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PROJECTIVE COCYCLES OVER $SL(2,\mathbb{R})$ ACTIONS: MEASURES INVARIANT UNDER THE UPPER TRIANGULAR GROUP

by

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To the memory of Jean-Christophe

Abstract. — We consider the action of $SL(2,\mathbb{R})$ on a vector bundle $\mathbf{H}$ preserving an ergodic probability measure ν on the base X. Under an irreducibility assumption on this action, we prove that if $\hat{\nu}$ is any lift of ν to a probability measure on the projectivized bunde $\mathbb{P}(\mathbf{H})$ that is invariant under the upper triangular subgroup, then $\hat{\nu}$ is supported in the projectivization $\mathbb{P}(\mathbf{E}_1)$ of the top Lyapunov subspace of the positive diagonal semigroup. We derive two applications. First, the Lyapunov exponents for the Kontsevich-Zorich cocycle depend continuously on affine measures, answering a question in **[57]**. Second, if $\mathbb{P}(\mathbb{V})$ is an irreducible, flat projective bundle over a compact hyperbolic surface Σ, with hyperbolic foliation $\mathscr{F}$ tangent to the flat connection, then the foliated horocycle flow on $T^1\mathscr{F}$ is uniquely ergodic if the top Lyapunov exponent of the foliated geodesic flow is simple. This generalizes results in **[13]** to arbitrary dimension.

Résumé **(Cocycles projectifs au-dessus d'actions de $SL(2,\mathbb{R})$: mesures invariantes au-dessus du groupe triangulaire supérieur)**

Nous considérons l'action de $SL(2,\mathbb{R})$ sur un fibré vectoriel H, préservant une mesure de probabilité ergodique ν sur la base X. Soit $\hat{\nu}$ un relevé quelconque de ν qui est une mesure de probabilité sur le fibré projectivisé $\mathbb{P}(H)$, invariante sous l'action du sous-groupe triangulaire supérieur. Sous une hypothèse d'irréductibilité de l'action, nous prouvons que toute mesure $\hat{\nu}$ comme ci-dessus est supportée par le projectivisé $\mathbb{P}(E_1)$ de l'espace de Lyapunov associé à l'exposant de Lyapunov le plus grand pour l'action du semi-groupe diagonal positif. Nous en déduisons deux applications : Premièrement, les exposants du cocycle de Kontsevich-Zorich dépendent continûment des mesures affines, ce qui répond à une question de **[57]**. Deuxièmement, soit $\mathbb{P}(V)$ un fibré projectif irréductible, plat, au dessus d'une surface fermée hyperbolique Σ, et soit $\mathscr{F}$ le feuilletage à feuilles hyperboliques, tangent à la connection plate ; alors le flot horocyclique sur $T^1(\mathscr{F})$ est uniquement ergodique sous l'hypothèse que le plus grand exposant de Lyapunov du flot géodésique est simple. Ceci généralise le résultat principal de **[13]** en dimension arbitraire.

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1. Introduction

Let G denote the group $SL(2,\mathbb{R})$, and for $t\in\mathbb{R}$, let

$$a^t=\begin{pmatrix} e^t & 0\\ 0 & e^{-t}\end{pmatrix},\qquad u_+^t=\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix},\quad \text{and } u_-^t=\begin{pmatrix} 1 & 0\\ t & 1\end{pmatrix}.$$

Let $A=\{a^t : t\in\mathbb{R}\}$, $U_+=\{u_+^t : t\in\mathbb{R}\}$, and $U_-=\{u_-^t : t\in\mathbb{R}\}$. The group G is generated by A, U_+ and U_-. We denote by $P=AU_+$ the group of upper triangular matrices, and we recall that the group P is solvable and hence *amenable*: any action of P on a compact metric space admits an invariant probability measure (see, e.g., [**22**, Section 8.4]). Recall that $PSL(2,\mathbb{R})$ admits a natural identification with the unit tangent bundle $T^1\mathbb{H}^2$ where $\mathbb{H}^2$ is the upper half plane endowed with the hyperbolic metric. Through this identification, the left action of $SL(2,\mathbb{R})$ on itself induces an identification of $A=\{a^t\}$ with the geodesic flow, and U_+ (resp. U_-) with the unstable (resp. stable) horocycle flow of $\mathbb{H}^2$.

In this paper, we consider the following situation. Suppose G acts on a separable metric space X, preserving an ergodic Borel probability measure ν. By the Mautner phenomenon (see e.g., [**64**]), ν is ergodic with respect to the action of the diagonal group A. Let $\mathbf{H}\to X$ be a continuous vector bundle over X with fiber a finite dimensional vector space H, and write $\mathbf{H}(x)$ for the fiber of $\mathbf{H}$ over $x\in X$. Suppose that G acts on $\mathbf{H}$ by linear automorphisms on the fibers and the given action on the base. We denote the action of $g\in G$ on $\mathbf{H}$ by g_*, so for $\mathbf{v}\in\mathbf{H}(x)$, we have $g_*\mathbf{v}\in\mathbf{H}(gx)$.

Assume that $\mathbf{H}$ is equipped with a Finsler structure (that is, a continuous choice of norm $\{\|\cdot\|_x : x\in X\}$ on the fibers of $\mathbf{H}$), and that with respect to this Finsler, the action of G on $\mathbf{H}$ satisfies the following integrability condition:

$$\int_X \sup_{t\in[-1,1]}\left(\log\|a_*^t\|_x\right)\,d\nu(x)<+\infty, \tag{1}$$

where, for $g\in G$, $\|g_*\|_x$ denotes the operator norm of the linear action of g_* on the fiber over x with respect to the Finsler structure:

$$\|g_*\|_x:=\sup_{\{\mathbf{v}\in H:\|\mathbf{v}\|_x=1\}}\|g_*\mathbf{v}\|_{gx}.$$

Since the A-action on X is ergodic with respect to ν, the Oseledets multiplicative ergodic theorem implies that there exists an a^t-equivariant splitting

$$\mathbf{H}(x)=\bigoplus_{j=1}^m \mathbf{E}_j(x), \tag{2}$$

defined for ν-almost every $x\in X$, and real numbers $\lambda_1>\cdots>\lambda_m$ (called the *Lyapunov exponents* of the A-action) such that for ν-a.e. $x\in X$ and all $\mathbf{v}\in\mathbf{E}_j(x)$,

$$\lim_{|t|\to\infty}\frac{1}{t}\log\frac{\|a_*^t\mathbf{v}\|_{a^tx}}{\|\mathbf{v}\|_x}=\lambda_j.$$

Definition 1.1 **(ν-measurable invariant subbbundle).** — A ν-measurable invariant subbundle for the G action on $\mathbf{H}$ is a measurable linear subbundle $\mathbf{W}$ of $\mathbf{H}$ with the property that $g_* (\mathbf{W}(x)) = \mathbf{W}(gx)$, for ν-a.e. $x \in X$ and every $g \in G$.

Definition 1.2 **(Irreducible).** — We say that the G action on $\mathbf{H}$ is *irreducible with respect to the G-invariant measure ν on X* if it does not admit ν-measurable invariant subbundles: that is, if $\mathbf{W}$ is a ν-measurable invariant subbundle, then either $\mathbf{W}(x) = \{0\}$, ν-a.e. or $\mathbf{W}(x) = \mathbf{H}(x)$, ν-a.e.

We note that the bundles $\mathbf{E}_j$ in the splitting (2) are not invariant in the sense of Definition 1.1, since they are (in general) equivariant only under the subgroup A and not under all of G.

Let $\mathbb{P}(H)$ be the projective space of H (i.e., the space of lines in H), and let $\mathbb{P}(\mathbf{H})$ be the projective bundle associated to $\mathbf{H}$ with fiber $\mathbb{P}(H)$. Then G also acts on $\mathbb{P}(\mathbf{H})$ via the induced projective action on the fibers. The space $\mathbb{P}(\mathbf{H})$ may not support a G-invariant measure, but since P is amenable and $\mathbb{P}(H)$ is compact, it will always support a P-invariant measure. In particular, for any P-invariant measure μ on X, there will be a P-invariant measure $\hat{\mu}$ on $\mathbb{P}(\mathbf{H})$ that projects to μ under the natural map $\mathbb{P}(\mathbf{H}) \to X$. For such a μ, denote by $\mathscr{M}^1_P(\mu)$ the (nonempty) set of all P-invariant Borel probability measures on $\mathbb{P}(\mathbf{H})$ projecting to μ on X.

We can now state our main theorem.

Theorem 1.3. — *Suppose that the G-action on $\pi\colon \mathbf{H} \to X$ is irreducible with respect to the G-invariant (and therefore P-invariant) measure ν on X, and let $\hat{\nu} \in \mathscr{M}^1_P(\nu)$. Disintegrating $\hat{\nu}$ along the fibers of $\mathbb{P}(\mathbf{H})$, write*

$$d\hat{\nu}([\mathbf{v}]) = d\eta_{\pi(\mathbf{v})}([\mathbf{v}])\, d\nu(\pi(\mathbf{v})),$$

where $[\mathbf{v}] \in \mathbb{P}(\mathbf{H})$ denotes the line determined by $\{0\} \neq \mathbf{v} \in \mathbf{H}$. Then for ν-a.e. $x \in X$, the measure η_x on $\mathbb{P}(\mathbf{H})(x)$ is supported on $\mathbb{P}(\mathbf{E}_1(x))$, where as in (2), $\mathbf{E}_1(x)$ is the Lyapunov subspace corresponding to the top Lyapunov exponent of the A-action.

In particular, if $\mathbf{E}_1$ is one-dimensional, then $\#\mathscr{M}^1_P(\nu) = 1$.

The same conclusions of Theorem 1.3 hold when $G = SL(2, \mathbb{R})$ is replaced by any rank 1 semisimple Lie group, and P denotes the minimal parabolic subgroup of G (i.e., the normalizer of the unipotent radical of G).

P-invariant measures are a natural object of study, as they are closely related to stationary measures of a G-action. Suppose that G acts on a space Ω, and let m be a Borel probability measure on G. Recall that a probability measure ρ on Ω is *m-stationary* if $m * \rho = \rho$, where for $A \subset \Omega$ measurable, we define

$$m * \rho(A) = \int_G \rho(gA)\, dm(g).$$

A compactly supported Borel probability measure m on G is *admissible* if the following two conditions hold: first, there exists a $k \geq 1$ such that the k-fold convolution $m^{\star k}$ is absolutely continuous with respect to Haar measure; second, $\mathrm{supp}(m)$ generates G

as a semigroup. Furstenberg ([**43**], [**42**], restated as [**66**, Theorem 1.4]), proved that there is a 1–1 correspondence between P-invariant measures on Ω and m-stationary measures for admissible m. In fact any m-stationary measure on Ω is of the form $\lambda * \hat{\nu}$ where λ is the unique m-stationary measure on the Furstenberg boundary of G (which is in our case the circle G/P) and $\hat{\nu}$ is a P-invariant measure on Ω.

Remark. — In light of the discussion above, for any admissible measure m on G, Theorem 1.3 also gives a classification of the m-stationary measures on $\mathbf{H}$ projecting to ν.

In the context where $X = SL(2,\mathbb{R})/\Gamma$, with Γ cocompact (or of finite covolume), $\mathbf{H}$ a flat H-bundle over X, and $H = \mathbb{R}^2$, $\mathbb{C}^2$ or $\mathbb{C}^3$, Theorem 1.3 was proved by Bonatti and Gomez-Mont [**13**], where irreducibility is replaced with the equivalent hypothesis that $\rho(\Gamma)$ is Zariski dense, where ρ is the monodromy representation.

Some constructions used in the proof of Theorem 1.3 are also used in [**29**] in their classification of $SL(2,\mathbb{R})$ invariant probability measures on moduli spaces.

2. Applications and the irreducibility criterion

The irreducibility hypothesis in Theorem 1.3 is not innocuous. Checking for the non-existence of invariant *measurable* subbundles is in general an impossible task, but there are two restricted contexts where it is feasible, on which we focus here:

— Suppose that $X = G/\Gamma$, for some discrete subgroup $\Gamma \subset G$, and $\mathbf{H}$ is a flat bundle over $X = G/\Gamma$ with monodromy representation $\rho\colon \Gamma \to GL(H)$. Then G acts transitively on X, the P-invariant measures on X are all algebraic by [**64**] (which uses Ratner's Theorem), and irreducibility is then equivalent to the condition that there are no invariant algebraic subbundles of $\mathbf{H}$. In the case where ν is Haar measure, irreducibility of the associated G-action reduces to the condition that ρ is an irreducible representation. In Subsection 2.1 we derive some consequences of Theorem 1.3 in this context.

— If the bundle $\mathbf{H}$ admits a Hodge structure (not necessarily G-invariant) then checking irreducibility can sometimes be reduced to showing that there are no invariant subbundles that are compatible with the Hodge structure, a much simpler task (since such subbundles must be real-analytic). In particular, the condition that G acts transitively on the base in the previous setting can be relaxed. This has been established rigorously by Simion Filip for the Kontsevich-Zorich action, and we use Theorem 1.3 in Subsection 2.2 to deduce further results in that context.

  The mantra here is that for such Hodge bundles whose base supports a G-invariant measure, any measurable G-invariant subbundle must come from algebraic geometry. For the Kontsevich-Zorich action, this has been established in [**23**].

We now describe the applications in more detail.