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Stable accessibility with 2-dimensional center

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STABLE ACCESSIBILITY WITH 2-DIMENSIONAL CENTER

by

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Abstract. — For partially hyperbolic diffeomorphisms with 2-dimensional center, accessibility is C^1 -stable. Moreover, for center bunched skew-products (stable) accessibility is C^∞ -dense.

Résumé (Accessibilité stable de dimension centrale 2). — L’accessibilité est une propriété C^1 -stable parmi les difféomorphismes partiellement hyperboliques à dimension centrale 2. De plus, l’accessibilité (stable) est une propriété C^1 -dense dans le domaine des produits gauches satisfaisant la condition de regroupement central (‘center bunching’).

1. Introduction

A diffeomorphism $f : M \rightarrow M$ of a compact manifold M is *partially hyperbolic* if there exist: a continuous splitting of the tangent bundle $TM = E^u \oplus E^c \oplus E^s$ invariant under the derivative Df (all three sub-bundles are assumed to have positive dimension); a Riemannian metric $\|\cdot\|$ on M ; and positive continuous functions ν , $\hat{\nu}$, γ , $\hat{\gamma}$ with ν , $\hat{\nu} < 1$ and $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$, such that

$$(1) \quad \begin{aligned} \|Df(p)v\| &< \nu(p) && \text{if } v \in E^s(p), \\ \gamma(p) &< \|Df(p)v\| < \hat{\gamma}(p)^{-1} && \text{if } v \in E^c(p), \\ \hat{\nu}(p)^{-1} &< \|Df(p)v\| && \text{if } v \in E^u(p) \end{aligned}$$

for any unit vector $v \in T_p M$. This is an open property in the space of C^1 diffeomorphisms. We will denote $d_* = \dim E^*$, for $*$ $\in \{u, c, s\}$, and $d = \dim M$.

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The stable bundle E^s and the unstable bundle E^u are uniquely integrable and their integral manifolds form two quasi transverse continuous foliations, $\mathcal{W}^u = \mathcal{W}_f^u$ and $\mathcal{W}^s = \mathcal{W}_f^s$, whose leaves are immersed submanifolds of the same class of differentiability as f . These are called the *strong unstable* and *strong stable* foliations of f . They are invariant under f , in the sense that $f(\mathcal{W}^*(x)) = \mathcal{W}^*(f(x))$ for any $x \in M$ and $* \in \{u, s\}$. Given $\epsilon > 0$ and $* \in \{u, s\}$, we represent by $\mathcal{W}_\epsilon^*(x) = \mathcal{W}_{f,\epsilon}^*(x)$ the ϵ -neighborhood of x inside $\mathcal{W}^*(x)$.

Given two points $x, y \in M$, we say that x is accessible from y if there exists a C^1 path that connects x to y and is tangent at every point to the union $E^u \cup E^s$. The equivalence classes of this (equivalence) relation are called *f-accessibility classes*. The diffeomorphism f is called *accessible* if there exists a unique *f-accessibility class*, namely, the ambient M . Moreover, f is called *stably accessible* if it admits a C^1 neighborhood \mathcal{U} such that every C^2 diffeomorphism $g \in \mathcal{U}$ is accessible.

For any $k \geq 1$, we denote by \mathcal{PH}^k the space of C^k partially hyperbolic diffeomorphisms in M . Most of our results concern the subspace \mathcal{PH}_2^k of diffeomorphisms $f \in \mathcal{PH}^k$ with 2-dimensional center bundle, that is, such that $d_c = 2$.

Theorem A. — *If $f \in \mathcal{PH}_2^1$ is accessible then f is stably accessible.*

We say that an *f-accessibility class* C is *stable* if for every compact set $K \subset C$ there exists a C^1 neighborhood $\mathcal{U} = \mathcal{U}_K$ of f such that K is contained in a unique *g-accessibility class* for every C^2 diffeomorphism $g \in \mathcal{U}$. In particular, f is stably accessible if, and only if, the ambient M is a stable *f-accessibility class*.

Stable accessibility classes are open sets. Indeed, let p and q be two distinct points in C (for instance, in the same stable manifold). For any $r \in M$ close to q , let $h : M \rightarrow M$ be a diffeomorphism C^∞ close to the identity, such that $h(p) = p$ and $h(r) = q$. Then $g = h \circ f \circ h^{-1}$ is close to f . Taking $K = \{p, q\}$, the assumption implies that p and q are in the same *g-accessibility class*. This means that p and q are in the same *f-accessibility class*, that is, $r \in C$. So, C contains a whole neighborhood of q .

Here we prove that the converse is also true, at least when the center bundle is 2-dimensional:

Theorem B. — *If $f \in \mathcal{PH}_2^1$ then any open *f-accessibility class* is stable.*

Theorem A is a direct consequence of Theorem B. The main technical step in the proof of Theorem B is a result on approximation of general paths in accessibility classes by a certain class of paths for which a continuation exists for every nearby diffeomorphism. This result is stated in Section 4 (Theorem 4.1), where we also explain how it leads to Theorem B.

In Sections 6–5 we state and prove a result about density of stable accessibility (Theorem 6.1), for a class of fibered partially hyperbolic diffeomorphisms with 2-dimensional center bundle. It contains a claim made in Section 7 of our paper [2], that

was used for proving Theorem H in that paper. After our research had been completed, we learned from V. Horita and M. Sambarino that they had independently obtained a similar result, in a paper that appeared in [8].

When the center dimension $d_c = 1$, the accessibility property is always stable [5]. The present work extends that fact to center dimension equal to 2. Recently, and also in the 2-dimensional case, J. Rodriguez-Hertz and C. Vasquez [13] proved that accessibility classes are immersed submanifolds, which implies Theorem A.

When the center bundle is one-dimensional, the (stable) accessibility property is known to be C^r dense among partially hyperbolic diffeomorphisms [3, 12]. Without any hypothesis on the dimension of the central bundle, Dolgopyat and Wilkinson [6] proved that stable accessibility is C^1 dense.

2. Deformations paths

In this section, all maps are assumed to be C^1 and proximity is always meant in the C^1 topology. We introduce a class of paths, that we call deformation paths, contained in accessibility classes and having a useful property of persistence under variation of the diffeomorphism and the base point. This also provides a kind of parametrization for accessibility classes:

Theorem 2.1. — *For every $f \in \mathcal{PH}^1$, there exist $k \geq 1$, a neighborhood \mathcal{V} of f and a sequence $P_l : \mathcal{V} \times M \times \mathbb{R}^{k(d_u+d_s)l} \rightarrow M$ of continuous maps such that, for any $g \in \mathcal{V}$,*

1. $P_m(g, \cdot, w) \circ P_l(g, \cdot, v) = P_{l+m}(g, \cdot, (v, w))$ for every $l \geq 1$ and $m \geq 1$ and $v \in \mathbb{R}^{k(d_u+d_s)l}$ and $w \in \mathbb{R}^{k(d_u+d_s)m}$;
2. $\zeta \mapsto P_l(g, \zeta, v)$ is a homeomorphism from M to M , with $P_l(g, \cdot, 0) = \text{id}$, for every $l \geq 1$ and $v \in \mathbb{R}^{k(d_u+d_s)l}$;
3. $\bigcup_{n \geq 1} P_n(\{(g, z)\} \times \mathbb{R}^{k(d_u+d_s)n})$ is the g -accessibility class of each $z \in M$.

A deformation path based on (f, z) is a (continuous) map $\gamma : [0, 1] \rightarrow M$ such that there exist $l \geq 1$ and a continuous map $\Gamma : [0, 1] \rightarrow \mathbb{R}^{k(d_u+d_s)l}$ satisfying $\gamma(t) = P_l(f, z, \Gamma(t))$. Notice that any deformation path based on (f, z) is contained in the f -accessibility class of z . It follows immediately from the definition that deformation paths are persistent, in the following sense:

Corollary 2.2. — *If $\gamma : [0, 1] \rightarrow M$ is a deformation path based on (f, z) then, for any g close to f and any w close to z , there exists a deformation path based on (g, w) that is uniformly close to γ .*

In the remainder of this section we prove Theorem 2.1. Let $I = [-1, 1]$. We need the following particular case of [7, Theorem 4.1]:

Lemma 2.3. — *For every $f \in \mathcal{PH}^1$ and $\zeta \in M$, there exists a neighborhood \mathcal{V} of f and a continuous map $\psi = \psi_{f,\zeta} : \mathcal{V} \times I^d \rightarrow M$ such that for every $g \in \mathcal{V}$,*

1. $\psi(g, 0) = \zeta$,

- 2. $x \mapsto \psi(g, x)$ is a homeomorphism,
- 3. $\psi(g, x, y) \in W_g^u(\psi(g, 0, y))$ for every $x \in I^{d_u}$ and $y \in I^{d-d_u}$.

Lemma 2.4. — For every $f \in \mathcal{P}\mathcal{S}\mathcal{H}^1$ there exist a neighborhood \mathcal{V} of f , numbers $k \geq 1$ and $\epsilon > 0$ and continuous maps $\Phi_u : \mathcal{V} \times M \times \mathbb{R}^{kd_u} \rightarrow M$ and $\Phi_s : \mathcal{V} \times M \times \mathbb{R}^{kd_s} \rightarrow M$ such that:

- 1. $x \mapsto \Phi_u(g, x, v)$ is a homeomorphism, for every $g \in \mathcal{V}$ and $v \in \mathbb{R}^{kd_u}$;
- 2. $\mathcal{W}_{g,\epsilon}^u(x) \subset \Phi_u(\{g\} \times \{x\} \times \mathbb{R}^{kd_u}) \subset W_g^u(x)$ for every $g \in \mathcal{V}$ and $x \in M$,

and analogously for Φ_s .

Proof. — We will only go through the details of the construction of Φ_u , the case of Φ_s being analogous. Let $h_t : I \rightarrow I$ be the flow satisfying $(dh_t/dt)(x) = 1 - h_t(x)^2$. Let $H : I^d \rightarrow I$ be given by $H(v) = (1 - v_{d_u+1}^2) \dots (1 - v_d^2)$. For $v \in \mathbb{R}^{d_u}$, let $h_v : I^d \rightarrow I^d$ be given by

$$h_v(x) = (h_{H(x)v_1}(x_1), \dots, h_{H(x)v_{d_u}}(x_{d_u}), x_{d_u+1}, \dots, x_d).$$

Pick points $\zeta_i \in M$, $1 \leq i \leq k$ so that the interiors of the images $\psi_i(\{f\} \times I^d)$ cover M , where $\psi_i = \psi_{f,\zeta_i} : \mathcal{V}_i \times I^d \rightarrow M$ are the maps given by Lemma 2.3. Let \mathcal{V} be a neighborhood of f contained in $\bigcap_i \mathcal{V}_i$ and ϵ be a positive number such that for every $g \in \mathcal{V}$ and $z \in M$ there exist i and y such that

$$W_{g,\epsilon}^u(z) \subset \psi_i(\{g\} \times \text{inter}(I^{d_u}) \times \{y\}).$$

Let $\Phi_i : \mathcal{V} \times M \times \mathbb{R}^{d_u} \rightarrow M$ be given by

$$\begin{aligned} \Phi_i(g, \psi_i(g, \zeta), v) &= \psi_i(g, h_v(\zeta)) \quad \text{for } \zeta \in I^d \\ \Phi_i(g, z, v) &= z \quad \text{if } z \notin \psi_i(\{g\} \times I^d). \end{aligned}$$

Then define $\Phi^{(i)} : \mathcal{V} \times M \times \mathbb{R}^{id_u} \rightarrow M$, $1 \leq i \leq k$ by

$$\Phi^{(1)} = \Phi_1 \quad \text{and} \quad \Phi^{(i+1)}(g, \cdot, (w_i, w)) = \Phi_{i+1}(g, \cdot, w) \circ \Phi^{(i)}(g, \cdot, w_i)$$

and take $\Phi_u = \Phi^{(k)}$. Claim (1) follows from part (2) of Lemma 2.3, by composition. The lower bound in claim (2) follows from the choice of ϵ and the upper bound is a consequence of part (3) of Lemma 2.3. □

Proof of Theorem 2.1. — Define $P_l : \mathcal{V} \times M \times \mathbb{R}^{k(d_u+d_s)l} \rightarrow M$, $l \in \mathbb{N}$ by letting

$$P_l(g, \cdot, (w_u, w_s)) = \Phi_s(g, \cdot, w_s) \circ \Phi_u(g, \cdot, w_u)$$

for $w_u \in \mathbb{R}^{kd_u}$ and $w_s \in \mathbb{R}^{kd_s}$ and

$$P_l(g, \cdot, (w_1, \dots, w_l)) = P_1(g, \cdot, w_l) \circ \dots \circ P_1(g, \cdot, w_1)$$

for $w_1, \dots, w_l \in \mathbb{R}^{k(d_u+d_s)}$. Property (1) in Theorem 2.1 is a direct consequence of this definition. Property (2) follows from part (2) of Lemma 2.3, by composition. Finally, Lemma 2.4 gives that $\bigcup_{n \geq 1} P_n(g, z, \mathbb{R}^{k(d_u+d_s)n})$ is the g -accessibility class of z , as claimed in part (3) of the theorem. □