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THREE-DIMENSIONAL ORBIFOLDS AND THEIR GEOMETRIC STRUCTURES

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This volume is dedicated to Professor Laurent Siebenmann for his 65th birthday

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Michel Boileau, Sylvain Maillot, Joan Porti

***Abstract.* —** Orbifolds locally look like quotients of manifolds by finite group actions. They play an important rôle in the study of proper actions of discrete groups on manifolds. This monograph presents recent fundamental results on the geometry and topology of 3-dimensional orbifolds, with an emphasis on their geometric properties.

***Résumé* (Les orbivariétés tridimensionnelles et leurs structures géométriques)**

Une orbivariété est localement le quotient d'une variété par un groupe fini. Cette notion joue un rôle important dans l'étude des actions propres de groupes discrets sur les variétés. Cette monographie présente des résultats fondamentaux récents sur la géométrie et la topologie des orbivariétés de dimension 3, en mettant l'accent sur leurs propriétés géométriques.

CONTENTS

INTRODUCTION

In this book, we present important recent results on the geometry and topology of 3-dimensional manifolds and orbifolds. Orbifolds are natural generalizations of manifolds, and can be roughly described as spaces which locally look like quotients of manifolds by finite group actions. They were introduced by I. Satake, under the name V-manifold, and their importance in dimension 3 emerged from the work of W. Thurston, who used them as tools for geometrizing 3-manifolds. Orbifolds occur in many contexts, for instance as orbit spaces of group actions on manifolds, or as leaf spaces of certain foliations.

A basic idea behind geometrization is the concept of *uniformization,* which for us means studying a manifold M by putting a structure on its universal cover $\widetilde{M}$ that is preserved by the action of the fundamental group $\pi_1 M$. If the structure is rigid enough, this gives information about M. More specifically, we shall call *geometry* a homogeneous, simply-connected, unimodular Riemannian manifold, and say that a manifold is *geometric* if it is diffeomorphic to the quotient of a geometry by a discrete subgroup of its isometry group.

It has been known since the beginning of the twentieth century that every compact surface is geometric: more precisely, it is either elliptic, Euclidean or hyperbolic, *i.e.* can be obtained as the quotient of the round 2-sphere $\mathbf{S}^2$, the Euclidean plane $\mathbf{E}^2$, or the hyperbolic plane $\mathbf{H}^2$ by a discrete group of isometries.

Some important properties of surfaces, *e.g.* linearity of the fundamental group, can be deduced from this fact. Geometric structures on surfaces can also be used to attack more difficult and subtle problems such as studying mapping class groups. Moreover, the Gauss-Bonnet formula provides a strong link between geometry and topology in dimension 2.

In dimension 3, it is fairly easy to see that not every manifold is geometric. However, it was W. Thurston's groundbreaking idea that the situation should be almost as nice: any compact 3-manifold should be uniquely decomposable along a finite collection of disjoint embedded surfaces into geometric pieces. This is the content of his Geometrization Conjecture, formulated in the mid seventies, and which we shall state more precisely in Chapter 1. Positive solutions of many important problems in 3-manifold topology, including the famous Poincaré Conjecture, as well the Universal Covering Conjecture, or residual finiteness of 3-manifold groups, would follow from the Geometrization Conjecture.

Thurston observed that there are only eight 3-dimensional geometries: those of constant curvature $\mathbf{S}^3$, $\mathbf{E}^3$, and $\mathbf{H}^3$; the product geometries $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^2 \times \mathbf{R}$; the twisted product geometries **Nil** and $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, and finally **Sol**. Among geometric manifolds, those modelled on $\mathbf{H}^3$ remain the most mysterious. Thus the Geometrization Conjecture reduces in principle any problem on 3-manifolds to combination theorems and understanding hyperbolic manifolds. Hence Thurston's work entailed a shift of emphasis from the purely topological (combinatorial) methods of the 50's and 60's toward geometric methods. It not only offers an approach to old topological problems, but also motivates the study of geometric ones. In particular, it renewed Kleinian group theory, which before Thurston was mainly considered from the point of view of complex analysis, by bringing hyperbolic geometry and topology into it. This is still an active field of research.

The Geometrization Conjecture is known to hold in various cases. The first breakthrough was Thurston's Hyperbolization Theorem, which covers an important and fairly general class of 3-manifolds called Haken manifolds. Since knot exteriors are included in this class, this result had spectacular applications to knot theory, leading for instance to the solution of the Smith Conjecture.

The Geometrization Conjecture is also true for prime 3-manifolds whose fundamental group contains a subgroup isomorphic to $\mathbf{Z}\times\mathbf{Z}$, by combining the result mentioned above with the full version of the Torus Theorem, including the solution of the Seifert Fiber Space Conjecture. Lastly, it is known for a class of 'manifolds with symmetries', *i.e.* manifolds with finite group actions satisfying certain properties. The geometrization of these manifolds is reduced to the geometrization of the quotient orbifolds, which is the content of the Orbifold Theorem.

The main purpose of this book is to present those results and some of the ideas and techniques involved in their proofs. Some parts are covered in detail, while others are only sketched. We have tried to give a hint of the various methods and of the various parts of mathematics they draw ideas from: this includes geometric topology, algebraic and differential geometry, and geometric group theory. At several points we indicate connections with other fields in the form of short surveys, references to the literature or open questions. We also supply some background material that is scattered in the literature or missing from it.

The classification of the eight homogeneous 3-dimensional geometries is given in Chapter 1. Chapter 2 provides background material for orbifold theory. The existence of the canonical decomposition is established in Chapter 3, while in Chapter 4 we present the fundamental properties of the class of Haken orbifolds. Chapter 5 is concerned with a homotopic characterization of Seifert fibered orbifolds, which is an important case of the Geometrization Conjecture. Chapter 6 is devoted to hyperbolic orbifolds and Thurston's Hyperbolization Theorem for Haken Orbifolds. In Chapter 7 we discuss the basic properties of representation varieties and the Culler-Shalen theory of ideal points of curves. Chapter 8 deals with Thurston's construction of hyperbolic manifolds by Dehn filling and the structure of the set of volumes of hyperbolic 3-orbifolds. Finally, a proof of the Orbifold Theorem in a special case is outlined in Chapter 9.

We do not present here G. Perelman's recent breakthrough in R. Hamilton's program for proving the Geometrization Conjecture using the so-called Ricci flow equation. This approach relies on techniques from differential geometry and global analysis which are outside the scope of this book.

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CHAPTER 1

THURSTON'S EIGHT GEOMETRIES

In this chapter we present Thurston's Geometrization Conjecture and explain its interaction with some important problems in the topology and geometry of 3-manifolds. We also give the classification of the eight homogeneous 3-dimensional geometries involved in the Geometrization Conjecture.

By Moise's Theorem [**147**] each topological 3-manifold admits a unique piecewise linear (PL) or smooth structure. Hence throughout this monograph we will work in the category of differentiable manifolds.

1.1. The Geometrization Conjecture

Recall that a Riemannian manifold X is called *homogeneous* if its isometry group $\mathrm{Isom}(X)$ acts transitively. We call X *unimodular* if it has a quotient of finite volume.

A *geometry* is a simply connected, complete, homogeneous, unimodular Riemannian manifold satisfying the following maximality condition: there is no $\mathrm{Isom}(X)$-invariant Riemannian metric on X whose isometry group is strictly larger than $\mathrm{Isom}(X)$. Two geometries X, X' are *equivalent* if there is a diffeomorphism $\phi : X \to X'$ conjugating $\mathrm{Isom}(X)$ and $\mathrm{Isom}(X')$. Notice that ϕ is not required to be an isometry, nor even a homothety.

Let X be a geometry. If Γ is a discrete subgroup of $\mathrm{Isom}(X)$ acting freely, then the quotient space X/Γ is a smooth manifold with a natural Riemannian metric which is locally isometric to X. If the action is not free, then the quotient has a natural *orbifold* structure, as we will see in Chapter 2.

Let M be a smooth manifold (possibly with boundary). We say that M *admits an X-structure* if $\mathrm{Int}\, M$ is diffeomorphic to some quotient X/Γ as above. A manifold is *geometric* if it admits an X-structure for some geometry X.

A geometry X is *isotropic* if $\mathrm{Isom}(X)$ acts transitively on the unit tangent bundle T_1X. Intuitively, this means that X looks the same in every direction. This condition is equivalent to requiring that X has constant sectional curvature. A classical result in Riemannian geometry (see *e.g.* [**220**]) asserts that in every dimension $n \geqslant 2$, there are exactly three isotropic geometries up to equivalence. These are the n-sphere $\mathbf{S}^n$, Euclidean n-space $\mathbf{E}^n$ and hyperbolic n-space $\mathbf{H}^n$, with constant sectional curvature equal to respectively $+1$, 0, and -1. The fact that these spaces are unimodular is obvious for $\mathbf{S}^n$ (which is compact) and $\mathbf{E}^n$ (which for each dimension n admits as compact quotient the n-torus $\mathbf{T}^n = \mathbf{E}^n/\mathbf{Z}^n$). This is nontrivial for $\mathbf{H}^n$ (see *e.g.* [**24, 210**]).

A n-manifold is called *spherical* (resp. *Euclidean*, resp. *hyperbolic*) if it has a $\mathbf{S}^n$-structure (resp. a $\mathbf{E}^n$-structure, resp. a $\mathbf{H}^n$-structure). For *closed* manifolds, these three situations are mutually exclusive. Indeed, if M is spherical, then $\pi_1 M$ is finite, so M cannot be Euclidean or hyperbolic. If M is Euclidean, then a theorem of Bieberbach (again see [**220**]) asserts that $\pi_1 M$ has an abelian subgroup of finite index, which implies that M cannot be hyperbolic.

The situation in dimension 2 is very special. Indeed, the three isotropic geometries are the only ones; furthermore, every closed surface is geometric, *i.e.* either spherical, Euclidean, or hyperbolic. This last fact can be proved by direct construction once one knows the classification of surfaces, or deduced from the Poincaré-Koebe Uniformization Theorem (see the discussion in [**20**]).

The fact that a closed surface F cannot have two structures modelled on inequivalent geometries admits a more elementary proof than the one quoted above for isotropic geometries in general dimension n. Indeed, it is a direct consequence of the Gauss-Bonnet formula $\chi(F) = \int_F K\,ds$, where $\chi(F)$ is the Euler characteristic. The situation is therefore particularly nice: F is elliptic if and only if $\chi(F) > 0$ (this gives $\mathbf{S}^2$ and $\mathbf{RP}^2$), Euclidean if and only if $\chi(F) = 0$ (this gives the 2-torus $\mathbf{T}^2$ and the Klein Bottle $\mathbf{K}^2$), and hyperbolic otherwise. We shall see in Chapter 2 a more general statement for 2-dimensional orbifolds (*cf.* Theorem 2.10).

In dimension 3 the situation is more complicated. Beside the three isotropic geometries $(\mathbf{S}^3, \mathbf{E}^3, \mathbf{H}^3)$, there are five anisotropic 3-dimensional geometries: four geometries are straight line bundles over $\mathbf{S}^2, \mathbf{E}^2$ or $\mathbf{H}^2$ $(\mathbf{S}^2 \times \mathbf{E}^1, \mathbf{Nil}, \mathbf{H}^2 \times \mathbf{E}^1, \widetilde{\mathrm{SL}_2(\mathbf{R})})$, and one geometry is modelled on the only simply connected unimodular Lie group **Sol** which is solvable, but not nilpotent. This classification is explained in Section 1.2.

Thurston's fundamental idea is that geometry should take a central part in the study of compact, orientable 3-dimensional manifolds, through decompositions of these manifolds into canonical geometric pieces. He proposed the following conjecture:

***Conjecture 1.1* (Geometrization Conjecture)**. — *The interior of any compact orientable* 3*-manifold can be split along a finite collection of essential disjoint embedded spheres and tori into a canonical collection of* 3*-submanifolds* $X_1, \dots, X_n$ *such that for each* i*, the manifold obtained from* X_i *by capping off all sphere components by balls is geometric.*

In the previous statement, an embedding of a closed connected surface in a compact orientable 3-manifold M is called *essential* if it induces an injective homomorphism of fundamental groups and if it does not bound a 3-ball nor cobounds a product with a connected component of ∂M.

A special case of the Geometrization Conjecture is the well-known Poincaré Conjecture. It claims the positive answer to a question raised by Poincaré in 1904 [**164**], and is one of the leading open problems in low dimensional topology.

***Conjecture 1.2* (Poincaré Conjecture)**. — *Any closed, simply-connected* 3*-manifold is homeomorphic to* $\mathbf{S}^3$.

More generally, the Geometrization Conjecture would imply that every closed, orientable, aspherical 3-manifold is determined, up to homeomorphism, by its fundamental group. This is a special case of the so-called Borel conjecture and will be discussed further in Section 4.4.

The groups which are fundamental groups of compact surfaces are known. The Poincaré-Koebe Uniformization Theorem shows that the fundamental group of a surface acts isometrically on the round sphere $\mathbf{S}^2$, the Euclidean plane $\mathbf{E}^2$ or the hyperbolic plane $\mathbf{H}^2$. This geometric action is reflected in algebraic properties of the group. For instance, it provides solutions of the word problem and the conjugacy problem. By contrast, any finitely presented group is the fundamental group of some compact 4-manifold.

Characterizing algebraically the class of fundamental groups of compact 3-manifolds is still an open problem. If M is a compact orientable 3-manifold satisfying the conclusion of the Geometrization Conjecture, then $\pi_1 M$ is the fundamental group of a graph of groups whose vertices are discrete subgroups of isometries of the 3-dimensional geometries above, and edges are trivial or isomorphic to $\mathbf{Z}^2$. One can deduce from this the solvability of the word and the conjugacy problems for these groups, see [**61, 168**]. In general these two questions are still unsolved for the fundamental group of a compact 3-manifold.

The topological background for Thurston's Geometrization Conjecture is given by a splitting of the compact, orientable 3-manifold along a finite collection of disjoint essential spheres and tori into canonical pieces. The existence of this decomposition is a central result in the study of 3-manifolds, which is presented in a more general context in Chapter 3.

An orientable 3-manifold M is *irreducible* if any embedding of the 2-sphere into M extends to an embedding of the 3-ball into M. This notion is crucial for the study of topological properties of 3-manifolds. The *connected sum* of two orientable 3-manifolds is the orientable 3-manifold obtained by pulling out the interior of a 3-ball in each manifold and gluing the remaining parts together along the boundary spheres.

The first stage of the decomposition, due to H. Kneser [**115**] and J. Milnor [**143**], expresses any compact, orientable 3-manifold M as the connected sum of 3-manifolds that are either homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^2$ or irreducible. Moreover, the connected summands are unique up to order and orientation-preserving homeomorphism.