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On quasi-invariant curves

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ON QUASI-INVARIANT CURVES

by

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Abstract. — Quasi-invariant curves are the fundamental tool for the study of hedgehog's dynamics. The Denjoy-Yoccoz lemma is the preliminary step for Yoccoz's complex renormalization techniques for the study of linearization of analytic circle diffeomorphisms. We give a new geometric interpretation of the Denjoy-Yoccoz lemma using the hyperbolic metric that gives a new direct construction of quasi-invariant curves without renormalization theory as in the original construction.

***Résumé* (Sur les courbes quasi-invariantes).** — Les courbes quasi-invariantes sont un outil fondamental dans l'étude de la dynamique des hérissons. Le lemme de Denjoy-Yoccoz est le premier pas dans la théorie de renormalisation de Yoccoz des difféomorphismes analytiques du cercle et l'étude de sa linéarisation. On donne une nouvelle version du lemme de Denjoy-Yoccoz en termes de métrique hyperbolique, ce qui fournit une nouvelle construction directe des courbes quasi-invariantes sans utiliser la renormalisation comme dans la construction originelle.

1. Introduction

Quasi-invariant curves and their properties were announced in 1995 in a Note to the Comptes Rendus of the Académie des Sciences [**6**] presented by J.-Ch. Yoccoz. Their construction using renormalization techniques was carried out in the unpublished manuscript [**8**]. The goal of the present article is to present a short and direct construction of quasi-invariant curves without renormalization theory. In particular, this is the first complete published construction of quasi-invariant curves.

***Theorem 1* (Quasi-invariant curves).** — *Let g be an analytic circle diffeomorphism with irrational rotation number α. Let $(p_n/q_n)_{n\geq 0}$ be the sequence of convergents of α given by the continued fraction algorithm.*

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Given $C_0 > 0$ there is $n_0 \geq 0$ large enough such that there is a sequence of Jordan curves $(\gamma_n)_{n \geq n_0}$, homotopic to $\mathbb{S}^1$ and exterior to $\overline{\mathbb{D}}$, such that all the iterates g^j, $0 \leq j \leq q_n$, are defined on a neighborhood of the closure of the annulus U_n bounded by $\mathbb{S}^1$ and γ_n, and we have

$$\mathscr{D}_P(g^j(\gamma_n), \gamma_n) \leq C_0,$$

where $\mathscr{D}_P$ is the Hausdorff distance between compact sets associated to d_P, the Poincaré distance in $\mathbb{C} - \overline{\mathbb{D}}$. We also have for any $z \in \gamma_n$, $d_P(g^{q_n}(z), z) \leq C_0$, that is,

$$||g^{q_n} - \mathrm{id}||_{C^0_P(\gamma_n)} \leq C_0.$$

The curves γ_n are called quasi-invariant curves for g.

The delicate, and useful, part of the construction of quasi-invariant curves is to obtain the estimates for the Poincaré metric, which is much harder and stronger than the estimates for the euclidean metric since the curves γ_n are close to $\mathbb{S}^1$. This is also what is needed for the application to hedgehog's dynamics. Hedgehogs are totally invariant continua associated to indifferent irrational non-linearizable fixed points discovered by the author in [7]. The dynamics in a neighborhood outside a hedgehog K is conjugated to the dynamics of an analytic circle diffeomorphism by the dictionary construction presented in [7]. Quasi-invariant curves with their Poincaré metric estimates can then be transported to osculating curves around the hedgehog and provide the tool to analyze the dynamics on the hedgehog. This is only possible thanks to the Poincaré estimates.

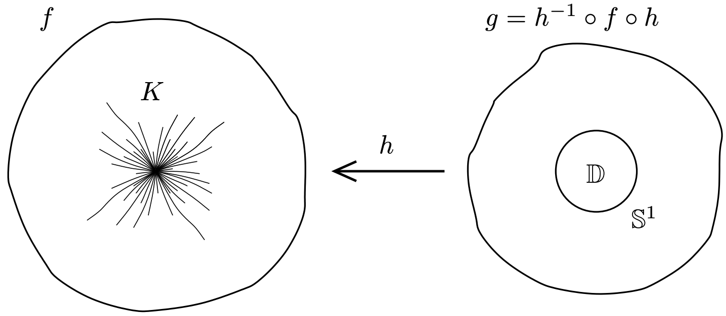

FIGURE 1. Dictionary of fixed points and circle maps using hedgehog K.

The new construction of quasi-invariant curves without renormalization is based on the key observation that the Denjoy-Yoccoz Lemma from [13] (Proposition 4.4 in Section 4.4) has a natural hyperbolic interpretation. First we carry out the construction in the situation where we assume that the non-linearity of g is small, that is $||D \log Dg||_{C^0}$ small. This case is enough for most of the applications, in particular for the solution in [9] of Briot and Bouquet problem from 1856 (see [1]). The general case reduces to this situation by the same arguments as in Section 3.6 of [13] that reduces by a sectorial return map the dynamics of a general analytic circle diffeomorphism to the dynamics of one with arbitrarily small non-linearity. The proof of the Denjoy-Yoccoz Lemma relies on real estimates for the iterates of circle diffeomorphisms that follow from classical work by M. Herman ([**4**]) and J.-Ch. Yoccoz ([**11**]).

2. Analytic circle diffeomorphisms

2.1. Notations. — We refer to the Thesis of M. Herman [**4**] for the classical theory and background on circle diffeomorphisms. We denote by $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ the abstract circle, and $\mathbb{S}^1=E(\mathbb{T})$ its embedding in the complex plane $\mathbb{C}$ given by the exponential mapping $E(x)=e^{2\pi i x}$.

We study analytic diffeomorphisms of the circle, but we prefer to work at the level of the universal covering, the real line, with its standard embedding $\mathbb{R}\subset\mathbb{C}$. We denote by $D^\omega(\mathbb{T})$ the space of increasing analytic diffeomorphisms g of the real line such that, for any $x\in\mathbb{R}$, $g(x+1)=g(x)+1$, i.e g commutes with $T(x)=x+1$, the generator of the group of deck transformations of the universal covering. Thanks to H. Poincaré [**10**], we know that an element of the space $D^\omega(\mathbb{T})$ has a well defined rotation number $\rho(g)\in\mathbb{R}$ which is the constant uniform limit of $\frac{1}{n}(g^n-id)$ when $n\to+\infty$. Thanks to A. Denjoy [**2**], we know that the order preserving diffeomorphism g is indeed conjugated to the rigid translation $T_{\rho(g)}: x\mapsto x+\rho(g)$, by an orientation preserving homeomorphism $h:\mathbb{R}\to\mathbb{R}$, such that $h(x+1)=h(x)+1$.

For $\Delta>0$, we note $B_\Delta=\{z\in\mathbb{C}; |\Im z|<\Delta\}$, and $A_\Delta=E(B_\Delta)$. The subspace $D^\omega(\mathbb{T},\Delta)\subset D^\omega(\mathbb{T})$ is composed by the elements of $D^\omega(\mathbb{T})$ which extend analytically to a holomorphic diffeomorphism, denoted again by g, such that g and g^{-1} are defined on B_Δ.

2.2. Real estimates. — We refer to J.-Ch. Yoccoz article [**13**] for the results on this section. We assume that the orientation preserving circle diffeomorphism g is C^3 and that the rotation number $\alpha=\rho(g)$ is irrational. We consider the convergents $(p_n/q_n)_{n\geq 0}$ of α obtained by the continued fraction algorithm (see [**3**] for notations and basic properties of continued fractions).

For $n\geq 0$, we define the map $g_n(x)=g^{q_n}(x)-p_n$ and the intervals $I_n(x)=[x,g_n(x)]$, $J_n(x)=I_n(x)\cup I_n(g_n^{-1}(x))=[g_n^{-1}(x),g_n(x)]$. Let $m_n(x)=g^{q_n}(x)-x-p_n=\pm|I_n(x)|$, $M_n=\sup_{\mathbb{R}}|m_n(x)|$, and $m_n=\min_{\mathbb{R}}|m_n(x)|$. Topological linearization obviously implies $\lim_{n\to+\infty}M_n=0$, since this holds for a rigid rotation, and is equivalent to this condition since then any orbit is dense modulo 1 and determines uniquely h modulo 1. This is always true for analytic diffeomorphisms by Denjoy's Theorem, that also holds for C^1 diffeomorphisms such that $\log Dg$ has bounded variation.

Since g is topologically linearizable, the combinatorics of the irrational translation, or the continued fraction algorithm, shows (see Lemma 3.7 [**13**]):

Lemma 2. — *Let $x\in\mathbb{R}$, $0\leq j<q_{n+1}$ and $k\in\mathbb{Z}$. The intervals $g^j\circ T^k(I_n(x))$ have disjoint interiors, and the intervals $g^j\circ T^k(J_n(x))$ cover $\mathbb{R}$ at most twice.*

We have the following fundamental estimate (see [4], [11] and, more precisely, Corollary 3.16 in [13]) on the Schwarzian derivatives of the iterates of f, for $0 \leq j \leq q_{n+1}$,

$$\left|Sg^j(x)\right| \leq \frac{M_n e^{2V} S}{|I_n(x)|^2},$$

with $S = ||Sg||_{C^0(\mathbb{R})}$ and $V = \operatorname{Var} \log Dg$.

These estimates imply a control of the non-linearity of the iterates (see Corollary 3.18 in [13]):

Proposition 3. — *For $0 \leq j \leq 2q_{n+1}$, $c = \sqrt{2S}e^V$, we have*

$$||D \log Dg^j||_{C^0(\mathbb{R})} \leq c \, \frac{M_n^{1/2}}{m_n}.$$

These estmates on the iterates of g give estimates on g_n. More precisely, we have (Corollary 3.20 in [13]):

Proposition 4. — *For some constant $C > 0$, we have*

$$|| \log Dg_n||_{C^0(\mathbb{R})} \leq C M_n^{1/2}.$$

Corollary 5. — *For any $\epsilon > 0$, there exists $n_0 \geq 1$ such that for $n \geq n_0$, we have*

$$||Dg_n - 1||_{C^0(\mathbb{R})} \leq \epsilon.$$

Proof. — Take $n_0 \geq 1$ large enough so that for $n \geq n_0$, $C M_n^{1/2} < \min(\frac{2}{3}\epsilon, \frac{1}{2})$, then use Proposition 4 and the estimate

$$|e^w - 1| \leq \frac{3}{2}|w|$$

for $|w| < 1/2$. □

Corollary 6. — *For any $\epsilon > 0$, there exists $n_0 \geq 1$ such that for $n \geq n_0$, for any $x \in \mathbb{R}$ and $y \in I_n(x)$ we have*

$$1 - \epsilon \leq \frac{m_n(y)}{m_n(x)} \leq 1 + \epsilon.$$

Proof. — We have $Dm_n(x) = Dg_n(x) - 1$, and

$$|m_n(y) - m_n(x)| \leq ||Dm_n||_{C^0(\mathbb{R})}|y - x| \leq ||Dg_n - 1||_{C^0(\mathbb{R})}|m_n(x)|.$$

We conclude using Corollary 5. □