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POSITIVE FIBERED LYAPUNOV EXPONENTS FOR SOME QUASI-PERIODICALLY DRIVEN CIRCLE ENDOMORPHISMS WITH CRITICAL POINTS

by

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Abstract. — In this paper we give examples of skew-product maps $T:\mathbb{T}^2\to\mathbb{T}^2$ of the form $T(x,y)=(x+\omega,x+f(y))$, where $f:\mathbb{T}\to\mathbb{T}$ is an explicit C^1-endomorphism of degree two with a unique critical point and ω belongs to a set of positive measure, for which the fibered Lyapunov exponent is positive for a.e. $(x,y)\in\mathbb{T}^2$. The critical point is of type $f'(\pm e^{-s})\approx e^{-\beta s/(\ln s)^2}$ for all large s, where $\beta>0$ is a small numerical constant.

Résumé **(Exposants de Lyapunov fibrés pour certains produits-croisés d'endomorphismes du cercle avec points critiques et force quasi-périodique)**

Dans cet article, nous donnons des exemples de produits croisés $T:\mathbb{T}^2\to\mathbb{T}^2$ de forme $T:(x,y)=(x+\omega,x+f(y))$, où $f:\mathbb{T}\to\mathbb{T}$ est un endomorphisme C^1 explicite de degré deux avec un point critique unique et où ω appartient à un ensemble de mesures positives, dont l'exposant de Lyapunov fibré est positif pour presque tout $(x,y)\in\mathbb{T}^2$. Le point critique est de type $f'(\pm e^{-s})\approx e^{-\frac{\beta s}{(\log s)^2}}$ pour tout s grand, où $\beta>0$ est une petite constante numérique.

1. Introduction

1.1. One dimensional models. — The investigation of the dynamics of the quadratic family $f_c(x)=cx(1-x)$, $c\in[0,4]$, has been a real success story since the seminal works by Jakobson [**5**] and by Benedicks and Carleson [**1**](see [**6**] for an excellent review of this development). We have seen that regular and chaotic dynamics is completely intertwined: for an open and dense set of parameters $c\in[0,4]$ the map f_c has an attracting periodic orbit which attracts a.e. $x\in[0,1]$; and for a set of positive Lebesgue measure of parameters c the map f_c has an absolutely continuous invariant probability measure, and the Lyapunov exponents $\lim_{n\to\infty}\log|(f^n)'(x)|/n$ are positive for a.e. $x\in[0,1]$. Furthermore, a.e. $c\in[0,1]$ falls into one of these two cases.

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Several other families of one-dimensional maps (with a single critical point) have been studied; e.g., the one by Collet-Eckmann [3]; and recently there has been interest in the family of circle endomorphisms $g_c(x) = 2x + c + \frac{1}{\pi}\sin 2\pi x$ (modification of the Arnold family); see, e.g., [4, 7]. The latter family have many similarities with the quadratic family, but also differences.

In the analysis of each of the above mentioned families, the orbit of the (unique) critical point, and it's recurrence to itself, plays a central role for the global dynamics. To control this dynamics one needs information about the the close returns of the iterates of the critical point. Such recurrence conditions, in turn, induces conditions on the parameters c. For example, note that by the chain rule we have $|(f^n)'(x)| = \prod_{j=0}^{n-1}|f'(x_j)|$, where $x_j = f^j(x)$. From this expression it is clear why the iterates x_j cannot be allowed to come too close to the critical point too fast, or too frequently make close returns, if one wants $|(f^n)'(x)|$ to grow with n.

1.2. Skew-products. — A natural question to ask is what can happen with the dynamics if we instead of iterating with the same map, iterate with different family members h_c along the orbit (i.e., consider orbits $(y_n)_{n\geq 0}$ where $y_n = h_{c_n}\circ\cdots\circ h_{c_1}(y)$). To investigate this, let $h_c : Y \to Y$, $c \in I$, be a one-parameter family of maps on $Y = [0,1]$ or $Y = \mathbb{T}$, and consider skew-product maps $T(x,y) = (g(x), h_{c(x)}(y))$ where $(X, \mathscr{B}, \mu)$ is a probability space, $g : X \to X$ a measure preserving transformation and $c : X \to I$. We define the fibered Lyapunov exponent by

$$L(x,y) = \varliminf_{n\to\infty}\frac{1}{n}\log|\partial y_n/\partial y| = \varliminf_{n\to\infty}\frac{1}{n}\log\left|\prod_{j=0}^{n-1}h'_{c(x_j)}(y_j)\right|,$$

where we use the notation $(x_n, y_n) = T^n(x,y)$.

In this article we are interested in the question when one can have $L(x,y) > 0$ for most (x,y), although each family member h_c has a critical point. A natural model is to let h_c be the quadratic family f_c above, and assume that $c(x) \leq 4$ is non-constant and close to 4. It is well-known that f_4 is chaotic (note that the critical $x = 1/2$ is mapped to the unstable fixed point $x = 0$), and it is known that for "most" (in measure sense) parameters c close to 4, the map f_c is chaotic. One could therefore expect that the chaotic behavior dominates over the regular behavior (recall that for an open and dense set of c, f_c is regular) and that typically $L(x,y) > 0$. However, proving that this is the case has shown to be very difficult (and the difficulty depends on the base dynamics g).

In the paper [9] Viana showed (among other things) that if h_c is the quadratic family, $g : \mathbb{T} \to \mathbb{T}$ is a strongly expanding map (like $g(x) = dx$, d a large integer) and $c(x) \leq 4$ is close to 4, then $L(x,y) > 0$ for a.e. $(x,y) \in \mathbb{T}^2$. A serious difficulty here, compared with the one-dimensional case, is that the we now have a whole "critical circle": $\mathbb{T}\times\{1/2\}$. And iterations of this circle will intersect itself. Thus one cannot avoid critical points returning to the critical circle. This is a major problem.

The strong expansion of g (chaoticity) is crucial for the approach in [9]. It is therefore interesting to ask what can happen if g is far from expanding (and non-periodic). In this article we are interested in the question when $L(x, y) > 0$ for "most" (x, y) for families h_c, where each member has a (unique) critical point, and when $g : \mathbb{T} \to \mathbb{T}$ is an irrational rotation $g(x) = x + \omega$. We shall give examples of families h_c for which we can give a positive answer (see the next subsection). The most natural model would be the setting of [9], replacing the base dynamics by $g(x) = x + \omega$; but unfortunately our techniques cannot handle this case due to the fact that the critical point of f_c (which is quadratic) is "too flat".

We end this subsection by mentioning that related problems have been investigated by Sester in [8].

1.3. Our model.

— To get the analysis as clean as possible, we follow [3] and assume that our fiber maps are affine outside a small region close to the critical point. More precisely, we consider families $h_x : \mathbb{T} \to \mathbb{T}$ of the form $h_x(y) = x + f(y)$ where $f : \mathbb{T} \to \mathbb{T}$ is C^1 and of degree 2, and satisfies the following (it is C^0-close to $y \mapsto 2y$): Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. Then $F(y+1) = F(y)+2$; the derivative F' is continuous and satisfies

$$F'(y) = \begin{cases} a, & \varepsilon \le |y| \le 1/2; \\ a\varepsilon^{-\beta}|y|^{\frac{\beta}{(\ln(\ln|y|/\ln\varepsilon))^2+1}}, & 0 < |y| \le \varepsilon; \\ 0, & y = 0. \end{cases}$$

Here $0 < \varepsilon \ll 1$. Since ε is small it follows that $a > 2$ and $a \approx 2$. The constant $\beta > 0$ is a small (absolute) numerical constant (see Lemma 2.2). We note that f has a unique critical point, located at $y = 0$. We also note that

$$F'(\pm\varepsilon^s) = a\varepsilon^{\beta(s/((\ln s)^2+1)-1)} \text{ for all } s \ge 1.$$

Our main object is the skew-product map $T : \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$T(x, y) = (x + \omega, x + f(y)). \tag{1.1}$$

The frequency $\omega \in \mathbb{T}$ will not be fixed from the beginning; instead we shall use ω as a parameter (like in, e.g., [2, 10]). Thus, $T = T_\omega$. Note that T can be viewed as a perturbation of the map $S(x, y) = (x + \omega, x + 2y)$.

1.4. Statement of results

Theorem 1. — *There exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there is a non-empty set $\Omega = \Omega(\varepsilon) \subset \mathbb{T}$ ($|\Omega| \to 1$ as $\varepsilon \searrow 0$) such that for each frequency $\omega \in \Omega$ the following holds for the map T in (1.1):*

$$L(x, y) > (\log a)/4 \textit{ for a.e. } (x, y) \in \mathbb{T}^2.$$

The strategy of the proof is to first show that we have good expansion under certain (abstract) assumptions on ω. This is done in Section 2. Then, in Section 3 we show that there indeed is a large set of ω satisfying all the assumptions needed in Section 2.

Remark 1. — a) We would like to be able to handle "flatter" critical points, like $f'(y) = a\varepsilon^{-\beta}|y|^{\beta}$ for $|y| \le \varepsilon$; and especially $\beta = 2$. Unfortunately the proof does not work in this case, even for very small $\beta > 0$.

b) In the setting of this article, a natural model to investigate would be the quasi-periodically driven double standard map: $(x, y) \mapsto (x + \omega, 2y + x + (\sin 2\pi y)/\pi)$.

2. Building expansion for good ω

We first observe that, from the assumptions on f, we always have expansion outside the strip $\mathbb{T} \times [-\varepsilon, \varepsilon]$:

Lemma 2.1. — *If (x, y) are such that $|y_j| \ge \varepsilon$ for all $0 \le j \le k$ then*

$$\prod_{j=0}^{k} f'(y_j) = a^{k+1}.$$

Proof. — Follows immediately from the assumption $f'(z) = a$ for all $|z| \ge \varepsilon$. □

2.1. The functions φ_k and the sets A_k^s. — Since it will be important for the analysis to keep track of the iterates of the critical circle $\mathbb{T} \times \{0\}$, as well as the sets of x where these iterates come back close to the critical circle, we define the functions

$$\varphi_k(x) = \pi_2(T^k(x, 0))$$

and the sets

$$A_k^s = \{x : |\varphi_k(x)| \le \varepsilon^s\}, \quad s > 0.$$

Clearly $A_k^t \subset A_k^s$ for all $k = 1, 2, \ldots$ if $t > s$.

Note that, by definition, we have $T^k(x, 0) = (x + k\omega, \varphi_k(x))$.

2.2. Scales and some arithmetics. — Let $s_0 = 1$ and $N_0 = 1$, and define, for each $k \ge 1$, the integers N_k and numbers s_k by:

$$\begin{aligned} \varepsilon^{-s_{k-1}/5}/a < a^{N_k} &\le \varepsilon^{-s_{k-1}/5} \\ s_k &= 2^k. \end{aligned} \tag{2.1}$$

Remark 2. — (1) Later in this section we will use the obvious fact that $a^{N_k}\varepsilon^{s_{k-1}} < \varepsilon^{3s_{k-1}/4}/2$. (2) We could take $s_k = e^{rk}$ for any $r > 0$; it only changes the value of β in the assumption on f'.

That the following estimates hold is central for the analysis (it is only here where assumptions on β is needed).

Lemma 2.2. — *For all sufficiently small $\varepsilon, \beta > 0$ the following holds: For each $n \ge 1$ we have*

$$a^{(1+1/n)(N-5)/2} f'(\varepsilon^{s_n})^5 > a^{(1+1/(n+1))N/2} \text{ for all } N \ge N_n/3.$$