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ASYMPTOTIC EXPANSION OF SMOOTH INTERVAL MAPS

by

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Abstract. — We associate to each non-degenerate smooth interval map a number measuring its global asymptotic expansion. We show that this number can be calculated in various different ways. A consequence is that several natural notions of nonuniform hyperbolicity coincide. In this way we obtain an extension to interval maps with an arbitrary number of critical points of the remarkable result of Nowicki and Sands characterizing the Collet-Eckmann condition for unimodal maps. This also solves a conjecture of Luzzatto in dimension 1.

Combined with a result of Nowicki and Przytycki, these considerations imply that several natural nonuniform hyperbolicity conditions are invariant under topological conjugacy. Another consequence is for the thermodynamic formalism: A non-degenerate smooth map has a high-temperature phase transition if and only if it is not Lyapunov hyperbolic.

***Résumé* (Expansion asymptotique des applications lisses d'intervalle).** — On associe à chaque application lisse et non dégénérée de l'intervalle un nombre measurant son expansion asymptotique globale. On montre que ce nombre peut être calculé de plusieurs façons distinctes. En conséquence, plusieurs notions d'hyperbolicité faible coïncident. De cette façon on obtient une extension aux applications de l'intervalle avec une nombre arbitraire de points critiques du fameux résultat de Nowicki et Sands caractérisant la condition de Collet-Eckmann pour les applications unimodales. Ceci résout aussi une conjecture de Luzzatto en dimensión 1. En combinaison avec un résultat de Nowicki et Przytycki, ces considérations entraînent que plusieurs notions d'hyperbolicité faible sont invariantes par conjugaison topologique. Une autre conséquence est pour le formalisme thermodynamique : une application lisse et non dégénérée de l'intervalle possède une transition de phase de haute temperature si et seulement si elle n'est pas Lyapunov hyperbolique.

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1. Introduction

In the last few decades, the statistical and stochastic properties of nonuniformly hyperbolic maps have been extensively studied in the one-dimensional setting, see for example [**6, 12, 16, 37, 39, 45**] and references therein. These maps are known to be abundant, see for example [**3, 5, 15, 10, 21, 42, 44**] for interval maps and [**2, 34, 40, 14**] for complex rational maps.

In this paper we associate to each non-degenerate smooth interval map a number measuring its global asymptotic expansion. Our main result is that this number can be calculated in various different ways. For example, it can be calculated using the Lyapunov exponents of periodic points or the Lyapunov exponents of invariant measures, and it can also be calculated using the exponential contraction rate of preimages of a small ball. This implies that several natural notions of nonuniform hyperbolicity coincide, including the existence of an absolutely continuous invariant probability (acip) that is exponentially mixing. In this way we obtain an extension to interval maps with an arbitrary number of critical points of the remarkable result of Nowicki and Sands characterizing the Collet-Eckmann condition for unicritical maps, see [**28**]. Moreover, this solves in the affirmative a conjecture of Luzzatto in dimension 1, see [**19**, Conjecture 1].

Combined with a result of Nowicki and Przytycki, we obtain that several natural notions of nonuniform hyperbolicity are invariant under topological conjugacy, see [**27**]. In particular, for non-degenerate smooth interval maps the existence of an exponentially mixing acip is invariant under topological conjugacy.

Combined with [**11, 22, 23, 43, 46**], these considerations imply that an arbitrary exponentially mixing acip satisfies strong statistical properties, such as the local central limit theorem and the vector-valued almost sure invariant principle. On the other hand, by [**37**] it follows that for some $p > 1$ the density of such a measure is in the space $L^p(\mathrm{Leb})$.

Our main result provides an important step in the study of the thermodynamic formalism of non-degenerate smooth interval maps in [**32**].(1) Combining our main result with [**32**, Theorem A], we obtain a characterization of those maps having a high-temperature phase transition.

We proceed to describe our results more precisely. To simplify the exposition, below we state our results in a more restricted setting than what we are able to handle. For general versions, see §4 and the remarks in §6.

1.1. Quantifying asymptotic expansion. — Let I be a compact interval and $f\colon I \to I$ a smooth map. A *critical point of* f is a point of I at which the derivative of f vanishes. The map f is *non-degenerate* if it is non-injective, if the number of its critical points is

(1) The proof of our Main Theorem applies without change to the more general class of maps considered in [**32**], see Theorem C of that paper. Note however that, although the proof in [**32**] follows the proof of our Main Theorem, it has a part that is different. This modified proof only gives a qualitative version of our Main Theorem, similar to Corollary A.

finite, and if at each critical point of f some higher order derivative of f is nonzero. A non-degenerate smooth interval map is *unicritical* if it has a unique critical point.[2]

Let $f\colon I \to I$ be a non-degenerate smooth map. For an integer $n \geq 1$, a periodic point p of f of period n is *hyperbolic repelling* if $|Df^n(p)| > 1$. In this case, denote by

$$\chi_p(f) := \frac{1}{n} \ln |Df^n(p)|$$

the Lyapunov exponent of p. Similarly, for a Borel probability measure ν on I that is invariant by f denote by

$$\chi_\nu(f) := \int \ln |Df| \, \mathrm{d}\nu$$

its Lyapunov exponent.

The following is our main result. A non-degenerate smooth map $f\colon I \to I$ is *topologically exact*, if for every open subset U of I there is an integer $n \geq 1$ such that $f^n(U) = I$.

Main Theorem. — *For a non-degenerate smooth map $f\colon I \to I$, the number*

$$\chi_{\mathrm{per}}(f) := \inf \{\chi_p(f) : p \textit{ hyperbolic repelling periodic point of } f\}$$

is equal to

$$\chi_{\mathrm{inf}}(f) := \inf \{\chi_\nu(f) : \nu \textit{ invariant probability measure of } f\}.$$

If in addition f is topologically exact, then there is $\delta > 0$ such that for every interval J contained in I that satisfies $|J| \leq \delta$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \ln \max \left\{|W| : W \textit{ connected component of } f^{-n}(J)\right\} = -\chi_{\mathrm{inf}}(f).$$

Moreover, for each point x_0 in I we have

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \min \left\{|Df^n(x)| : x \in f^{-n}(x_0)\right\} \leq \chi_{\mathrm{inf}}(f), \tag{1.1}$$

and there is a subset E of I of zero Hausdorff dimension such that for each point x_0 in $I \setminus E$ the lim sup *above is a limit and the inequality an equality.*

Except for the equality $\chi_{\mathrm{inf}}(f) = \chi_{\mathrm{per}}(f)$, the hypothesis that f is topologically exact is necessary, see §1.6.

The result above suggests that for a non-degenerate smooth map f the number $\chi_{\mathrm{per}}(f)$ (equal to $\chi_{\mathrm{inf}}(f)$) is a natural measure of the asymptotic expansion of f. In fact, $\chi_{\mathrm{inf}}(f)$ gives a lower bound for the (lower) Lyapunov exponent of every point in a set of total probability. This motivates the following definition.

Definition 1.1. — A non-degenerate smooth map f is *Lyapunov hyperbolic* if $\chi_{\mathrm{inf}}(f) > 0$. In this case, we call $\chi_{\mathrm{inf}}(f)$ the *total Lyapunov exponent of f*.

[2] Note that every unicritical map is unimodal, but not conversely.

Lyapunov hyperbolicity can be regarded as a strong form of nonuniform hyperbolicity in the sense of Pesin. A consequence of the Main Theorem is that Lyapunov hyperbolicity coincides with several natural nonuniform hyperbolicity conditions, see §1.2.

When restricted to the case where f is unicritical, the Main Theorem gives a quantified version of the fundamental part of [**28**, Theorem A]. In [**28**, Theorem A], property (1.1) was only considered in the case where x_0 is the critical point of f; so the assertions concerning (1.1) in the Main Theorem are new, even when restricted to the case where f is unicritical. The proof of [**28**, Theorem A] relies heavily on delicate combinatorial arguments that are specific to unicritical maps. As is, it does not extend to interval maps with several critical points. When restricted to unicritical maps, our argument is substantially simpler than that of [**28**].

When f is a complex rational map, the Main Theorem is the essence of [**33**, Main Theorem]. The proof in [**33**, Main Theorem] does not extend to interval maps, because at a key point it relies on the fact that a complex rational map is open as a map of the Riemann sphere to itself. Our argument allows us to deal with the fact that a non-degenerate smooth interval map is not an open map in general, see §1.7 for further details.

1.2. Nonuniformly hyperbolic interval maps. — We introduce some terminology to state a consequence of the Main Theorem about the equivalence of various nonuniform hyperbolicity conditions.

Let (X, dist) be a compact metric space, $T\colon X \to X$ a continuous map and ν a Borel probability measure that is invariant by T. Then ν is *exponentially mixing* or *has exponential decay of correlations*, if there are constants $C > 0$ and ρ in $(0,1)$ such that for every continuous function $\varphi\colon X \to \mathbb{R}$ and every Lipschitz continuous function $\psi\colon X \to \mathbb{R}$ we have for every integer $n \ge 1$

$$\left| \int_X \varphi \circ f^n \cdot \psi \, \mathrm{d}\nu - \int_X \varphi \, \mathrm{d}\nu \int_X \psi \, \mathrm{d}\nu \right| \le C \left(\sup_X |\varphi| \right) \|\psi\|_{\mathrm{Lip}} \rho^n,$$

where $\|\psi\|_{\mathrm{Lip}} := \sup_{x, x' \in X, x \neq x'} \frac{|\psi(x) - \psi(x')|}{\mathrm{dist}(x, x')}$.

We denote by Leb the Lebesgue measure on $\mathbb{R}$. For a non-degenerate smooth map $f\colon I \to I$, we use *acip* to refer to a Borel probability measure on I that is absolutely continuous with respect Leb and that is invariant by f.

A non-degenerate smooth map $f\colon I \to I$ has *Uniform Hyperbolicity on Periodic Orbits*, if $\chi_{\mathrm{per}}(f) > 0$. Moreover, f satisfies the:

— *Collet-Eckmann condition*, if all the periodic points of f are hyperbolic repelling and if for every critical value v of f we have

$$\liminf_{n \to +\infty} \frac{1}{n} \ln |Df^n(v)| > 0.$$

— *Backward* or *Second Collet-Eckmann condition at a point x of I*, if there are constants $C > 0$ and $\lambda > 1$, such that for every integer $n \ge 1$ and every point y of $f^{-n}(x)$ we have $|Df^n(y)| \ge C\lambda^n$.