \mathbb{Q}_p$, nor set-theoretic map $\mathbb{Q}_p \to F$, only the associated diamond.) Weil's simple proof **[34]** of the Riemann hypothesis for a curve $\mathcal{C}$ over a finite field $\mathbb{F}_q$ crucially depends on the geometry of the surface $\mathcal{C} \times_{\mathbb{F}_q} \mathcal{C}$, and a well-known philosophy predicts that there should exist a similar object "$\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$" in arithmetic geometry. Diamonds appear to provide this object p-adically. (We emphasize that this is not an empty philosophy: the "shtukas" of Drinfeld **[9]** which are central in the geometric Langlands correspondence for $\mathcal{C}$ also involve $\mathcal{C} \times_{\mathbb{F}_q} \mathcal{C}$, and Fargues and Scholze's ongoing work on arithmetic local Langlands uses diamonds to develop an analogous theory over $\mathbb{Q}_p$ **[18, 32]**.)

In Section 4 we attempt to explain these ideas more precisely by defining the category of diamonds. Scholze associates to any reasonable adic space X (e.g., the analytification of a variety over a non-archimedean field such as $\mathbb{Q}_p$, $\mathbb{C}_p$, or F) a diamond which classifies certain untilts of perfectoid spaces. For example, $\mathbb{Q}_p$ and F themselves give rise to diamonds $\mathrm{Spd}(\mathbb{Q}_p)$ and $\mathrm{Spd}(F)$ and, as we suggested in the previous paragraph, morphisms of diamonds $\mathrm{Spd}(F) \to \mathrm{Spd}(\mathbb{Q}_p)$ are exactly the untilts of F (the morphism has changed direction, as usual when passing from algebra to geometry). From the point of view of diamonds, the Fargues-Fontaine curve gains the following beautiful interpretation:

THEOREM 1.3 (Scholze). — *The diamond associated to the Fargues-Fontaine curve X_F^{FF} is naturally isomorphic to the product*

$$\mathrm{Spd}(F)/\varphi^{\mathbb{Z}} \times \mathrm{Spd}(\mathbb{Q}_p).$$

Finally, in Section 5 we give a detailed sketch of the construction of the Fargues-Fontaine curve; this is necessarily slightly technical (though, in principal, it only requires some elementary algebraic geometry and some comfort manipulating large p-adic algebras) and may be safely ignored by readers uninterested in the actual construction. It begins by observing that Fontaine's infinitesimal period ring $A_{\mathrm{inf},F} := W(\mathcal{O}_F)$ (i.e., Witt vectors of the ring of integers of F) may be naturally viewed as a ring of functions on the set $|Y_F|$. Fargues and Fontaine substantially develop this point of view by introducing a topological structure on $|Y_F|$ and replacing $A_{\mathrm{inf},F}$ by a larger ring of functions B_F; this is the largest reasonable ring of continuous functions on $|Y_F|$ in the sense that $y \mapsto \{f \in B_F : f(y) = 0\}$ identifies $|Y_F|$ with the closed maximal ideals of B_F (see Prop. 5.4). Moreover, each of these ideals is principal, generated by a so-called primitive element of degree one, indicating that $|Y_F|$ is one-dimensional in some sense.

The Frobenius action on $|Y_F|$ from before Theorem 1.1 turns out to be properly discontinuous, whence $|Y_F|/\varphi^{\mathbb{Z}}$ inherits a topology making it locally homeomorphic to $|Y_F|$. The next step is to construct functions on $|Y_F|/\varphi^{\mathbb{Z}}$. Unfortunately, the only φ-invariant functions on $|Y_F|$ are constant. Instead, Fargues and Fontaine develop a theory of Weierstrass products to construct, for each point $y \in |Y_F|$, a function $t_y \in B_F$ satisfying $\varphi(t_y) = pt_y$ and with a simple zero at each point of the discrete set $\varphi^{\mathbb{Z}}(y) \subseteq |Y_F|$ and no other zeros or poles. So, given any other function $g \in B_F$ satisfying $\varphi(g) = pg$, we obtain a meromorphic function g/t_y on $|Y_F|/\varphi^{\mathbb{Z}}$ which is regular away from the image of y. Fargues and Fontaine prove that this process generates all functions on $|Y_F|/\varphi^{\mathbb{Z}}$ or rather, in more precise algebro-geometric language:

THEOREM 1.4 (Fargues-Fontaine). — 1. *The graded ring* $\bigoplus_{k\geq 0} B_F^{\varphi=p^k}$ *is graded factorial, with irreducible elements of degree one.*

2. *The closed points of the scheme* $\mathrm{Proj}(\bigoplus_{k\geq 0} B_F^{\varphi=p^k})$ *canonically identify with the set* $|Y_F|/\varphi^{\mathbb{Z}}$*.*

Part (1) of Theorem 1.4 is a central result in the entire theory; in particular, it more or less formally implies that the Fargues-Fontaine curve, which we may now define, really is a curve:

Definition 1.5. — The Fargues-Fontaine curve is

$$X_F^{\mathrm{FF}} := \mathrm{Proj}(\bigoplus_{k\geq 0} B_F^{\varphi=p^k}).$$

We will see earlier in Section 2.3 a similar (and ultimately equivalent) definition of X_F^{FF} in terms of the crystalline period ring.