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A CLOSING LEMMA FOR POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

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Abstract. — We prove that for a dissipative polynomial diffeomorphism of $\mathbb{C}^2$, the support of any invariant measure is, apart from a few well-understood cases, contained in the closure of the set of saddle periodic points.

Résumé **(Un lemme de fermeture pour les automorphismes polynomiaux de $\mathbb{C}^2$)**
Nous montrons que pour un automorphisme polynomial dissipatif de $\mathbb{C}^2$, le support de toute mesure invariante est contenu dans l'adhérence de l'ensemble des points selles, à l'exception de quelques cas bien compris.

1. Introduction and results

Let f be a polynomial diffeomorphism of $\mathbb{C}^2$ with non-trivial dynamics. This non-triviality can be expressed in a variety of ways, for instance it is equivalent to the exponential growth of the algebraic degrees of the iterates f^n or to the positivity of topological entropy. The dynamics of such transformations has attracted a lot of attention in the past few decades (the reader can consult e.g., **[1]** for basic facts and references).

In this paper we make the standing assumption that f is dissipative, i.e., that the (constant) Jacobian of f satisfies $|\mathrm{Jac}(f)| < 1$.

We denote by J^+ the forward Julia set, which can be classically characterized in terms of normal families, or by saying that $J^+ = \partial K^+$, where K^+ is the set of points with bounded forward orbits. Reasoning analogously for backward iteration gives the backward Julia set $J^- = \partial K^-$. Thus the 2-sided Julia set is naturally defined by $J = J^+ \cap J^-$. Another interesting dynamically defined subset is the closure J^* of

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the set of saddle periodic points (which is also the support of the unique entropy maximizing measure by [**2**]).

The inclusion $J^* \subset J$ is obvious. It is a major open question in this area of research whether the converse inclusion holds. Partial answers have been given in [**3, 5, 7, 13, 11**].

Let ν be an ergodic f-invariant probability measure. If ν is hyperbolic, that is, its two Lyapunov exponents [(1)] are non-zero and of opposite sign, then the so-called Katok closing lemma [**12**] implies that $\mathrm{Supp}(\nu) \subset J^*$. It may also be the case that ν is supported in the Fatou set: then from the classification of recurrent Fatou components in [**4**], this happens if and only if ν is supported on an attracting or semi-Siegel periodic orbit, or is the Haar measure on a cycle of k circles along which f^k is conjugate to an irrational rotation (recall that f is assumed dissipative). Here by semi-Siegel periodic orbit, we mean a linearizable periodic orbit with one attracting and one irrationally indifferent multipliers.

The following "ergodic closing lemma" is the main result of this note:

Theorem 1.1. — *Let f be a dissipative polynomial diffeomorphism of $\mathbb{C}^2$ with non-trivial dynamics, and ν be any invariant measure supported on J. Then* $\mathrm{Supp}(\nu)$ *is contained in J^*.*

A consequence is that if $J \setminus J^*$ happens to be non-empty, then the dynamics on $J \setminus J^*$ is "transient" in a measure-theoretic sense. Indeed, if $x \in J$, we can form an invariant probability measure by taking a cluster limit of $\frac{1}{n} \sum_{k=0}^{n} \delta_{f^k(x)}$ and the theorem says that any such invariant measure will be concentrated on J^*. More generally the same argument implies:

Corollary 1.2. — *Under the assumptions of the theorem, if $x \in J^+$, then $\omega(x) \cap J^* \neq \emptyset$.*

Here as usual $\omega(x)$ denotes the ω-limit set of x. Note that for $x \in J^+$ it is obvious that $\omega(x) \subset J$. It would be interesting to know whether the conclusion of the corollary can be replaced by the sharper one: $\omega(x) \subset J^*$.

Theorem 1.1 can be formulated slightly more precisely as follows.

Theorem 1.3. — *Let f be a dissipative polynomial diffeomorphism of $\mathbb{C}^2$ with non-trivial dynamics, and ν be any ergodic invariant probability measure. Then one of the following situations holds:*

(i) *either ν is atomic and supported on an attracting or semi-Siegel cycle;*

(ii) *or ν is the Haar measure on an invariant cycle of circles contained in a periodic rotation domain;*

(iii) *or* $\mathrm{Supp}(\nu) \subset J^*$.

(1) Recall that in holomorphic dynamics, Lyapunov exponents always have even multiplicity.

Note that the additional ergodicity assumption on ν is harmless since any invariant measure is an integral of ergodic ones. The only new ingredient with respect to Theorem 1.1 is the fact that measures supported on periodic orbits that do not fall in case (i), that is, are either semi-parabolic or semi-Cremer, are supported on J^*. For semi-parabolic points this is certainly known to the experts although apparently not available in print. For semi-Cremer points this follows from the hedgehog construction of Firsova, Lyubich, Radu and Tanase (see [**14**]). For completeness we give complete proofs below.

A final comment on the dissipativity assumption. Of course Theorem 1.1 also holds if $|\mathrm{Jac}(f)| > 1$ by simply replacing f by f^{-1}. On the other hand our methods break down completely when f is conservative ($|\mathrm{Jac}(f)| = 1$), since they are based on the analysis of strong stable manifolds.

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2. Proofs

In this section we prove Theorem 1.3 by dealing separately with the atomic and the non-atomic case. Theorem 1.1 follows immediately. Recall that f denotes a dissipative polynomial diffeomorphism with non trivial dynamics and ν an f-invariant ergodic probability measure.

2.1. Preliminaries. — Using the theory of laminar currents, it was shown in [**2**] that any saddle periodic point belongs to J^*. More generally, if p and q are saddle points, then $J^* = \overline{W^u(p) \cap W^u(q)}$ (see Theorems 9.6 and 9.9 in [**2**]). This result was generalized in [**8**] as follows. If p is any saddle point and $X \subset W^u(p)$, we respectively denote by $\mathrm{Int}_i X$, $\mathrm{cl}_i X$, $\partial_i X$ the interior, closure and boundary of X relative to the intrinsic topology of $W^u(p)$, that is the topology induced by the biholomorphism $W^u(p) \simeq \mathbb{C}$.

Lemma 2.1 ([**8**, Lemma 5.1]). — *Let p be a saddle periodic point. Relative to the intrinsic topology in $W^u(p)$, $\partial_i(W^u(p) \cap K^+)$ is contained in the closure of the set of transverse homoclinic intersections. In particular $\partial_i(W^u(p) \cap K^+) \subset J^*$.*

Here is another statement along the same lines, which can easily be extracted from [**2**].

Lemma 2.2. — *Let $\psi : \mathbb{C} \to \mathbb{C}^2$ be an entire curve such that $\psi(\mathbb{C}) \subset K^+$. Then for any saddle point p, $\psi(\mathbb{C})$ admits transverse intersections with $W^u(p)$.*

Proof. — This is identical to the first half of the proof of [**8**, Lemma 5.4]. □

We will repeatedly use the following alternative which follows from the combination of the two previous lemmas. Recall that a Fatou disk is a holomorphic disk along which the iterates $(f^n)_{n\geq 0}$ form a normal family.

Lemma 2.3. — *Let $\mathscr{E}$ be an entire curve contained in K^+, p be any saddle point, and t be a transverse intersection point between $\mathscr{E}$ and $W^u(p)$. Then either $t \in J^*$ or there is a Fatou disk $\Delta \subset W^u(p)$ containing t.*

Proof. — Indeed, either t is in $\partial_i(W^u(p) \cap K^+)$ so by Lemma 2.1, t belongs to J^*, or t is in $\mathrm{Int}_i(W^u(p) \cap K^+)$. In the latter case, pick any open disk $\Delta \subset \mathrm{Int}_i(W^u(p) \cap K^+)$ containing t. Since Δ is contained in K^+, its forward iterates remain bounded so it is a Fatou disk. □

2.2. The atomic case. — Here we prove Theorem 1.3 when ν is atomic. By ergodicity, this implies that ν is concentrated on a single periodic orbit. Replacing f by an iterate we may assume that it is concentrated on a fixed point. Since f is dissipative there must be an attracting eigenvalue. A first possibility is that this fixed point is attracting or semi-Siegel. Then we are in case (i) and there is nothing to say. Otherwise p is of saddle, semi-parabolic or semi-Cremer type and we must show that $p \in J^*$. The case of saddles was treated in [**2**, Thm 9.2]. In both remaining cases, p admits a strong stable manifold $W^{ss}(p)$ associated to the contracting eigenvalue, which is biholomorphic to $\mathbb{C}$ by a theorem of Poincaré. Let q be a saddle periodic point and t be a point of transverse intersection between $W^{ss}(p)$ and $W^u(q)$. If $t \in J^*$, then since $f^n(t)$ converges to p as $n \to \infty$ we are done. Otherwise there is a non-trivial Fatou disk Δ transverse to $W^{ss}(p)$ at t. Let us show that this is contradictory.

In the semi-parabolic case, this is classical. A short argument goes as follows (compare [**18**, Prop. 7.2]). Replace f by an iterate so that the neutral eigenvalue is equal to 1. Since f has no curve of fixed points there are local coordinates (x, y) near p in which $p = (0, 0)$, $W^{ss}_{\mathrm{loc}}(p)$ is the y-axis $\{x = 0\}$ and f takes the form

$$(x, y) \longmapsto (x + x^{k+1} + h.o.t., by + h.o.t.) \ ,$$

with $|b| < 1$ (see [**18**, §6]). Then f^n is of the form

$$(x, y) \longmapsto (x + nx^{k+1} + h.o.t., b^n y + h.o.t.) \ ,$$

from which it follows that f^n cannot be normal along any disk transverse to the y axis, so we are done.

In the semi-Cremer case we rely on the hedgehog theory of [**10, 14**]. Let $\phi : \mathbb{D} \to \Delta$ be any parameterization, and fix local coordinates (x, y) as before in which $p = (0, 0)$, $W^{ss}_{\mathrm{loc}}(p)$ is the y-axis and f takes the form

$$(x, y) \longmapsto (e^{i2\pi\theta}x, by) + h.o.t.$$

Let B be a small neighborhood of the origin in which the hedgehog $\mathscr{H}$ is well-defined. Reducing Δ and iterating a few times if necessary, we can assume that for all $k \geq 0$, $f^k(\Delta) \subset B$ and ϕ is of the form $s \mapsto (s, \phi_2(s))$. Then the first coordinate of $f^n \circ \phi$ is of the form $s \mapsto e^{i2n\pi\theta}s + h.o.t.$ If $(n_j)_{j\geq 0}$ is a subsequence such that $f^{n_j} \circ \phi$