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Time-changes of Heisenberg nilflows

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TIME-CHANGES OF HEISENBERG NILFLOWS

by

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In memory of Jean-Christophe Yoccoz,
il miglior fabbro.

Abstract. — We consider the three dimensional Heisenberg nilflows. Under a full measure set Diophantine condition on the generator of the flow we construct *Bufetov functionals* which are asymptotic to ergodic integrals for sufficiently smooth functions, have a modular property and scale exactly under the renormalization dynamics. By the asymptotic property we derive results on limit distributions, which generalize earlier work of Griffin and Marklof [**17**] and Cellarosi and Marklof [**8**]. We then prove analyticity of the functionals in the transverse directions to the flow. As a consequence of this analyticity property we derive that there exists a full measure set of nilflows such that *generic* (non-trivial) time-changes are mixing and moreover have a "stretched polynomial" decay of correlations for sufficiently smooth functions (this strengthens a result of Avila, Forni, and Ulcigrai [**2**]). Moreover we also prove that there exists a full Hausdorff dimension set of nilflows such that generic non-trivial time-changes have polynomial decay of correlations.

***Résumé* (Changements de temps des flots nilpotents d'Heisenberg).** — Nous étudions les flots nilpotents de Heisenberg en dimension trois. Sous une condition Diophantienne de mesure pleine sur le générateur du flot, nous montrons l'existence de *fonctionnelles de Bufetov*, qui sont asymptotiques aux intégrales ergodiques pour toutes les fonctions suffisamment différentiables, qui ont une propriété modulaire, et satisfont une identité de changement d'échelle sous la dynamique de renormalisation. De la propriété asymptotique, nous dérivons des résultats sur les distributions limites des moyennes ergodiques, qui généralisent les travaux de Griffin et Marklof [**17**], et Cellarosi et Marklof [**8**]. Ensuite nous montrons une propriété d'analyticité des fonctionnelles dans les directions transverses au flot. Comme conséquence de cette propriété d'analyticité, nous dérivons l'existence d'un ensemble de mesure pleine de flots nilpotents dont les changements de temps *génériques* (non-triviaux) sont mélangeant, et de plus ont une vitesse de mélange « polynomiale étirée » pour toutes les fonctions suffisamment différentiables (cela améliore un résultat de Avila, Forni,

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et Ulcigrai [2]). De plus, nous montrons qu'il existe un ensemble de dimension de Hausdorff maximale de flots nilpotents tels que les changements de temps génériques non-triviaux ont une vitesse de mélange polynomiale.

1. Introduction

This paper concerns the smooth ergodic theory of parabolic flows, that is, flows characterized by polynomial (sub-exponential) divergence of nearby orbits. In particular we prove results on limit distributions of Heisenberg nilflows and on the decay of correlations of their non-trivial reparametrizations (time-changes). Our approach is based on the construction of *finitely additive Hölder measures* and *Hölder cocycles* for Heisenberg nilflows, asymptotic to ergodic integrals, following the work of A. Bufetov [6] on translation flows and of Bufetov and G. Forni [5], [9] on horocycle flows. Hölder cocycles for translation flows are closely related to "limit shapes" of ergodic sums for Interval Exchange Transformations, studied in the work of S. Marmi, P. Moussa and J.-C. Yoccoz [22] on wandering intervals for affine Interval Exchange Transformations. In fact, roughly speaking, "limit shapes" are related to graphs of Hölder cocycles as functions of time.

We recall that the mixing property for generic, non-trivial time-changes of Heisenberg nilflows was proved by A. Avila, G. Forni and C. Ulcigrai [2]. The main result of that paper was that for uniquely ergodic Heisenberg nilflow all non-trivial time-changes, within a dense subspace of time-changes, are mixing. Under a Diophantine condition the set of trivial time-changes has countable codimension and can be explicitly described in terms of invariant distributions for the nilflow.

Results on limit theorems for skew-translations, which appear as return maps (with constant return time) of Heisenberg nilflows, limited however to a single character function, have more recently been proved by J. Griffin and J. Marklof [17] and refined by F. Cellarosi and Marklof [8] by an approach based on theta functions. Their worked raised the question of possible relations between theta functions and Bufetov's Hölder cocycles, developed for other analogous dynamical systems in [6] (translation flows), [5] (horocycle flows), [7] (tilings), as a formalism to derive asymptotic theorem for ergodic averages and prove limit theorems.

In this paper we generalize the results of Griffin and Marklof [17] on limit distributions, proving in particular that almost all limits of ergodic averages of arbitrary sufficiently smooth functions are distributions of Hölder continuous functions on the Heisenberg nilmanifold, hence in particular they have compact support. Our main results, however, are on the decay of correlations of smooth functions for time-changes: we prove that it has polynomial (power law) speed for all non-trivial smooth time-changes of Heisenberg nilflows of bounded type, within a generic subspace of time-changes. As mentioned above, the study of limit distributions for parabolic flows has been developed only in recent years after Bufetov's work [6] on translations flows (and

Interval Exchange Transformations). A comprehensive study of spatial and temporal limit theorems for dynamical systems of different type has been more recently carried out in the work of D. Dolgopyat and O. Sarig [**10**].

The study of mixing properties of elliptic and parabolic flows and their time-changes has a longer history. For instance, mixing properties of suspension flows over rotations and Interval Exchange Transformations have been investigated in depth (see for instance [**18**], [**19**], [**27**], [**26**],[**28**], [**29**] and reference therein), mixing for reparametrizations of linear toral flows were investigated by B. Fayad (see for instance [**12**]), finally mixing for time-changes of classical horocycle flows was proved in a classical paper of B. Marcus [**21**] after a partial result of Kushnirenko [**20**]. As mentioned above mixing for time-changes of Heisenberg nilflows was investigated in [**2**]. Work of Avila, Forni, Ravotti and Ulcigrai [**1**] extends the methods developed there to prove mixing for a dense set of nontrivial time-change for any uniquely ergodic nilflows. Ravotti's paper [**25**] was a first step in that direction. It should be remarked that there is an important difference between time-changes of linear toral flows and parabolic flows. In the parabolic case there are often countably many obstructions to triviality of time-changes for Diophantine flows, while in the elliptic case of linear toral flows non-trivial time-changes can exist only in the Liouvillean case.

Estimates on the decay of correlations of smooth functions for non-homogenous elliptic or parabolic flows are harder to come by and there are much fewer results in the literature. A classical paper of M. Ratner [**23**] established the decay rate for classical horocycle flows (as well as geodesic flows) on surfaces of constant negative curvature. This result was generalized to sufficiently smooth time-changes of horocycle flows by Forni and Ulcigrai [**15**], who also proved that the spectrum remains Lebesgue. Fayad [**11**] proved polynomial decay for a class of Kochergin-type flows on the 2-torus and only recently, in [**13**], it was shown that there exists a class of Kochergin flows on the 2-torus with countable Lebesgue spectrum. For locally Hamiltonian flows with a saddle loop on surfaces (or, more generally, for suspension flows over Interval Exchange Transformations with asymmetric logarithmic singularities of the roof function), Ravotti [**24**] was able to prove (logarithmic) estimates on decay of correlations. For these flows mixing was proved by Khanin and Sinai [**27**] in the toral case, and by C. Ulcigrai [**28**] for suspension flows over Interval Echange Transformations in the significant special case of roof functions with a single asymmetric logarithmic singularity.

We expect non-trivial time-changes of nilflows to have polynomial decay of correlations. However, we are able to prove this result only for Heisenberg nilflows of bounded type. Our methods do not generalize to higher step nilflows, since they are based on the renormalization dynamics introduced by L. Flaminio and G. Forni in [**14**], which has no known generalization to the higher step case. We are also unable to decide whether the spectral measures of time-changes of Heisenberg nilflows are absolutely continuous with respect to Lebesgue. Indeed, the approach of [**15**], considerably refined in [**13**] , fails since the "stretching of Birkhoff sums" is at best borderline square integrable (it grows at most as the square root of the time, up to logarithmic terms). In fact, our bounds on the decay of correlations are significantly worse than that, and we have no control on the size of the exponent. This follows from the general principle that proving "lower bounds" on Birkhoff sums or ergodic integrals is much harder than proving "upper bounds". In our case we are able to prove polynomial (power-law) lower bounds outside appropriate sublevel sets of Bufetov's Hölder cocycles, which are asymptotic to ergodic integrals up to a well-controlled error. Polynomial estimates on the measure of such sublevel sets (for small parameter values) are derived from general results (see **[3]**, **[4]**) on the measure of the sublevel sets of *analytic functions*. In fact, at the core of our argument we establish the real analyticity of the Bufetov cocycles along the leaves of a foliation transverse to the flow.

This outline is different from the proof of mixing in **[2]**. In that paper the stretching of Birkhoff sums for Heisenberg nilflows was derived from a more general result on the growth of Birkhoff sums of functions which are not coboundaries with *measurable* transfer function, essentially based on a measurable Gottschalk-Hedlund theorem, and on the parabolic divergence of orbits. However, it is completely unclear whether it is possible to prove an effective version of this argument. For this reason we have followed here a different approach.

Outline of the paper. — In Section 2 we give basic definitions on Heisenberg nilflows, the Heisenberg moduli space, renormalization flow and Sobolev spaces. Finally we state two main theorems. In Section 3 we recall some basic results in representation theory of Heisenberg group. In Section 4 we compute the stretching of arcs (in the central direction) under the reparametrized flow. Sections 5 and 6 are crucial since *Bufetov functionals* are constructed and their main properties are studied. In particular we prove the expected asymptotic formula according to which Bufetov fuctionals control orbital integrals. In Section 7 we derive from the asymptotic formula results on limit distributions of ergodic integrals for Diophantine Heisenberg niflows, following the method developed in **[6]**, **[5]**. We also give an alternative proof, based on representation theory, of a substantial part of the work of Griffin and Marklof **[17]** on limit theorems for skew-shifts of the 2-torus, and generalize most of their conclusions to arbitrary smooth functions. Our approach also naturally gives results on the regularity of limit distributions, in particular their Hölder property (with exponent $1/2-$) first derived for quadratic Weyl sums in the work of Cellarosi and Marklof **[8]**.

In Section 8 we prove sharp square mean lower bounds for Bufetov functionals along the leaves of a one-dimensional foliation transverse to the flow. Our aim is to prove measure estimates for the sub-level sets of Bufetov functionals, a key result in establishing the stretching of ergodic integrals outside sets of small measure. For that we prove in Section 9 that Bufetov functionals are *real analytic* on the leaves of a 2-dimensional foliation (the weak-stable foliation of the renormalization dynamics on the Heisenberg nilmanifold). We then recall in Section 10 a result of A. Brudnyi **[3]** on the measure of the sub-level sets of real analytic functions. These estimates depend on the so-called *Chebyshev degree* and *valency* of the function. We prove that under