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Joel W. Fish & Helmut Hofer

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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EXHAUSTIVE GROMOV COMPACTNESS FOR PSEUDOHOLOMORPHIC CURVES

by

Joel W. Fish & Helmut Hofer

Dedicated to the memory of Jean-Christophe Yoccoz

Abstract. — Here we extend the notion of target-local Gromov convergence of pseudoholomorphic curves to the case in which the target manifold is not compact, but rather is exhausted by compact neighborhoods. Under the assumption that the curves in question have uniformly bounded area and genus on each of the compact regions (but not necessarily global bounds), we prove a subsequence converges in an exhaustive Gromov sense.

***Résumé* (Compacité de Gromov pour courbes pseudoholomorphes au sens exhaustif)**
Considérant la convergence de courbes pseudoholomorphiques vers une variété dans un sens « local » de Gromov, nous étendons cette notion de convergence au cas où la variété ciblée n'est pas nécessairement compacte, mais recouvrable par une suite de voisinages compacts. Sous l'hypothèse que les courbes en question ont une aire et un genre borné sur chacun des compacts (mais pas forcément globalement), nous prouvons l'existence d'une sous-suite convergente au sens « exhaustif » de Gromov.

1. Introduction

In his celebrated 1985 paper, **[10]**, Gromov introduced the notion of a pseudoholomorphic curve, and provided an accompanying compactness theorem. His idea was to generalize the notion of an algebraic curve in, say, a complex projective variety to that of a pseudoholomorphic curve in a symplectically tamed almost complex manifold, and he showed that families of such curves are analogously compact. In the decades since, pseudoholomorphic curves have played a fundamental role in the development of symplectic geometry and topology as well as Hamiltonian dynamics,

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and a variety generalizing compactness theorems have been established. These tend to proceed along two general paths.

The first approach is exemplified by Rugang Ye [**20**], Floer [**9**], Hofer [**12**], and the SFT compactness paper [**2**], in that each of these treat closed or punctured curves from a global perspective. Besides additional ingredients dealing with the analysis near punctures and the necks, see for example [**13**], the analysis proceeds rather analogously to that for families of harmonic maps, which we outline as follows.

1. Obtain convergence of underlying Riemann surfaces.
2. With respect to a constant curvature metric guaranteed by the Uniformization Theorem, show that gradient bounds imply $\mathcal{C}^\infty$ bounds.
3. Employ bubbling analysis at points of gradient blow-up, and show that only finitely many bubbles appear due to energy bounds and an energy threshold.
4. Use $\mathcal{C}^\infty$ bounds and Arzelà-Ascoli to pass to a further subsequence which converges in $\mathcal{C}^\infty$.
5. Verify that bubbles connect via, say, a monotonicity lemma.

This approach is most applicable when one has genus bounds, energy bounds, some global control over the entirety of each curve in the family, and when one already has a good idea of what types of curves should arise in the limits of such families. For completeness, we also mention Hummel [**14**], and for a more classical viewpoint, [**1**].

The second approach, typified by Taubes via Proposition 3.3 in [**19**], treats curves as sets and integral currents, and proves compactness from a more measure theoretic perspective. Roughly speaking, area bounds, a monotonicity lemma, and some measure theory yield a compactness theorem, however some additional work is necessary to show the limit is rectifiable, or rather that the measure theoretic limit has the structure of a weighted union of images of pseudoholomorphic curves. This approach is quite natural from the perspective of Seiberg-Witten theory, particularly when employing a Taubes-like degeneration to obtain pseudoholomorphic curves. The result has also been used extensively in Embedded Contact Homology, introduced by Hutchings in [**15**]; see also [**17**] and [**16**]. More generally, the technique is applicable when one has little more than area bounds – indeed, one does not need genus bounds on the sequence of curves. However, this can also be a weakness, in that genus cannot be detected a priori by these techniques. For example, one can construct degree-two holomorphic branched coverings of the unit disk with arbitrarily large genus, but from the integral current perspective, all such objects are indistinguishable.

The purpose of this manuscript is to further develop a less used third approach, introduced by the first author in [**3**] and streamlined in [**4**] and [**5**]. This is the so-called target-local Gromov compactness result; for a restatement, see Theorem 2 below. The basic idea was to follow the Taubes approach to studying curves locally in the target and allowing a free boundary (including arbitrarily many boundary components), but also demanding a genus bound, and then extracting a subsequence which converges in the Gromov-topology, rather than the substantially weaker topology of integral-currents. In some sense, the target-local compactness theorem says that if W is a smooth compact manifold with boundary, and $u_k : (S_k, \partial S_k) \to (W, \partial W)$ is a sequence of pseudoholomorphic maps with genus bounds and area bounds, then after trimming the curves near ∂W a subsequence converges in a Gromov sense. Our main result, stated below as Theorem 1, extends this to the case that W is no longer compact, but instead is exhausted by compact manifolds with smooth boundary:

$$W_1 \subset W_2 \subset W_3 \subset \cdots \bigcup_{\ell \in \mathbb{N}} W_\ell = W.$$

Here the key assumptions on the curves are that

$$\mathrm{Area}(u_k^{-1}(W_\ell)) \leq C_\ell \qquad \text{and} \qquad \mathrm{Genus}(u_k^{-1}(W_\ell)) \leq C_\ell.$$

In other words, this means that the curves in question may have infinite area and genus, however on each compact W_ℓ (the union of which exhaust W) one has area and genus bounds for the portions of curves in that region.

It is important to mention that our main result here, stated as Theorem 1 below, is not a needless extension of Theorem 2, proved in [**5**], but rather it plays a foundational role in two forthcoming papers. The first was announced in [**7**] and will appear in [**8**] in which we prove that no regular energy level of a proper Hamiltonian function on $(\mathbb{R}^4, \omega_{\mathrm{std}})$ has a minimal Hamiltonian flow, which answers a question for the case $n = 2$ raised by Herman in his 1998 ICM address; see [**11**]. The idea is to use neck-stretching techniques to study pseudoholomorphic curves in the symplectization of framed Hamiltonian manifolds. The tremendous difficulty is that such curves will lack a priori energy bounds like those that appear in Symplectic Field Theory, and thus the global techniques employed in [**2**] fail quickly and completely. Moreover, the Taubes approach of [**19**] also fails, precisely because the topology used to obtain compactness is simply too coarse. Indeed, the genus bounds and curvature properties that follow from the Gromov topology (but not the integral current topology) which are guaranteed by Theorem 1 play crucial roles in the proofs of the main results in [**8**]. In essence, our main result here strikes the perfect balance between the flexibility of the integral-current approach with the strength of the Gromov topology, and this balance is then heavily exploited in [**8**] to first find a limit curve (which might be wildly complicated), and then to use the Gromov topology and a posteriori analysis on the limit curve to show that it has a surprising number of unexpected properties, which are necessary to establish the non-minimality of the hypersurfaces.

The second result relying on Theorem 1 is the so called sideways stretching compactness results developed by the first author; see [**6**]. Here the idea is that Symplectic Field Theory is something akin to a TQFT for symplectic manifolds, and an extended TQFT would be akin to an extended Symplectic Field Theory in which one could independently stretch the neck along two transverse contact hypersurfaces. This has been carried out by the first author in certain sub-critical cases, and will appear in a forthcoming paper. Again though, the idea is similar: use a sequence of expanding domains in the target manifold, on which one has successively increasing area bounds to obtain a preliminary compactness result from Theorem 1; then use a posteriori analysis and the Gromov topology to improve properties of both the limit and precision of the convergence; then iterate this procedure to develop a full extended Symplectic Field Theory style compactness theorem.

More generally still, it is not difficult to imagine a wide range of applications of Theorem 1. Indeed, consider any symplectic manifold W, and any compact set $K \subset W$ which has empty interior. Then consider any sequence of almost complex structures which are tame, but degenerate along K. That is, the J_k converge in $\mathcal{C}^\infty_{\rm loc}(W \setminus K)$ to an almost complex structure on $W \setminus K$, which is uniformly tame on each $W \setminus \mathcal{O}(K)$, but not uniformly tame on $W \setminus K$. Then consider a sequence of closed pseudoholomorphic curves in a fixed homology class in W which have bounded genus (e.g., only spheres). Theorem 1 immediately guarantees that a subsequence converges, in an exhaustive Gromov sense (see Definition 2.21), to a pseudoholomorphic curve in $W \setminus K$. Such a curve may have wildly complicated behavior – and yet a posteriori analysis can be employed which exploits the particular features of the K and J_k in question. Considering the ubiquitous use of pseudoholomorphic curves in symplectic geometry, topology, and Hamiltonian dynamics, such a result would seem potentially quite useful.

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2. Preliminaries

This section is devoted to presenting some preliminary concepts and small supporting results. In Section 2.1 we introduce the notion of an embedding diagram and direct limit manifolds. This is meant to generalize the notion of an exhausting sequence of regions:

$$W_1 \subset W_2 \subset W_3 \subset \cdots \subset \bigcup_{k\in\mathbb{N}} W_k =: \overline{W},$$

and is necessary for constructing the domain of the limit curve we must later produce. In Section 2.2 we review the basic definitions of Riemann surfaces, as well as additional structures like marked points, nodal points, decorations, arithmetic genus, etc. In Section 2.3, we recall the definition of pseudoholomorphic curves and some related concepts, like stability, boundary-immersed maps, generally immersed maps, and area. Finally, in Section 2.4 we provide a number of definitions of convergence for pseudoholomorphic curves. We note that the key notion of Section 2 is given in Definition 2.21, which is the novel definition of convergence of pseudoholomorphic curves in an exhaustive Gromov sense. Finally we note that here and throughout, in the case that a domain is non-compact, $\mathcal{C}^\infty$ convergence will mean $\mathcal{C}^\infty_{\rm loc}$ convergence.