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Changjian SU & Gufang ZHAO & Changlong ZHONG

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

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ON THE K-THEORY STABLE BASES OF THE SPRINGER RESOLUTION

BY CHANGJIAN SU, GUFANG ZHAO AND CHANGLONG ZHONG

ABSTRACT. – Cohomological and K-theoretic stable bases originated from the study of quantum cohomology and quantum K-theory. Restriction formula for cohomological stable bases played an important role in computing the quantum connection of cotangent bundle of partial flag varieties. In this paper we study the K-theoretic stable bases of cotangent bundles of flag varieties. We describe these bases in terms of the action of the affine Hecke algebra and the twisted group algebra of Kostant-Kumar. Using this algebraic description and the method of root polynomials, we give a restriction formula of the stable bases. We apply it to obtain the restriction formula for partial flag varieties. We also build a relation between the stable basis and the Casselman basis in the principal series representations of the Langlands dual group. As an application, we give a closed formula for the transition matrix between Casselman basis and the characteristic functions.

RÉSUMÉ. – Les bases stables cohomologiques et K-théoriques proviennent de l'étude de la cohomologie quantique et de la K-théorie quantique. La formule de restriction pour les bases stables cohomologiques a joué un rôle important dans le calcul de la connexion quantique du fibré cotangent de variétés de drapeaux partielles. Dans cet article, nous étudions les bases stables K-théoriques de fibré cotangents des variétés de drapeaux. Nous décrivons ces bases en fonction de l'action de l'algèbre de Hecke affine et de l'algèbre de Kostant-Kumar. En utilisant cette description algébrique et la méthode des polynômes de racine, nous donnons une formule de restriction des bases stables. Nous l'appliquons pour obtenir la formule de restriction pour les variétés de drapeaux partielles. Nous construisons également une relation entre la base stable et la base de Casselman dans les représentations de la série principale du groupe dual de Langlands p-adique. Comme une application, nous donnons une formule close pour la matrice de transition entre la base de Casselman et les fonctions caractéristiques.

1. Introduction

In [30], Maulik-Okounkov defined the cohomological stable envelope for symplectic resolutions (see also [10]). The image of certain cohomology classes under the stable envelope map are called the cohomological stable bases. The stable envelope is used to construct a quantum group action on the cohomology of quiver varieties, and to compute the quantum

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connection of quiver varieties. Moreover, Nakajima gave a sheaf theoretic definition of the stable envelope [33]. We refer the readers to [8, 32, 44, 46] for other applications.

The K-theoretic stable envelope is defined in [31] (see also [36, 45, 40]). It is constructed in [31] and used to define a quantum group action on the equivariant K-theory of quiver varieties [45]. Based on that, in [36], difference equations in quantum K-theory of quiver varieties are constructed geometrically, which are further identified algebraically with the quantum Knizhnik-Zamolodchikov equations [17, 36] and quantum Weyl group actions [45]. The monodromy of these difference equations is studied in [1] using the elliptic stable envelope. The K-theoretic stable bases for Hilbert scheme of points on $\mathbb{C}^2$ are studied in [35] and [18].

Stable bases for cotangent bundle of flag varieties and partial flag varieties are also of interest. The cohomological stable bases for $T^*(G/B)$ turn out to be the characteristic cycles of certain D-modules on the flag variety G/B. Pulling it back to G/B, we get the Chern-Schwartz-MacPherson classes for the Schubert cells [3, 42]. Moreover, for cohomological stable bases of the cotangent bundle $T^*(G/P)$, in [48], the first-named author obtained their restriction formula, which played an important role in computing the quantum connection of $T^*(G/P)$ in [47].

The goal of the present paper is to study the K-theory stable bases of cotangent bundle of flag varieties, and to find a restriction formula for the K-theoretic stable bases, formula expressing the stable bases in terms of the torus fixed point basis in $T^*(G/B)$. For each choice of a Weyl chamber, there is a set of stable bases, labeled by Weyl group elements $w \in W$. For the positive/negative Weyl chambers, the stable basis will be denoted by $\{\mathrm{stab}_\pm(w) \mid w \in W\}$. (There are other choices involved in the definition. See § 4.2 for the detail.) In the special cases when $w \in W$ is the identity e or the longest element $w_0 \in W$, $\mathrm{stab}_+(e)$ and $\mathrm{stab}_-(w_0)$ are equal to the structure sheaves of the corresponding fixed points, up to a factor.

Let Z be the Steinberg variety and A be the maximal torus of G. The convolution algebra $K_{G\times\mathbb{C}^*}(Z)$, which is isomorphic to the affine Hecke algebra by a well known theorem of Kazhdan-Lusztig and Ginzburg ([22, 16]), acts on $K_{A\times\mathbb{C}^*}(T^*G/B)$ on the left and on the right. Under these two actions, the Demazure-Lusztig operators corresponding to simple root α are denoted respectively by T_α and T'_α. Our first main result is the following:

THEOREM 1.1 (Theorem 4.5). – *The elements* $\mathrm{stab}_\pm(w)$ *are generated by the action of* $K_{G\times\mathbb{C}^*}(Z)$. *More precisely,*

$$\mathrm{stab}_+(w) = q^{-\ell(w)/2} T'_{w^{-1}}(\mathrm{stab}_+(e)), \quad \mathrm{stab}_-(w) = q^{\ell(w_0 w)/2} (T_{w_0 w})^{-1}(\mathrm{stab}_-(w_0)).$$

In the proof of this theorem, we use the *rigidity* technique (see § 3) to calculate the affine Hecke algebra actions on the stable bases in Proposition 4.3.

Theorem 1.1 allows us to give a purely algebraic definition of the stable bases (Definition 6.3), involving only the affine Hecke algebra, the twisted group algebra of Kostant-Kumar and its dual. The study of properties of the stable bases boils down to combinatorics of the twisted group algebra.

We use Theorem 1.1 and the root polynomial method to find a restriction formula of stable bases. Such polynomials for cohomology and K-theory of flag varieties were studied by Billey, Graham, and Willems [7, 19, 51], and then generalized by Lenart-Zainoulline [25].

In this method, a formula of the Schubert classes in terms of classes of torus fixed points is determined by the coefficients of root polynomials (see Theorem 7.3). Generalizing the root polynomial method, we obtain our second main result. For the cotangent bundle of partial flag varieties in type A, this is also obtained by Rimányi, Tarasov and Varchenko using weight functions in [40, 41]. In a work in progress of Knutson-Zinn-Justin, K-theory stable basis is also studied from the point of view of integrable systems.

THEOREM 1.2 (Theorem 7.5). – *With* $a^+_{w,v}$ *(resp.* $K^\tau_{w,v}$*) defined in Lemma 5.2 (resp. §7.3), we have*

$$\begin{aligned} \mathrm{stab}_+(w)|_v &= q^{-\ell(w)/2} v(a^+_{w^{-1},v^{-1}}) \prod_{\alpha>0}(1-e^\alpha). \\ \mathrm{stab}_-(w)|_v &= q^{\ell(w)/2} K^\tau_{w,v}[\prod_{\alpha>0, v^{-1}\alpha>0}(1-qe^{-\alpha})]\cdot[\prod_{\alpha>0, v^{-1}\alpha<0}(1-e^\alpha)]. \end{aligned}$$

We also give some applications of the above theorems in § 8. We obtain the restriction formula for stable bases in $K_T(T^*G/P_J)$ in Theorem 8.6. This is done by showing that the stable bases coincide with the image of $\mathrm{stab}_\pm(w) \in K_T(T^*G/B)$ via the Lagrange correspondence from T^*G/B to T^*G/P_J.

As an application, we study the relation between K-theory of the Springer resolution and the principal series representations of p-adic groups.

In Theorem 9.4, we relate the T-equivariant K-theory of the Springer resolution to the bases in the Iwahori invariants of an unramified principle series [14, 37]. Such an isomorphism has been well-known, and has been studied by Lusztig [28] and Braverman-Kazhdan [9] from different points of view. However, the present paper explicitly identifies different bases from K-theory and from p-adic representation theory, which had been previously unknown. In particular, the K-theory stable basis is identified with the characteristic functions on certain semi-infinite orbits; the T-fixed-point basis is identified with the Casselman basis. Consequently, Theorem 1.2 also gives a closed formula for the transition matrix between these characteristic functions and the Casselman basis. A formula for the generating function of the matrix coefficients has been previously achieved by Reeder via a different approach [37, Proposition 5.2].

Under the isomorphism in Theorem 9.4, various structures from the p-adic representations, e.g., the intertwiners, Macdonald's formula for the spherical functions [29, 14], and the Casselman-Shalika formula for Whittaker functions [15], have meanings in terms of K-theory. Although this isomorphism is well-known, the K-theory interpretation of these structures is not well-documented. For the convenience of the readers, we also spell these out in § 9.

The results in the present paper also provide a way to study the transition matrix between stable bases and the Schubert classes of $K_T(G/B)$, as will be explained in a future publication. Such transition matrix is related with [24] which studies the (spherical) Whittaker functions of p-adic groups. It is also shadowed by the two geometric realizations of the affine Hecke algebras [5] and the periodic modules [9, 27, 28]. The cohomological analogue of this transition matrix, i.e., the transition matrix from cohomological Schubert classes to the cohomological stable bases, is of independent interest. It was proved in [3] that cohomological stable bases can be identified with Chern-Schwartz-MacPherson classes. In [2],