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SMOOTH SIEGEL DISKS EVERYWHERE

by

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Abstract. — We prove the existence of Siegel disks with smooth boundaries in most families of holomorphic maps fixing the origin. The method can also yield other types of regularity conditions for the boundary. The family is required to have an indifferent fixed point at 0, to be parameterized by the rotation number α, to depend on α in a Lipschitz-continuous way, and to be non-degenerate. A degenerate family is one for which the set of non-linearizable maps is not dense. We give a characterization of degenerate families, which proves that they are quite exceptional.

***Résumé* (De l'ubiquité des disques de Siegel à bord lisse).** — Nous démontrons l'existence de disques de Siegel à bord lisse dans la plupart des familles de fonctions holomorphes fixant l'origine. La méthode peut également donner d'autres types de régularité pour le bord. On demande à la famille d'avoir un point fixe indifférent en 0, d'être paramétrisée par le nombre de rotation α, de dépendre d'α de façon Lipschitz-continue et d'être non-dégénérée. Une famille est dite dégénérée si l'ensemble de ses applications non-linéarisables n'est pas dense. Nous donnons une caractérisation des familles dégénérées, qui prouve qu'elles sont assez exceptionnelles.

Introduction

In [**24**], Pérez-Marco was the first to prove the existence of univalent maps $f : \mathbb{D} \to \mathbb{C}$ having Siegel disks compactly contained in $\mathbb{D}$ and with smooth (C^∞) boundaries. The methods in [**24**] can in fact give any class of regularity below analytic, in particular quasi-analytic classes, but also maps that are C^α for a prescribed α but for no bigger α, etc. However the maps produced in [**24**] do not a priori have an extension to an entire map, let alone polynomial. In [**3**], we were able to adapt some of these techniques and show the existence of quadratic polynomials having Siegel disks with smooth boundaries (for a simplification of the proof, and an extension to unisingular meromorphic maps, see [**17**]). In [**11**], two authors of the present article

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proved the existence of quadratic polynomials having Siegel disks whose boundaries have any prescribed regularity between C^0 and analytic (excluded).

In this article, we generalize these results to most families of maps having a non persistent indifferent cycle.

Definition 1 (Non-degenerate families). — Assume $I \subset \mathbb{R}$ is an open interval. Consider a family of holomorphic maps $f_\alpha : \mathbb{D} \to \mathbb{C}$ parameterized by $\alpha \in I$, with

$$f_\alpha(z) = e^{2\pi i\alpha} z + \mathcal{O}(z^2)$$

and assume that f_α depends continously [(1)] on α. We say that the family is *degenerate* if the set $\{\alpha \in I\ ;\ f_\alpha \text{ is not linearizable}\}$ is not dense in I. Otherwise it is called *non-degenerate.*

This definition is purely local so if we are given a holomorphic dynamical system on a Riemann surface, we can extend the definition above by working in a chart and restricting the map to a neighborhood of the fixed point.

For example, if f_α is a family of rational maps of the same degree $d \geq 2$, then it is automatically non-degenerate. Indeed, a fixed point of a rational map of degree ≥ 2 whose multiplier is a root of unity is never linearizable. [(2)]

In Appendix A we characterize degenerate families in the case where the dependence with respect to the parameter α is analytic.

Notation 2. — Assume $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic map having an indifferent fixed point at 0. We write

- $K(f)$ the set of points in $\mathbb{D}$ whose forward orbit remains in $\mathbb{D}$ and
- $\Delta(f)$ the connected component of the interior of $K(f)$ that contains 0; $\Delta(f) = \varnothing$ if there is no such component.

Remark (Siegel disks). — If $\Delta(f) \neq \varnothing$ it is known that $\Delta(f)$ is simply connected [(3)] and that the restriction $f : \Delta(f) \to \Delta(f)$ is analytically conjugate to a rotation via a conformal bijection between $\Delta(f)$ and $\mathbb{D}$ sending 0 to 0, see Section 1.3. The set $\Delta(f)$ is usually called a *Siegel disk* in the case $\alpha \notin \mathbb{Q}$ and we will use the same terminology in this article for the case $\alpha \in \mathbb{Q}$, though subtleties arise. See Section 1.1

We prove here that the main theorem in **[11]** holds for a non-degenerate family under the assumption that the dependence $\alpha \mapsto f_\alpha$ is Lipschitz. [(4)] We thus get in particular (see Appendix B for the general statement):

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(1) This means: $(\alpha, z) \mapsto f_\alpha(z)$ is continuous.

(2) This is a simple and classical fact, that seems difficult to find in written form. If an iterate of f is conjugate on an open set U to a finite order rotation then a further iterate of f is the identity on U. Since f^n is holomorphic on the Riemann sphere, it is the identity everywhere. This contradicts the fact that f^n has degree $d^n > 1$.

(3) This is another classical fact. See Footnote 8 in Section 1.1.

(4) By this we mean: $(\exists C > 0)$ $(\forall \alpha \in I,\, \alpha' \in I,\, z \in \mathbb{D})$, $|f_{\alpha'}(z) - f_\alpha(z)| \leq C|\alpha' - \alpha|$.

Theorem 3. — *Under the non-degeneracy assumptions of Definition 1, if moreover the dependence $\alpha \mapsto f_\alpha$ is Lipschitz then*
- — $\exists \alpha \in I \setminus \mathbb{Q}$ *such that* $\Delta(f_\alpha)$ *is compactly contained in* $\mathbb{D}$ *and* $\partial\Delta(f_\alpha)$ *is a* C^∞ *Jordan curve;*
- — $\exists \alpha \in I \setminus \mathbb{Q}$ *such that* $\Delta(f_\alpha)$ *is compactly contained in* $\mathbb{D}$ *and* $\partial\Delta(f_\alpha)$ *is a Jordan curve but is not a quasicircle;*
- — $\forall n \geq 0$, $\exists \alpha \in I \setminus \mathbb{Q}$, $\Delta(f_\alpha)$ *is compactly contained in* $\mathbb{D}$ *and* $\partial\Delta(f_\alpha)$ *is a Jordan curve which is* C^n *but not* C^{n+1}.

A family satisfying the assumptions on the interval I also satisfies them on every sub-interval. It follows that the parameters α in the theorem above are in fact dense in I.

Remark. — If f_α is a restriction of another map g_α and $\Delta(f_\alpha) \Subset \mathbb{D}$, then $\Delta(f_\alpha) = \Delta(g_\alpha)$, see Section 1.1. So the result gives information on the Siegel disks not only of maps from $\mathbb{D}$ to $\mathbb{C}$ but in fact of any kind of analytic maps, for instance polynomials $\mathbb{C} \to \mathbb{C}$, rational maps $\mathbb{S} \to \mathbb{S}$, entire maps $\mathbb{C} \to \mathbb{C}$, ...

Remark. — The main tool for Theorem 3 is Yoccoz's sector renormalization as in several of our previous works (except [**2**] who uses Risler's work instead [**27**]). In [**2**], [**3**] and [**11**] it was crucial to have a family for which it is known that f_α is linearizable if and only if α is a Brjuno number. The progress here is to get rid of this assumption. [(5)]

1. Conformal radius, wild combs and the general construction.

The method that Buff and Chéritat first developped to get smooth Siegel disks is one of the offsprings of a fine control, initiated in [**14**], on the periodic cycles that arise when one perturbs parabolic fixed points. Still today we can only make it work in specific contexts, which includes quadratic polynomials for instance. With the smooth Siegel disk objective in mind, Avila was able in [**2**] to identify essential sufficient properties so as to allow for a partial generalization, and also pointed to the bottleneck for a complete generalization. In this section, we essentially follow the presentation in [**2**]. We also mention a connection with continuum theory.

In this whole section, except Section 1.1, we consider a non-empty open interval $I \subset \mathbb{R}$ and a continuous family of analytic maps $f_\alpha : \mathbb{D} \to \mathbb{C}$ parameterized by $\alpha \in I$ with $f_\alpha(z) = e^{2\pi i \alpha} z + \mathcal{O}(z^2)$.

1.1. Siegel disks and restrictions. — Given a one dimensional complex manifold S and a holomorphic map

$$f : U \to S$$

(5) Note that in [**17**], optimality of Brjuno's condition was not required. However, it is assumed that f has a meromorphic extension to $\mathbb{C}$ that has only one non-zero critical or asymptotic value.

defined on an open subset U of S, assume there is a neutral fixed point $a \in U$ of multiplier $e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R}$. Call *rotation domain* any open set containing a on which the map is analytically conjugate to a rotation on a Euclidean disk or on the plane or on the Riemann sphere. [(6)] If $\alpha \notin \mathbb{Q}$ then the rotation domains are totally ordered by inclusion. This is *never* the case if $\alpha \in \mathbb{Q}$. If $\alpha \notin \mathbb{Q}$ there is a maximal element for inclusion, called *the* Siegel disk [(7)] of f at point a. If $\alpha \in \mathbb{Q}$ existence of a maximal element may also fail, depending on the situation. If $\alpha \notin \mathbb{Q}$ the Siegel disk of a restriction is automatically a subset of the original Siegel disk. If $\alpha \in \mathbb{Q}$ this may fail.

Remark. — The right approach in the general case is probably to use the *Fatou set*. Here is not the place for such a treatment, so we only give results specific to our situation

In the sequel we assume $S = \mathbb{C}$ and U is bounded and simply connected.

Following **[32]**, Section 2.4, we let K be the set of points whose orbit is defined for all times. The set $K \subset U$ is not necessarily closed in U. We let $\Delta(f) = \Delta \subset U$ where Δ is the connected component containing a of the interior K°, or $\Delta = \varnothing$ if $a \notin K^\circ$. Then Δ is necessarily simply connected: this is one classical application of the maximum principle. [(8)] Any rotation domain for a is necessarily contained in K. It follows that any rotation domain for a is in fact contained in Δ. Moreover, let us prove that Δ itself is a rotation domain:

Proof. — First note that we have $f(\Delta) \subset \Delta$. The set Δ is conformally equivalent to the unit disk $\mathbb{D}$. Conjugate f by a conformal map from Δ to $\mathbb{D}$ sending a to 0. We get a holomorphic self-map of $\mathbb{D}$ with a neutral fixed point at its center. By the case of equality in Schwarz's lemma this self-map is a rotation. □

Corollary 4. — *(We do not make an assumption on α.) Let U' be an open subset of $\mathbb{C}$. Let $g : U' \to \mathbb{C}$ be holomorphic with a neutral fixed point a. Assume U is an open subset of U' containing a and let f be the restriction of g to U. Then $\Delta(f) \subset \Delta(g)$. If moreover U and U' are simply connected and if $\Delta(f)$ is compactly contained in the domain of definition of f then $\Delta(g) = \Delta(f)$.*

Proof. — The first claim is immediate. For the second claim when $\Delta(f)$ is compactly contained in U, consider the image of $\Delta(f)$ by the uniformization $(\Delta(g), 0) \to (\mathbb{D}, 0)$: we get a simply connected subset A of $\mathbb{D}$, invariant by the rotation. In the case $\alpha \notin \mathbb{Q}$

(6) The last case is extremely specific, for we must have $U = S$ isomorphic to the Riemann sphere and f is a rotation.

(7) In the case of a rotation on the Riemann sphere this name is not appropriate since the disk is a sphere...

(8) If Δ would not be simply connected then there would exist a bounded closed set $C \neq \varnothing$ (not necessarily connected) such that $\Delta \cap C = \varnothing$ and such that $\Delta' := \Delta(f) \cup C$ is open and connected (this is a theorem in planar topology). By the maximum principle, $f^k(\Delta') \subset \mathbb{D}$ for all k. Hence Δ' would be an open subset of $K(f)$, contradicting the definition of Δ.