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ON THE TOPOLOGY OF A REAL ANALYTIC CURVE IN THE NEIGHBORHOOD OF A SINGULAR POINT

by

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Dedicated to the memory of Jean-Christophe Yoccoz

Abstract. — The purpose of this paper is to describe the topology of real analytic planar curves in the neighborhood of a singular point. Locally, such a curve consists of a number of branches that intersect a small circle centered on the singularity at two points. The local topology is described by a chord diagram: an even number of points on a circle, associated two by two. We show that most chord diagrams do not come from singularities. When this is the case, we call them analytical diagrams. First, we propose a recursive description of analytical diagrams. Then we characterize these analytical diagrams as those that do not contain as subdiagrams those which belong to a collection that we describe explicitly.

Résumé **(Sur la topologie des courbes analytiques réelles au voisinage des points singuliers)**
Le but de cet article est de décrire la topologie des courbes analytiques réelles planes au voisinage d'un point singulier. Localement, une telle courbe est constituée d'un certain nombre de branches qui coupent un petit cercle centré sur la singularité en deux points. La topologie locale est décrite par un diagramme de cordes : un nombre pair de points sur un cercle, associés deux par deux. Nous montrons que la plupart des diagrammes de cordes ne proviennent pas de singularités. Quand c'est le cas nous les qualifions d'analytiques. Nous proposons d'abord une description récursive des diagrammes analytiques. Puis nous caractérisons ces diagrammes analytiques comme étant ceux ne contenant pas comme sous-diagramme ceux qui appartiennent à une famille que nous décrivons explicitement.

Statement of the main result

In this paper, we propose a complete description of the topology of *real analytic planar curves* in the neighborhood of a singular point.

Denote by $\mathbb{R}\{x, y\}$ the factorial ring of germs of real analytic functions defined in some neighborhood of $(0, 0) \in \mathbb{R}^2$. The germ of a real analytic planar curve is defined

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by an equation $F(x, y) = 0$, where $F \in \mathbb{R}\{x, y\}$ vanishes at the origin. If F is an irreducible element in this ring, the topology of the curve $\mathcal{C}_F$ defined by F is well known. Either it only contains the origin (as for $x^2 + y^2 = 0$) or there is a *local homeomorphism* of the plane, mapping $\mathcal{C}_F$ to (the germ of) a straight line (as for instance with $x^3 - y^2 = 0$). In this second case, $\mathcal{C}_F$ intersects small circles centered at the origin in exactly two points.

In general F is a product $F_1^{n_1} \cdots F_k^{n_k}$ of irreducible non-associated factors F_i. Our curve $\mathcal{C}_F$ is therefore the union of the $\mathcal{C}_{F_i}$, which are usually called the *branches* of $\mathcal{C}_F$. Since we are only interested in the topology of $\mathcal{C}_F$, we can discard those F_i's such that $\mathcal{C}_{F_i}$ only contains the origin. Two distinct factors F_i yield two branches which only intersect at the origin. Hence the analytic curve $\mathcal{C}_F$ intersects small circles centered at the origin in an even number of points, grouped in pairs, each pair being associated to a branch. This yields a *chord diagram* which is by definition an even number of distinct points on the circle, grouped in pairs, up to an orientation preserving homeomorphism of the circle. Such a diagram is pictured by a certain number of chords with distinct endpoints in a circle. See for instance [**4**] for a detailed description of the role of chord diagrams in topology.

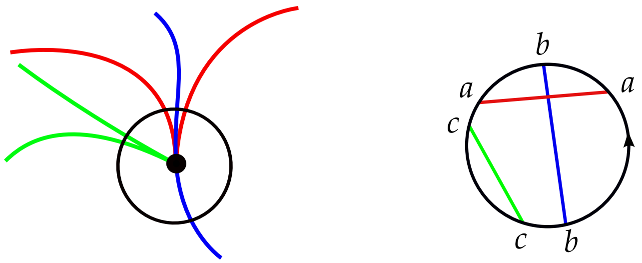

FIGURE 1. A curve with three branches and its associated chord diagram

The main theorem of this paper characterizes the chord diagrams arising from some analytic curve $\mathcal{C}_F$.

Theorem. — *A chord diagram is associated to some analytic curve if and only if it does not contain one of the "forbidden diagrams" shown in Figure 2 as a sub-chord diagram.*

The genesis of this paper

Consider four distinct polynomials P_1, P_2, P_3, P_4 in $\mathbb{R}[x]$. Order them in such a way that $P_1(x) < P_2(x) < P_3(x) < P_4(x)$ for small *negative* values of x. Then define the permutation π on $\{1, 2, 3, 4\}$ such that $P_{\pi(1)}(x) < P_{\pi(2)}(x) < P_{\pi(3)}(x) < P_{\pi(4)}(x)$ for small *positive* values of x. In 2009, Maxim Kontsevich explained to the first author EG that among the 24 permutations on $\{1, 2, 3, 4\}$ exactly two cannot be obtained

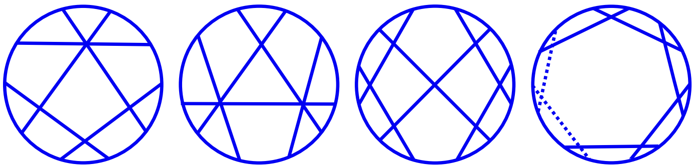

FIGURE 2. Forbidden diagrams: , , , and C_n ($n \geq 5$)

by this construction: $(1,2,3,4) \mapsto (2,4,1,3)$ or $(3,1,4,2)$. EG easily generalized this to any number of polynomials and proved that a permutation on $\{1,\dots,n\}$ can be obtained from n polynomials if and only if it does not "contain" one of Kontsevich's permutations. This was published as an elementary paper [**7**]. We will give a different proof later in Section 2.

It was then very natural to look at the topological configurations of the branches of a real analytic curve in the neighborhood of a singular point. Trying to solve this problem, EG found an explicit algorithm determining if a given chord diagram is *analytic*, i.e., is associated to the branches of some real analytic singular point. In particular, it followed that the above forbidden chord diagrams were indeed not analytic. One can always delete some branches of an analytic curve, so that a sub-chord diagram of an analytic diagram is of course analytic. In particular, a diagram containing one of the forbidden examples is non-analytic. The question of knowing whether these examples were the only "minimal" forbidden configurations remained open.

Since this proof was enjoyable and involved classical methods, EG decided to write a book proposing a leisurely promenade towards this partial result, intended for undergraduate students. The second author CS was such a student and read a preliminary draft of that book. He proposed to look at the problem from another side, explained below, and this new point of view enabled both authors to complete the proof of the above theorem in a joint effort. Therefore the final version of the book contains an additional chapter, describing this result [**8**].

The present paper contains two sections. The first provides an algorithmic description of the analytic chord diagrams and the second uses the first to prove the main result. This paper is very close to the corresponding chapters of the book. We essentially "compressed" these chapters in order to get more efficiently to the main goal.

1. Analytic chord diagrams: an algorithm

In this section, we get an algorithmic description of the analytic chord diagrams, that we defined as those which are determined by the branches of planar real analytic curves.

1.1. Polynomial interchanges: algorithmic description. — The only purpose of this subsection is to discuss quickly the much simpler situation of permutations arising from polynomials in $\mathbb{R}[x]$ which were the starting point of this paper. This serves as a motivation and gives a pattern for the general strategy, somewhat different from that in [7].

Let π be a permutation of $\{1, \dots, n\}$ $(n \geq 2)$. We say that π is a *polynomial interchange* if there exist n polynomials $P_1, \dots, P_n$ in $\mathbb{R}[x]$ such that

$$P_1(x) < P_2(x) < \cdots < P_n(x)$$

for small negative x and

$$P_{\pi(1)}(x) < P_{\pi(2)}(x) < \cdots < P_{\pi(n)}(x)$$

for small positive x.

We describe an elementary algorithm that determines if a given permutation is a polynomial interchange. In the next section, we will characterize polynomial interchanges as those permutations which do not contain the two forbidden Kontsevich permutations.

Lemma. — *For any polynomial interchange, at least two consecutive integers have consecutive images.*

The proof is easy. Denote by $v(P) \in \mathbb{N} \cup \{\infty\}$ the valuation (at 0) of a polynomial $P \in \mathbb{R}[x]$, i.e., the lowest degree of a non zero monomial in P (and ∞ if $P = 0$). Choose polynomials $P_1, \cdots, P_n$ as above. For every integer N, the relation $v(P_i - P_j) \geq N$ is an equivalence relation $\mathscr{R}_N$ on $\{1, \dots, n\}$. Each equivalence class $I \subset \{1, \dots, n\}$ is an interval. Indeed, suppose that $i < j < k$ and that $i, k \in I$. We know that $P_i(x) < P_j(x) < P_k(x)$ for small negative x. It follows that $v(P_j - P_i) \geq v(P_k - P_i) \geq N$ so that $j \in I$. The same argument, for small positive x, implies that $\pi(I)$ is also an interval. Let N_0 be the largest value of N for which equivalence classes of $\mathscr{R}_N$ are not reduced to singletons. Let I be an equivalence class of $\mathscr{R}_{N_0}$ with at least two elements. Since all the valuations $v(P_i - P_j)$ are equal to N_0 for i, j in I, the permutation π is either increasing or decreasing from I to $\pi(I)$, depending on the parity of N_0. The lemma follows if one chooses two consecutive elements in I. $\square$

Note in particular that the two permutations $(1, 2, 3, 4) \mapsto (2, 4, 1, 3)$ or $(3, 1, 4, 2)$ are not polynomial interchanges.

Theorem. — *The following algorithm decides if a permutation π is a polynomial interchange:*

1. *If no pair of consecutive integers have consecutive images then π is* not *a polynomial interchange.*
2. *If there is such a pair, merge it to a singleton. This produces a permutation with one object less. Continue.*
3. *If you end up with the trivial permutation on one object, then the original permutation was a polynomial interchange.*