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FIXED POINTS OF NILPOTENT ACTIONS ON $\mathbb{R}^2$

by

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En hommage à Jean-Christophe Yoccoz

Abstract. — We show several results providing global fixed points for nilpotent groups of orientation-preserving C^1 diffeomorphisms of the plane $\mathbb{R}^2$. The main cases are namely groups of diffeomorphisms of the sphere such that ∞ is a global fixed point, groups of diffeomorphisms preserving a non-empty compact set and finally groups of diffeomorphisms preserving a probability measure.

Résumé **(Points fixes des actions nilpotentes de $\mathbb{R}^2$).** — Nous montrons plusieurs résultats d'existence de points fixes pour des groupes nilpotents de difféomorphismes de classe C^1 du plan $\mathbb{R}^2$. Les cas principaux sont ceux de groupes de dfféomorphismes de la sphère fixant le point à l'infini, de groupes de difféomorphismes fixant un compact donné du plan, et finalement de groupes de difféomorphismes préservant une mesure de probabilité.

1. Introduction

We present here some results of existence of a global fixed point for a nilpotent group G of orientation preserving plane diffeomorphisms, which means a point fixed by every element of G. We denote $\mathrm{Diff}^1_+(\mathbb{R}^2)$ the group of orientation preserving diffeomorphisms of class C^1 of $\mathbb{R}^2$.

Let us state the key result:

Theorem 1. — *Let G be a nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ that preserves a non-empty compact set. Then G has a global fixed point.*

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A global fixed point is a common fixed point for all elements of the group G. Theorem 1 was proved by Franks, Handel and Parwani [**6**] in the particular case of a finitely generated abelian group and extended by the third author [**10**] to the case of a finitely generated nilpotent group. We will see here that the finiteness condition is not necessary.

The group G being nilpotent, it preserves a Borel probability measure (which means that the measure is invariant by every element of G) if it preserves a non-empty compact set. Does the conclusion of the theorem still holds supposing this weaker condition? We will see that the answer is yes supposing an extra property introduced in [**2**] that we will explain now. Every orientation preserving homeomorphism ϕ of $\mathbb{R}^2$ is isotopic to the identity. Fix an isotopy $I = (\phi_t)_{t\in[0,1]}$ from $\phi_0 = \mathrm{Id}$ to $\phi_1 = \phi$ and note $I_z : t \mapsto \phi_t(z)$ the trajectory of a point $z \in \mathbb{R}^2$ along the isotopy. One can define the linking number $\mathrm{Link}_I(z, z') \in \mathbb{R}$ of two different points of ϕ by setting

$$\mathrm{Link}_I(z, z') = \int_{I_z - I_{z'}} d\theta,$$

where $d\theta = \frac{1}{2\pi}\frac{xdy-ydx}{x^2+y^2}$ is the usual angular form (x and y being the cartesian coordinates on $\mathbb{R}^2$) and $I_z - I_{z'} : [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ is the path defined by $(I_z - I_{z'})(t) = \phi_t(z) - \phi_t(z')$. If I' is another isotopy from Id to ϕ, there exists an integer $k \in \mathbb{Z}$ such that $\mathrm{Link}_{I'}(z, z') = \mathrm{Link}_I(z, z') + k$ for every pair of distinct points. Note that $\mathrm{Link}_I(z, z') \in \mathbb{Z}$ if z and z' are fixed points of ϕ because $I_z - I_{z'}$ is a closed path in that case. We denote $\Delta = \{(z, z') \in \mathbb{R}^2 \times \mathbb{R}^2 \mid z = z'\}$ and consider the following condition, that depends only on ϕ:

(P1) the map Link_I is uniformly bounded on $(\mathrm{Fix}(\phi) \times \mathrm{Fix}(\phi)) \setminus \Delta$.

The notation $\mathrm{Fix}(\phi)$ stands for the fixed point set of ϕ. The second theorem is the following:

Theorem 2. — *Let G be a nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ that preserves a Borel probability measure μ. Suppose that ϕ satisfies* **(P1)** *for any $\phi \in G$. Then G has a global fixed point.*

This theorem was proved by Béguin, Le Calvez, Firmo and Miernowski [**2**] under three additional hypotheses, namely:
— G is abelian,
— G is finitely generated,
— every element of G satisfies the property **(P2)**.
This last property can be stated as follows

(P2) the function $\mathrm{Turn}_I : z \mapsto \int_{I_z} d\theta$ is constant in $W \cap \mathrm{Fix}(\phi)$ for some neighborhood W of ∞.

Their proof shows the existence of a bounded G-orbit if the support of the measure is not contained in the set $\mathrm{Fix}(G) := \bigcap_{\phi\in G} \mathrm{Fix}(\phi)$ of global fixed points of G. It remains to apply the version of Theorem 1 proved by Franks, Handel and Parwani in the case of a finitely generated abelian subgroup. Of course, to get Theorem 2 we will benefit from Theorem 1 but we will have to replace **(P2)** with a weaker condition **(P2)**$'$ that we use to show that there exists a bounded G-orbit if the support of μ is not contained in $\mathrm{Fix}(G)$.

An interesting situation where properties **(P1)** and **(P2)** are satified is the case where ϕ extends to a diffeomorphism $\overline{\phi}$ of the 2-sphere. Set $\mathbb{S}^2 = \mathbb{R}^2 \sqcup \{\infty\}$ and write $\mathrm{Diff}^1_+(\mathbb{R}^2, \infty)$ for the subgroup of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ that consists of diffeomorphisms whose natural extension to $\mathbb{S}^2$ is a diffeomorphism of class C^1. More precisely, ϕ belongs to $\mathrm{Diff}^1_+(\mathbb{R}^2, \infty)$ if the map $z \mapsto 1/\phi(1/z)$, defined if z is close to 0, extends to a diffeomorphism in a neighborhood of 0. One can blow-up ∞ adding to $\mathbb{R}^2$ the circle S_∞ of half lines of the tangent plane $T_\infty\mathbb{S}^2$ in such a way that every $\phi \in \mathrm{Diff}^1_+(\mathbb{R}^2, \infty)$ extends to the compact space $\mathbb{R}^2 \sqcup \mathbb{S}_\infty$ in a homeomorphism that coincides with the natural action of $D\overline{\phi}(\infty)$ on S_∞. If the extension has no fixed point on S_∞, the fixed point set of ϕ is a compact subset of $\mathbb{R}^2$ and **(P1)** and **(P2)** are obviously satisfied. If it has a fixed point, every isotopy in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ from Id to ϕ is homotopic (relative to the ends) to an isotopy in $\mathrm{Diff}^1_+(\mathbb{R}^2, \infty)$. One gets a natural isotopy on S_∞ defined by the action of $D\overline{\phi}(\infty)$. Say that I is adapted if the time one map of the lift of this isotopy to the universal covering of S_∞ has a fixed point (or equivalently if the real rotation number of the time one map is equal to 0). Say that an isotopy in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ from Id to ϕ is adapted if it is homotopic (relative to the ends) to such an isotopy in $\mathscr{D}$, where $\mathscr{D}$ is the set of diffeomorphisms $\phi \in \mathrm{Diff}^1_+(\mathbb{S}^2, \infty)$ such that $D\overline{\phi}(\infty)$ has real positive eigenvalues. The function Link_I does not depend on the choice of the adapted isotopy I and will be written Link_ϕ. If z is a fixed point of $\phi \in \mathrm{Diff}^1_+(\mathbb{R}^2)$, every isotopy in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ from Id to ϕ is homotopic (relative to the ends) to an isotopy $I = (\phi_t)_{t\in[0,1]}$ in $\mathrm{Diff}^1_+(\mathbb{R}^2, \infty)$ that fixes z. Similarly, one can blow up z adding to $\mathbb{R}^2$ the circle S_z of half lines of the tangent plane $T_z\mathbb{R}^2$ and look at the natural action of $D\overline{\phi}_t(z)$ on S_z. One denotes $\tau_I(z)$ the real rotation number of the time one map. It depends only on the homotopy class of I in $\mathrm{Diff}^1_+(\mathbb{R}^2)$. If ϕ belongs to $\mathscr{D}$ and $\overline{\phi}$ has a fixed point on S_∞ we will write $\tau_\phi(z) = \tau_I(z)$ if I is adapted. The following result gives more information than Theorem 2 in case we are working in $\mathrm{Diff}^1_+(\mathbb{R}^2, \infty)$. If not stated exactly as below, it was already shown in **[2]** in the case of a finitely generated abelian group. By definition a fixed point free group is a group with no global fixed points.

Theorem 3. — *Let* $G \subset \mathrm{Diff}^1_+(\mathbb{R}^2, \infty)$ *be a fixed point free nilpotent group and* $\overline{G}$ *the group of extensions to* $\mathbb{R}^2 \sqcup \mathbb{S}_\infty$. *Then:*

1. $\overline{G}$ *has a fixed point on* S_∞*;*
2. *if* $\phi \in G$, *then* $\mathrm{Fix}(\phi)$ *is either empty or unbounded;*
3. *every finite invariant measure of* $\phi \in G$ *is supported on* $\mathrm{Fix}(\phi)$ *and consequently every periodic point of* ϕ *is fixed;*
4. *for every* $\phi \in G$ *and every* $z \in \mathrm{Fix}(\phi)$, *one has* $\tau_\phi(z) = 0$*;*
5. *every* $\phi \in G$ *is isotopic to the identity relative to* $\mathrm{Fix}(\phi)$*;*

6. *for every* $\phi \in G$*, the function* $\frac{1}{n}\sum_{k=0}^{n-1} \mathrm{Link}_\phi \circ (\phi^k \times \phi^k)$ *converges uniformly to* 0 *on* $(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta$.

Let us explain now the plan of the article. We will prove Theorem 3 in Section 2 assuming Theorem 1. Among the three theorems, this is the less technical one. Up to a slight modification of the developed mathematical objects, the arguments used in [**2**] for the abelian case have a natural extension to the nilpotent case.

Theorem 1 will be proved in Sections 3 and 4. As said before, it was proved for finitely generated groups [**10**], as the following simple corollary using the finite intersection property for compact sets, where the hypothesis on the finite generation of G is replaced by a compactness property. This result will be used very often in the article and we recall the proof.

Theorem 4 **([10]).** — *Let* G *be a nilpotent subgroup of* $\mathrm{Diff}^1_+(\mathbb{R}^2)$*. Suppose there exists* $\phi \in G$ *such that* $\mathrm{Fix}(\phi)$ *is a non-empty compact set. Then* G *has a global fixed point.*

Proof. — Let us begin by proving the theorem in case G is finitely generated. Write

$$Z^{(0)}(G) = \{\mathrm{Id}\} \lhd Z^{(1)}(G) \lhd \cdots \lhd Z^{(r-1)}(G) \lhd Z^{(r)}(G) = G$$

for the ascending central series, defined inductively by the property that $Z^{(k+1)}(G)/Z^{(k)}(G)$ is the center of $G/Z^{(k)}(G)$, the integer r being the nilpotency class of G. Consider the following subnormal series

$$\langle \phi \rangle = \langle \phi, Z^{(0)}(G)\rangle \lhd \langle \phi, Z^{(1)}(G)\rangle \lhd \cdots \lhd \langle \phi, Z^{(r-1)}(G)\rangle \lhd \langle \phi, Z^{(r)}(G)\rangle = G.$$

To get the result, let us prove inductively on $s \in \{0, \dots, r\}$ that $\mathrm{Fix}\langle \phi, Z^{(s)}(G)\rangle \neq \emptyset$. By assumption the property is true for $s = 0$. Let us suppose that it is true for $s < r$. The set $\mathrm{Fix}\langle \phi, Z^{(s)}(G)\rangle$ is invariant by $\langle \phi, Z^{(s+1)}(G)\rangle$ because $\langle \phi, Z^{(s)}(G)\rangle$ is a normal subgroup of $\langle \phi, Z^{(s+1)}(G)\rangle$. Moreover $\mathrm{Fix}\langle \phi, Z^{(s)}(G)\rangle$ is non-empty by hypothesis, and compact because is included in $\mathrm{Fix}(\phi)$. Since every subgroup of a finitely generated nilpotent group is finitely generated (cf. [**11**, 5.2.17]), it follows that the group $\langle \phi, Z^{(s+1)}(G)\rangle$ is finitely generated. Since $\langle \phi, Z^{(s+1)}(G)\rangle$ is also nilpotent and preserves a non-empty compact set, it has a global fixed point by Theorem 1 for the finitely generated case [**10**].

In the general case the first part of the proof implies that $\mathrm{Fix}(\Gamma)$ is a non-empty compact set for any finitely generated subgroup Γ of G containing ϕ. Given finitely generated subgroups $\Gamma_1, \dots, \Gamma_m$ of G containing ϕ, the intersection $\bigcap_{i=1}^n \mathrm{Fix}(\Gamma_i)$ is the fixed point set of the finitely generated group $\langle \bigcup_{i=1}^n \Gamma_i \rangle$ and hence non-empty. Since $\mathrm{Fix}(\phi)$ is compact and the family $\{\mathrm{Fix}\langle \phi, f\rangle\}_{f \in G}$ has the finite intersection property, it follows that $\mathrm{Fix}(G) = \bigcap_{f \in G} \mathrm{Fix}\langle \phi, f\rangle$ is a non-empty compact subset of $\mathrm{Fix}(\phi)$. □

We will see later that we can replace the hypothesis on $\mathrm{Fix}(\phi)$ in Theorem 4 by a weaker compactness property. More precisely, in order to prove Theorem 1 it suffices to show the following theorem: