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HEDGEHOGS IN HIGHER DIMENSIONS AND THEIR APPLICATIONS

by

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Abstract. — In this paper we study the dynamics of germs of holomorphic diffeomorphisms of $(\mathbb{C}^n, 0)$ with a fixed point at the origin with exactly one neutral eigenvalue. We prove that the map on any local center manifold of 0 is quasiconformally conjugate to a holomorphic map and use this to transport results from one complex dimension to higher dimensions.

***Résumé* (Hérissons en dimension supérieure et leurs applications).** — Dans cet article, on étudie la dynamique des germes de difféomorphismes holomorphes de $(\mathbb{C}^n, 0)$ ayant un point fixe à l'origine avec exactement une valeur propre neutre. Nous prouvons que la fonction sur n'importe quelle variété centrale locale de 0 est quasiconformément conjuguée à une fonction holomorphe et utilisons ce théorème pour adapter des résultats en dimension une complexe aux dimensions supérieures.

1. Introduction

Let f be a germ of a holomorphic diffeomorphism of $(\mathbb{C}^2, 0)$ with a fixed point at the origin with eigenvalues λ and μ, where $|\lambda| = 1$ and $|\mu| < 1$. Following the terminology from one-dimensional dynamics, the fixed point is called *semi-neutral* or *semi-indifferent*.

The crude analysis of the local dynamics of the semi-indifferent fixed point exhibits the existence of an analytic strong stable manifold $W^{\mathrm{ss}}(0)$ corresponding to the dissipative eigenvalue μ and a not necessarily unique local center manifold $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ corresponding to the neutral eigenvalue λ. The center manifolds can be made C^r-smooth for any $r \geq 1$ by possibly restricting to smaller neighborhoods of the origin [**14**], however they are generally not analytic, or even C^∞-smooth [**32**]. In this paper, we

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show how to modify the complex structure on the center manifold $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ so that the restriction of the map f to the center manifold becomes analytic.

Theorem A. — *Let f be a germ of a holomorphic diffeomorphism of $(\mathbb{C}^2, 0)$ with a semi-neutral fixed point at the origin with eigenvalues λ and μ, where $|\lambda| = 1$ and $|\mu| < 1$. Consider $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ a C^1-smooth local center manifold of the fixed point 0. There exist neighborhoods W, W' of the origin inside $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ such that $f : W \to W'$ is quasiconformally conjugate to a holomorphic diffeomorphism $h : (\Omega, 0) \to (\Omega', 0)$, $h(z) = \lambda z + \mathcal{O}(z^2)$, where $\Omega, \Omega' \subset \mathbb{C}$.*

The conjugacy map is holomorphic on the interior of Λ rel $W^{\mathrm{c}}_{\mathrm{loc}}(0)$, where Λ is the set of points that stay in W under all backward iterations of f.

Theorem A generalizes to the case of germs of holomorphic diffeomorphisms of $(\mathbb{C}^n, 0)$, for $n > 2$, which have a fixed point at the origin with exactly one eigenvalue on the unit circle. The details are given in Section 6.

In dimension one, linearization properties and dynamics of germs of univalent holomorphic maps of $(\mathbb{C}, 0)$ with an indifferent fixed point at the origin have been extensively studied (**[39]**, **[38]**, **[23]**, **[22]**, **[24]**, and many more). Theorem A has important consequences and enables to us to transport results from one complex variable to $\mathbb{C}^2$. In Section 2 we examine the results of Pérez-Marco about the hedgehog dynamics, and in Section 5 we show how to extend them to $\mathbb{C}^2$ using Theorem A.

Suppose that the neutral eigenvalue λ of the semi-neutral fixed point of the germ f is $\lambda = e^{2\pi i \alpha}$. If the origin is an isolated fixed point of f and $\alpha \in \mathbb{Q}$, then the fixed point is called *semi-parabolic.* In the case when $\alpha \notin \mathbb{Q}$, the fixed point is called *irrational semi-indifferent.* We can further classify irrational semi-indifferent fixed points as semi-Siegel or semi-Cremer, as follows: if there exists an injective holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^2$ such that $f(\varphi(\xi)) = \varphi(\lambda\xi)$, for $\xi \in \mathbb{D}$, then the fixed point is called *semi-Siegel*, otherwise it is called *semi-Cremer.* Theorem D below motivates the following equivalent definition: if f is analytically conjugate to $(x, y) \mapsto (\lambda x, \mu(x)y)$, where $\mu(x) = \mu + \mathcal{O}(x^2)$ is a holomorphic function, then the fixed point is *semi-Siegel*; otherwise, the fixed point is *semi-Cremer.* In particular, when λ satisfies the Brjuno condition **[6]** and $|\mu| < 1$, the map f is linearizable (i.e., conjugate by a holomorphic change of variables to its linear part), so the fixed point is semi-Siegel.

If f is a germ of a holomorphic diffeomorphism of $(\mathbb{C}^2, 0)$ with a semi-indifferent fixed point at the origin, then there exists a domain B containing 0 such that f is partially hyperbolic on a neighborhood of $\overline{B}$. The concept of partial hyperbolicity is explained in the introductory part of Section 3. In **[9]** we have shown the existence of non-trivial compact invariant sets, called *hedgehogs*, for germs with semi-indifferent fixed points, using topological tools.

Theorem 1.1 (**[9]**). — *Let f be a germ of a holomorphic diffeomorphism of $(\mathbb{C}^2, 0)$ with a semi-indifferent fixed point at 0 with eigenvalues λ and μ, where $|\lambda| = 1$ and $|\mu| < 1$. There exists a neighborhood $B' \subset \mathbb{C}^2$ of 0 on which f is partially hyperbolic such that* *for any open ball* $B \Subset B'$ *centered at* 0 *there exists a set* $\mathscr{H} \subset \overline{B}$ *with the following properties:*

a) $\mathscr{H} \Subset W^{\mathrm{c}}_{\mathrm{loc}}(0)$*, where* $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ *is any local center manifold of the fixed point* 0*, constructed relative to* B'*.*

b) $\mathscr{H}$ *is compact, connected, completely invariant, and full.*

c) $0 \in \mathscr{H}$, $\mathscr{H} \cap \partial B \neq \emptyset$.

d) *Every point* $x \in \mathscr{H}$ *has a well defined local strong stable manifold* $W^{\mathrm{ss}}_{\mathrm{loc}}(x)$*, consisting of points from* B *whose orbits converge exponentially fast to the orbit of* x*, at a rate* $\asymp \mu^n$*. The strong stable set of* $\mathscr{H}$ *is laminated by vertical-like holomorphic disks.*

We distinguish between a *parabolic hedgehog*, a *Siegel hedgehog*, or a *Cremer hedgehog* (also called *non-linearizable hedgehog*), depending whether the fixed point is semi-parabolic, semi-Siegel, or semi-Cremer. In this paper we will explore the dynamical properties of hedgehogs. Let us start by noticing that in the irrationally semi-indifferent case the hedgehog is locally unique. There exists a neighborhood B of the origin in $\mathbb{C}^2$ such that the hedgehog $\mathscr{H}$ associated to f and B is unique and equal to the connected component containing 0 of the set $\{z \in \overline{B} : f^n(z) \in \overline{B}\ \forall n \in \mathbb{Z}\}$. This is a direct consequence of Theorem A and Theorem 2.2 of Pérez-Marco.

The next two theorems and the subsequent corollaries deal with Cremer hedgehogs (see Figure 1).

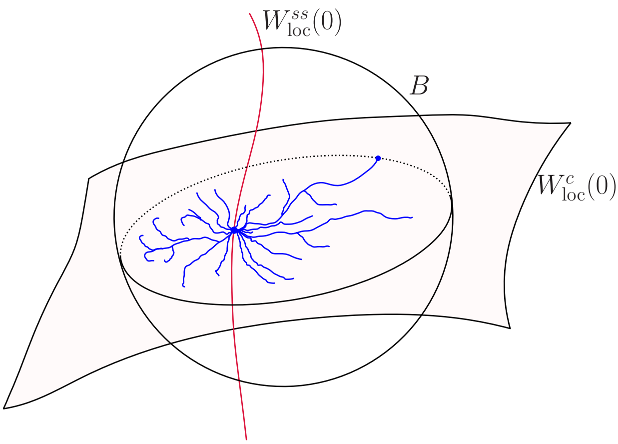

FIGURE 1. A Cremer hedgehog inside a center manifold.

Theorem B. — *Let* f *be a germ of a dissipative holomorphic diffeomorphism of* $(\mathbb{C}^2, 0)$ *with a semi-Cremer fixed point at* 0 *with an eigenvalue* $\lambda = e^{2\pi i\alpha}$*. Let* $\mathscr{H}$ *be a hedgehog for* f*. Let* $(p_n/q_n)_{n\geq 1}$ *be the convergents of the continued fraction of* α*. Then the iterates* $(f^{q_n})_{n\geq 1}$ *converge uniformly on* $\mathscr{H}$ *to the identity.*

Corollary B.1. — *The dynamics on the hedgehog $\mathcal{H}$ is recurrent. The hedgehog does not contain other periodic points except* 0.

Denote by $\omega(x)$ and $\alpha(x)$ the ω-limit, respectively α-limit set of x.

Theorem C. — *There exists a neighborhood $B \subset \mathbb{C}^2$ of the origin with the following properties. Let $\mathcal{H}$ be a Cremer hedgehog for f and B.*

a) *Let $x \in B \setminus W^{\mathrm{ss}}_{\mathrm{loc}}(\mathcal{H})$. If the forward iterates $f^n(x) \in B$ for all $n \geq 0$, then $\omega(x) \cap \mathcal{H} = \emptyset$.*

b) *Let $x \in B \setminus \mathcal{H}$. If the backward iterates $f^{-n}(x) \in B$ for all $n \geq 0$, then $\alpha(x) \cap \mathcal{H} = \emptyset$.*

Theorem C immediately implies the following corollaries.

Corollary C.1. — *If $x \notin W^{\mathrm{ss}}(0)$ then the orbit of x does not converge to* 0.

Let $\mathcal{H}$ be a hedgehog for a germ f with a semi-Cremer fixed point and denote by $W^s_{\mathrm{loc}}(\mathcal{H})$ the local stable set of $\mathcal{H}$, consisting of points that converge to the hedgehog. Let $W^{\mathrm{ss}}_{\mathrm{loc}}(\mathcal{H})$ be the local strong stable stable of $\mathcal{H}$, consisting of points which converge asymptotically exponentially fast to the hedgehog.

Corollary C.2. — $W^{\mathrm{ss}}_{\mathrm{loc}}(\mathcal{H}) = W^s_{\mathrm{loc}}(\mathcal{H})$.

Clearly, the set $\mathcal{H}$ has no interior in $\mathbb{C}^2$ since it lives in a center manifold. Let $\mathrm{int}^{\mathrm{c}}(\mathcal{H})$ denote the interior of $\mathcal{H}$ relative to a center manifold.

Theorem D. — *Let f be a germ of a holomorphic diffeomorphism of $(\mathbb{C}^2, 0)$ with an isolated semi-neutral fixed point at the origin. Let $\mathcal{H}$ be a hedgehog for f. Then $0 \in \mathrm{int}^{\mathrm{c}}(\mathcal{H})$ if and only if f is analytically conjugate to a linear cocycle $\tilde{f}$ given by*

$$\tilde{f}(x, y) = (\lambda x, \mu(x) y),$$

where $\mu(x) = \mu + \mathcal{O}(x)$ is a holomorphic function.

Corollary D.1. — *Let f be a dissipative polynomial diffeomorphism of $\mathbb{C}^2$ with a semi-neutral fixed point at the origin. Then $0 \in \mathrm{int}^{\mathrm{c}}(\mathcal{H})$ if and only if f is linearizable.*

For a polynomial automorphism f of $\mathbb{C}^2$ we define the sets $K^{\pm}$ of points that do not escape to ∞ under forward/backward iterations. Denote by $J^{\pm}$ the topological boundaries of $K^{\pm}$ in $\mathbb{C}^2$. The set $J = J^- \cap J^+$ is called the Julia set. Let J^* be the closure of the saddle periodic points of f. The set J^* is a subset of J, but it is an open question whether they are always equal. We obtain the following result.

Theorem E. — *Let f be a dissipative polynomial diffeomorphism of $\mathbb{C}^2$ with an irrationally semi-indifferent fixed point at 0. Suppose f is not linearizable in a neighborhood of the origin. Let $\mathcal{H}$ be a hedgehog for f. Then $\mathcal{H} \subset J^*$ and there are no wandering domains converging to $\mathcal{H}$.*