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CONTRIBUTION OF ONE-CYLINDER SQUARE-TILED SURFACES TO MASUR-VEECH VOLUMES

by

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with an appendix by Philip Engel

In memory of Jean-Christophe Yoccoz

Abstract. — We compute explicitly the *absolute* contribution of square-tiled surfaces having a single horizontal cylinder to the Masur-Veech volume of any ambient stratum of Abelian differentials. The resulting count is particularly simple and efficient in the large genus asymptotics. Using the recent results of Aggarwal and of Chen-Möller-Zagier on the long-standing conjecture about the large genus asymptotics of Masur-Veech volumes, we derive that the *relative* contribution is asymptotically of the order $1/d$, where d is the dimension of the stratum.

Similarly, we evaluate the contribution of one-cylinder square-tiled surfaces to Masur-Veech volumes of low-dimensional strata in the moduli space of quadratic differentials. We combine this count with our recent result on equidistribution of one-cylinder square-tiled surfaces translated to the language of interval exchange transformations to compute empirically approximate values of the Masur-Veech volumes of strata of quadratic differentials of all small dimensions.

Résumé **(Contribution des surfaces à petits carreaux à un cylindre aux volumes de Masur-Veech)**

Nous établissons une formule pour la contribution des surface à petits carreaux formées d'un seul cylindre horizontal au volume de Masur-Veech des strates de différentielles abéliennes. Nous en déduisons le comportement asymptotique lorsque le genre des surfaces grandit. À la lumière des résultats récents de Aggarwal et Chen-Möller-Zagier sur l'asymptotique des volumes de Masur-Veech, nous en déduisons que la contribution relative est de l'ordre de $1/d$ où d est la dimension de la strate. De manière similaire, nous donnons une formule pour la contribution des surfaces à petits carreaux formées d'un seul cylindre horizontal au volume de Masur-Veech des strates de différentielles quadratiques. En combinant cette formule avec nos résultats

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récents sur l'équidistribution des surface à un cylindre horizontal, nous proposons une méthode empirique pour le calcul des volumes de Masur-Veech des strates de différentielles quadratiques. Cette dernière s'avère être efficace en petites dimensions.

Introduction

Siegel-Veech constants and Masur-Veech volumes. — One of the most powerful tools in the study of billiards in rational polygons (including "wind-tree" billiards with periodic obstacles in the plane), of interval exchange transformations and of measured foliations on surfaces is renormalization. More precisely, to describe fine geometric and dynamical properties of the initial billiard, interval exchange transformation or measured foliation, one has to find the $\mathrm{GL}^+(2,\mathbb{R})$-orbit closure of the associated translation surface in the moduli space of Abelian (or quadratic) differentials, and study its geometry. This approach, initiated by H. Masur and W. Veech four decades ago became particularly powerful recently due to the breakthrough theorems of Eskin-Mirzakhani-Mohammadi **[17]** and **[18]** that ensure that such $\mathrm{GL}^+(2,\mathbb{R})$-orbit closure is linear.

The moduli space of Abelian (or quadratic) differentials is stratified by the degrees of zeroes of the Abelian (or quadratic) differential. Each stratum is endowed with a natural measure, the *Masur-Veech* measure, that is preserved by the $\mathrm{SL}(2,\mathbb{R})$-action (the action by scalar matrices rescales the volumes and only preserves the projective class of the measure).

The Masur-Veech measure of each connected component of a stratum is infinite. However, passing to a level hypersurface of the function $\frac{i}{2}\int_C \omega\wedge\bar{\omega}$, where ω is an Abelian differential, and C is the undelying complex curve (respectively to the level hypersurface of the function $\int_C |q|$, where q is a quadratic differential), the Masur-Veech measure induces an $\mathrm{SL}(2,\mathbb{R})$-invariant measure which by the results of Masur **[32]** and **[36]** is finite and ergodic.

In many important situations the $\mathrm{GL}^+(2,\mathbb{R})$-orbit closure of a translation surface is an entire connected component of a stratum. In order to count the growth rate for the number of closed geodesics on a translation surface as in **[15]**, or to describe the deviation spectrum of a measured foliation as in **[23]**, **[41]**, or to count the diffusion rate of a wind-tree as in **[12]**, **[13]**, one has to compute the corresponding *Siegel-Veech constants*, see **[37]**, and the Lyapunov exponents of the Hodge bundle over the connected component of stratum. Both quantities are expressed by explicit combinatorial formulas in terms of the Masur-Veech volumes of the strata, see **[16]**, **[14]**, **[3]**, **[26]**.

Equidistribution of square-tiled surfaces. — The Masur-Veech volumes of strata of Abelian differentials and of meromorphic quadratic differentials with at most simple poles were computed in **[19]**, **[20]**, and **[21]**. The underlying idea (see also **[42]**) was a computation of the asymptotic number of "integer points" (the ones having coordinates in $\mathbb{Z}\oplus i\mathbb{Z}$ in period coordinates) in appropriate bounded domains exhausting the stratum. Such integer points are represented by square-tiled surfaces. In the case of Abelian (respectively quadratic differentials), a square-tiled is a surface tiled by 1×1 unit squares (resp. $1/2\times 1/2$ unit squares). In the Abelian case, such surface can equivalently be viewed as a ramified cover over the square torus ramified only over $\{0\}$ and the degree of the cover correponds to the number of squares. In the quadratic case, a square tiled surface is a covering of the pillowcase in $\mathbb{C}\mathrm{P}^1$ ramified over four points but the degree does not coincide with the number of squares in general (there might be a factor 1, 2 or 4). Rescaling square-tiled surfaes by ε we get a sequence of grids that equidistribute towards the Masur-Veech measure.

Each square-tiled surface carries interesting combinatorial geometry, for example, the decomposition into maximal flat horizontal cylinders. We recall in Theorem 1.1 of Section 1 our recent result from **[11]** telling that square-tiled surfaces having fixed combinatorics of horizontal cylinder decomposition and tiled with squares of size ε become asymptotically equidistributed in the ambient stratum as ε tends to zero. This result gives sense to the notion of (asymptotic) probability P_k for a "random" square-tiled surface in a given stratum to have a fixed number $k\in\{1,2,\ldots,g+r-1\}$ of maximal cylinders in its horizontal decomposition, where g is the genus of the surface and r is the number of conical singularities.

An interval exchange transformation (or linear involution) is called *rational* if all its intervals under exchange have rational lengths. All orbits of such interval exchange transformation are periodic. We state in Theorem 1.2 an analogous equidistribution statement for rational interval exchange transformations (see **[11]** for the proof) and the proportions that appear in this context are the same as the ones for square-tiled surfaces. The (asymptotic) probability that a "random" rational interval exchange transformation with a given permutation has k maximal bands of fellow-traveling closed trajectories is P_k.

Contribution of 1*-cylinder square-tiled surfaces and large genus asymptotics of Masur-Veech volumes.* — The only currently known computation of Masur-Veech volumes of strata of Abelian differentials is based on counting square-tiled surfaces. In Section 2 we compute the *absolute* contribution $c_1(\mathcal{L})$ of 1-cylinder square-tiled surfaces to the Masur-Veech volume of a stratum $\mathcal{L}$, where $c_1(\mathcal{L}) := \mathrm{P}_1(\mathcal{L})\cdot\operatorname{Vol}\mathcal{L}$. We define $c_k(\mathcal{L})$ similarly for the absolute contribution of k-cylinders square-tiled surfaces. By definition, $\operatorname{Vol}\mathcal{L} = c_1(\mathcal{L}) + c_2(\mathcal{L}) + \cdots + c_{g+r-1}(\mathcal{L})$. We give simple close exact formulas for the contribution $c_1(\mathcal{L})$ to the volumes $\operatorname{Vol}\mathcal{H}(2g-2)$ and $\operatorname{Vol}\mathcal{H}(1,\ldots,1)$ of minimal and principal strata of Abelian differentials. We also provide sharp upper and lower bounds for contributions of 1-cylinder square-tiled surfaces to the Masur-Veech volumes of any stratum of Abelian differential. The ratio of the upper and lower bounds tends to 1 as $g\to+\infty$ uniformly for all strata in genus g, so the bounds are particularly efficient in large genus asymptotics.

Using the result [7] of Chen-Möller-Zagier and more general result [1] of Aggarwal on the Masur-Veech volume asymptotics conjectured in [22] we prove that the corresponding *relative* contribution $\mathrm{P}_1(\mathcal{L})$ of 1-cylinder square-tiled surfaces to the Masur-Veech volume $\mathrm{Vol}\,\mathcal{L}$ of any stratum $\mathcal{L}$ of Abelian differentials is asymptotically of the order $1/d$ as g (equivalently d) tends to infinity. Here d is the dimension $d = \dim_{\mathbb{C}}(\mathcal{L})$ of the stratum $\mathcal{L}$.

Siegel-Veech constants and Masur-Veech volumes of strata of meromorphic quadratic differentials. — The Masur-Veech volumes of any connected component of stratum of Abelian differentials in genus g has the form $s \cdot \pi^{2g}$, where s is some rational number [19]. The generating functions in [19] were translated by A. Eskin into computer code, which allowed to evaluate explicitly volumes of all connected components of all strata of Abelian differentials in genera up to $g = 10$ (that is, to compute explicitly the corresponding rational numbers s), and for some strata up to $g = 60$. The recent results of D. Chen, M. Möller and D. Zagier [7] allows to compute s for the principal stratum up to genus $g = 2000$ and higher.

In the *quadratic* case, the Masur-Veech volume still has the same arithmetic form $s \cdot \pi^{2\widehat{g}}$ where $\widehat{g}$ is the so-called effective genus [20], [21]. The computation of s in the quadratic case had to wait for a decade to be translated into tables of numbers. One of the reasons for such a delay is a more involved combinatorics and multitude of various conventions and normalizations required in volume computations (which is a common source of mistakes in normalization factors like powers of 2). This is why it is necessary to test theoretical predictions on some table of volumes obtained by an independent method. In the case of Abelian differentials, the volumes of several low-dimensional strata were computed by a direct combinatorial method elaborated by A. Eskin, M. Kontsevich and A. Zorich; this approach is described in [42]. Another, even more reliable test was provided by computer simulations of Lyapunov exponents and their ties with volumes through Siegel-Veech constants. In the case of quadratic differentials, explicit values of volumes of the strata in genus zero were conjectured by M. Kontsevich about fifteen years ago. The conjecture was proved in recent papers [2] and [3]. Further explicit values of volumes of all low-dimensional strata up to dimension 11 were obtained in [26].

Our counting results combined with the equidistribution Theorems 1.1 and 1.2 allow to compute approximate values of volumes of the strata. The idea is to evaluate experimentally the approximate value of the probability $\mathrm{P}_1(\mathcal{L})$ to get a 1-cylinder square-tiled surface taking a "random" square-tiled surface in a given stratum $\mathcal{L}$ of quadratic differentials. Then we compute rigorously the absolute contribution $c_1(\mathcal{L})$ of 1-cylinder square-tiled surfaces to the Masur-Veech volume $\mathrm{Vol}\,\mathcal{L}$ of the stratum. The relation $c_1(\mathcal{L}) = \mathrm{P}_1(\mathcal{L}) \cdot \mathrm{Vol}\,\mathcal{L}$ now provides the approximate value of the Masur-Veech volume $\mathrm{Vol}\,\mathcal{L}$ of the stratum $\mathcal{L}$ of quadratic differentials.

This approach is completely independent of the one of A. Eskin and A. Okounkov based on the representation theory of the symmetric group. The approximate data based on this approach were used for "debugging" rigorous formulas in [25] and [26].