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RECURRENCE ON INFINITE CYCLIC COVERINGS

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Abstract. — We study the chain-recurrence properties of the lift of a homeomorphism to an infinite cyclic cover. This is related to the Poincaré-Birkhoff theorem, see the work of John Franks [**2, 3**].

***Résumé* (Récurrence sur les revêtements infinis cycliques).** — Nous étudions les propriétés de récurrence par chaînes du relevé d'un homéomorphisme à un revêtement infini cyclique. Cette étude est connectée au théorème de Poincaré-Birkhoff, voir les travaux de John Franks [**2, 3**].

1. Introduction

Although our results are valid in a more general setting, we will restrict to the manifold case.

In this work, we fix a compact connected manifold M (possibly with boundary). We will suppose that $\pi : \tilde{M} \to M$ is a given infinite cyclic covering. To distinguish between objects related to M and $\tilde{M}$, we will us a $\sim$ on top of objects related to $\tilde{M}$. For example, a point in M will be denoted by x, and a point in $\tilde{M}$ denoted by $\tilde{x}$.

We denote the $\mathbb{Z}$ action on $\tilde{M}$ given by deck transformations of the infinite cyclic covering $\pi : \tilde{M} \to M$ by $(n, \tilde{x}) \mapsto \tilde{x} + n$, whith $n \in \mathbb{Z}$ and $\tilde{x} \in \tilde{M}$.

We will assume that d is a distance on M which comes from a Riemannian metric g on M. The distance $\tilde{d}$ on $\tilde{M}$ is the distance obtained from the Riemannian metric $\tilde{g}$ on $\tilde{M}$, where $\tilde{g}$ is the lift of g by the covering map $\pi : \tilde{M} \to M$. On the compact

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manifold M, all Riemannian metrics are equivalent. Therefore the distances d on M obtained from Riemannian metrics are all Lipschitz equivalent, and all distances $\tilde{d}$ on $\tilde{M}$ from lifts of Riemannian metrics on M are also Lipschitz equivalent.

Like any infinite cyclic covering, the covering $\pi : \tilde{M} \to M$ is classified by a map $\varphi : M \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$, or, in other words, the covering $\pi : \tilde{M} \to M$ is the pullback by φ of the standard infinite cyclic covering $\mathbb{R} \to \mathbb{T}$. The function $\varphi : M \to \mathbb{T}$ lifts to a continuous function $\tilde{\varphi} : \tilde{M} \to \mathbb{R}$. The function $\tilde{\varphi}$ satisfies

$$\tilde{\varphi}(\tilde{x} + n) = \tilde{\varphi}(\tilde{x}) + n, \text{ for all } \tilde{x} \in \tilde{M} \text{ and all } n \in \mathbb{Z}. \tag{1.1}$$

Standing Assumptions 1.1. — We assume that the we have fixed a continuous function $\tilde{\varphi} : \tilde{M} \to \mathbb{R}$ satisfying (1.1). We also assume that $h : M \to M$ is a homeomorphism of M that lifts to a homeomorphism $\tilde{h} : \tilde{M} \to \tilde{M}$, with $\tilde{h}$ commuting with the deck transformations of the covering $\pi : \tilde{M} \to M$, i.e., we have

$$\tilde{h}(\tilde{x} + n) = \tilde{h}(\tilde{x}) + n, \text{ for all } \tilde{x} \in \tilde{M} \text{ and all } n \in \mathbb{Z}.$$

We would like to compare the recurrence properties of h and that of $\tilde{h}$. Our motivation comes from the work of John Franks **[3, 2]** relating the Poincaré-Birkhoff fixed point for twist maps of the annulus to chain-recurrence properties of the lift to the universal cover, using Brouwer's plane translation theorem. In dimension 2, the results mentioned in this paper are essentially contained in the work of John Franks.

If $f : X \to X$ is a map of the topological space X, recall that the ω-limit set (for f) of a point $x \in X$ is the set $\omega_f(x)$ of all possible accumulation points in X of the forward orbit $(f^n(x))_{n\in\mathbb{N}}$ of x. When X is compact, we have $\omega_f(x) \neq \emptyset$, for every $x \in X$. When X is not compact, it is quite possible that $\omega_f(x) = \emptyset$, for every $x \in X$. This is the case, for example, for a non-zero translation in an Euclidean space.

Note that every $\omega_f(x)$ is contained in $\Omega(f)$, where $\Omega(f)$ is defined as the set of $x \in X$, such that $U \cap (\bigcup_{n\geq 1} f^n U) \neq \emptyset$, for every neighborhood of x.

A first result for recurrence properties of $\tilde{h}$ is given in the following well-known simple proposition (for a proof see the end of §2).

Proposition 1.2. — *Under the standing assumptions 1.1, for every $\tilde{x} \in \tilde{M}$, one of the following statements hold:*

1) $\omega_{\tilde{h}}(\tilde{x}) \neq \emptyset$;
2) $\lim_{n\to+\infty} \tilde{\varphi}(\tilde{h}^n(\tilde{x})) = +\infty$;
3) $\lim_{n\to+\infty} \tilde{\varphi}(\tilde{h}^n(\tilde{x})) = -\infty$.

In particular, if the non-wandering set $\Omega(\tilde{h})$ is empty, only 2) *and* 3) *can happen.*

We remain now with the following problem:

Can we still have some recurrence property for $\tilde{h}$ when $\Omega(\tilde{h}) = \emptyset$?

An answer to this question involves chain-recurrence.

Recall that, for a continuous map f of the metric space (X, d), an ϵ-chain, with $\epsilon > 0$, is a sequence of points $x_0, \dots, x_n$, with $n \geq 1$, such that $d(f(x_i), x_{i+1}) < \epsilon$.

A point $x \in X$ is chain-recurrent for f if for every $\epsilon > 0$, we can find an ϵ-chain $x_0, \ldots, x_n$, for some $n \geq 1$, with $x_0 = x_n = x$. The set of chain-recurrent points for f is denoted by $\mathscr{R}(f)$. This set $\mathscr{R}(f)$ depends on the choice of the metric d, but two uniformly equivalent metrics give rise to the same chain-recurrent set. In particular, if X is compact, the chain-recurrent set is independent of the choice of the metric d defining the topology on X. A companion concept is chain-transitive. The map f is said to be chain-transitive if for every $x, y \in X$ and every $\epsilon > 0$, we can find a ϵ-chain $x_0, \ldots, x_n$, for some $n \geq 1$, with $x_0 = x_n = y$. Every chain-transitive map is chain-recurrent. Conversely, *if X is connected* every chain-recurrent map is chain-transitive. We also recall that there exists an equivalence relation $x \sim y$ on $\mathscr{R}(f)$. We say that $x \sim y$ if for for every $\epsilon > 0$ there exists an ϵ-chain $x_0, \ldots, x_n$, for some $n \geq 1$, with with $x_0 = x, x_n = y$ and an ϵ-chain $y_0, \ldots, y_m$, for some $m \geq 1$, with with $y_0 = y, y_m = x$. The equivalence classes of this equivalence relation are called the chain-recurrent (or Conley) classes of f.

Since the manifold M is compact, the chain-recurrent set $\mathscr{R}(h)$ is well-defined. Moreover, since the distances $\tilde{d}$ that we use on $\tilde{M}$ are all coming from the lift of Riemannian metrics on M, they are all Lipschitz equivalent. Therefore the chain-recurrent set of $\tilde{h}$ is independent of the choice of such a distance $\tilde{d}$. We will denote this chain-recurrent set by $\tilde{\mathscr{R}}(\tilde{h})$ to emphasize that it depends on the uniform class of the distances $\tilde{d}$.

Theorem 1.3. — *Under the standing assumptions 1.1, assume the homeomorphism $h : M \to M$ is chain-transitive. Then one of the following holds:*

(i) *the lift $\tilde{h}$ is chain-recurrent (i.e., $\tilde{\mathscr{R}}(\tilde{h}) = \tilde{M}$);*

(ii) *there exists $K > -\infty$ such that $\tilde{\varphi}\tilde{h}^n(\tilde{x}) - \tilde{\varphi}(\tilde{x}) \geq K$, for every $\tilde{x} \in \tilde{M}$, and every $n \geq 1$;*

(iii) *there exists $K < +\infty$ such that $\tilde{\varphi}\tilde{h}^n(\tilde{x}) - \tilde{\varphi}(\tilde{x}) \leq K$, for every $\tilde{x} \in \tilde{M}$, and every $n \geq 1$.*

The following corollary is now immediate.

Corollary 1.4. — *With the assumptions 1.1, assume the homeomorphism $h : M \to M$ is chain-transitive. If there exists $\tilde{x}_+, \tilde{x}_- \in \tilde{M}$ with*

$$\limsup_{n\to+\infty} \tilde{\varphi}\tilde{h}^n(\tilde{x}_+) = +\infty \text{ and } \liminf_{n\to+\infty} \tilde{\varphi}\tilde{h}^n(\tilde{x}_-) = -\infty,$$

then $\tilde{h}$ is chain-recurrent.

As we will show below, the next corollary also follows from Theorem 1.3.

Corollary 1.5. — *Under the standing assumptions 1.1, assume the homeomorphism $h : M \to M$ is chain-transitive. Then one of the following holds:*

(i) *$\tilde{h}$ has a forward bounded orbit, therefore the non-wandering set $\Omega(\tilde{h})$ of $\tilde{h}$ is not empty;*

(ii) $\tilde{h}$ *is chain-recurrent;*

(iii) $\tilde{\varphi}\tilde{h}^n - \tilde{\varphi} \to +\infty$ *uniformly on* $\tilde{M}$*;*

(iv) $\tilde{\varphi}\tilde{h}^n - \tilde{\varphi} \to -\infty$ *uniformly on* $\tilde{M}$.

We can also give a characterization of $\tilde{\mathscr{R}}(\tilde{h}) \neq \emptyset$ using invariant measure with rotation number 0—see §2 for the definition of rotation number. The proof will be given in §6 below.

Corollary 1.6. — *Under the standing assumptions 1.1, the set* $\tilde{\mathscr{R}}(\tilde{h})$ *is not empty if and only if there exists an* h*-invariant probability measure* μ *on* M*, with* $\mathrm{Rot}(\mu) = 0$*, whose support is contained in a unique chain-recurrent component of* h.

Remark 1.7. — 1) As we already said, most of our results hold for an infinite cyclic covering $\pi : \tilde{X} \to \tilde{X}$, where X is any compact metric space.

2) There is also a version of this work that addresses the relation between the Aubry set of a Tonelli Lagrangian on M and the Aubry set of the lift of the Tonelli Lagrangian on $\tilde{M}$.

3) We will give an example of a chain-recurrent homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$, hence chain-transitive, which has a lift $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ with no chain-recurrent point, for which we can find an h-invariant probability measure μ such that $\mathrm{Rot}(\mu) = 0$. Here $\mathrm{Rot}(\mu) \in \mathbb{R}^2$ is the average on $\mathbb{T}^2$ (for its Haar measure) of the $\mathbb{Z}^2$ invariant function $\tilde{h} - \mathrm{Id}_{\mathbb{R}^2}$. This shows that Corollary 1.6 does not necessarily hold even for Abelian Galois coverings which are not cyclic.

2. The displacement function

Proposition 2.1. — *Under the standing assumptions 1.1, there exists a continuous function* $\theta : M \to \mathbb{R}$*, such that*

$$\tilde{\varphi}\tilde{h}(\tilde{x}) - \tilde{\varphi}(\tilde{x}) = \theta(\pi(\tilde{x})).$$

In particular $\sup_{\tilde{x}\in\tilde{M}}|\tilde{\varphi}\tilde{h}(x) - \tilde{\varphi}(x)| < +\infty$.

Moreover, we have

$$\tilde{\varphi}\tilde{h}^n - \tilde{\varphi} = \left(\sum_{i=0}^{n-1} \theta h^i\right) \circ \pi.$$

Proof. — To show the existence of θ, it suffices to show that the function $\tilde{\varphi}\tilde{h} - \tilde{\varphi}$ on $\tilde{M}$ is invariant under deck transformations. For this we note that, for $\tilde{y} \in \tilde{M}, n \in \mathbb{Z}$, we have

$$\begin{aligned}(\tilde{\varphi}\tilde{h} - \tilde{\varphi})(\tilde{y} + n) &= \tilde{\varphi}(\tilde{h}(\tilde{y} + n)) - \tilde{\varphi}(\tilde{y} + n)\\ &= \tilde{\varphi}(\tilde{h}(\tilde{y}) + n) - (\tilde{\varphi}(\tilde{y}) + n)\\ &= \tilde{\varphi}(\tilde{h}(\tilde{y})) + n - (\tilde{\varphi}(\tilde{y}) + n)\\ &= (\tilde{\varphi}\tilde{h} - \tilde{\varphi})(\tilde{y}).\end{aligned}$$