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RATIONAL REPRESENTATIONS, THE STEENROD ALGEBRA AND FUNCTOR HOMOLOGY

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***Abstract.* —** The book presents aspects of homological algebra in functor categories, with emphasis on polynomial functors between vector spaces over a finite field. With these foundations in place, the book presents applications to representation theory, algebraic topology and K-theory. As these applications reveal, functor categories offer powerful computational techniques and theoretical insights.

T. Pirashvili sets the stage with a discussion of foundations. E. Friedlander then presents applications to the rational representations of general linear groups. L. Schwartz emphasizes the relation of functor categories to the Steenrod algebra. Finally, V. Franjou and T. Pirashivili present A. Scorichenko's understanding of the stable K-theory of rings as functor homology.

***Résumé* (Représentations rationnelles, algèbre de Steenrod et homologie des foncteurs)**

Ce livre traite d'algèbre homologique dans les catégories de foncteurs, avec une attention particulière pour les foncteurs polynomiaux entre espaces vectoriels sur un corps fini. Il en présente des applications dans trois domaines des mathématiques : la théorie des représentations, la topologie algébrique et la K-théorie. À chacune de ces applications, les catégories de foncteurs apportent des avancées théoriques et des outils de calcul puissants.

D'abord, T. Pirashvili expose les bases de la théorie. E. Friedlander l'applique alors aux représentations rationnelles des groupes linéaires. L. Schwartz établit les relations de l'algèbre de Steenrod avec les catégories de foncteurs. Enfin, V. Franjou et T. Pirashvili présentent un théorème de Scorichenko : la K-théorie stable est l'homologie des foncteurs.

CONTENTS

ABSTRACTS

The aim of these notes is to provide the reader with the computational tools of functor homology. We do not assume any prior knowledge in the subject. Therefore we recall not only the basic notions on functors, but also Koszul and de Rham complexes, Cartier's homomorphism etc. We introduce two versions of polynomial functors over finite fields and explain the computation of Ext-groups between the identity functor and (twisted) symmetric powers in both categories, due to Franjou-Lannes-Schwartz and Friedlander-Suslin repectively.

We provide an introduction to the cohomology of finite group schemes, a class of objects which includes finite groups and p-restricted Lie algebras. Various qualitative results, known earlier for finite groups by work of Quillen and others, are extended to this general context. Various computational techniques which arise from classical homological algebra are recalled. We then proceed to discuss the essential role of strict polynomial functors in the proof of the fundamental theorem which asserts that the cohomology of a finite group scheme is finitely generated.

This text gives an introduction to the algebraic properties of the Steenrod algebra and of the category of unstable modules. The link with the category of polynomial functors is described.

One recalls in this note the construction of the Steenrod algebra in homotopy theory as algebra of natural stable transformations. One gives its natural properties.

For many rings R, the homology with coefficients of the infinite general linear group $\mathrm{GL}(R)$ is the tensor product of its homology with trivial coefficients with another term, which has been identified as the stable K-theory of the ring. Scorichenko's theorem states that stable K-theory is functor homology.

RÉSUMÉS DES ARTICLES

Ces notes se proposent de donner au lecteur des outils pour le calcul de l'homologie des foncteurs. Comme on ne lui suppose aucune connaissance préalable du sujet, on rappelle les notions de base, ainsi que les complexes de Koszul et de de Rham, l'isomorphisme de Cartier, etc. Nous introduisons deux notions de foncteur polynomial sur un corps fini, et nous expliquons pour chacune le calcul des groupes d'extensions entre le foncteur identité et les puissances symétriques, dû respectivement à Franjou-Lannes-Schwartz et Friedlander-Suslin.

Ce texte est une introduction à la cohomologie des schémas en groupes finis. Cette classe d'objets contient les groupes finis et les algèbres de Lie restreintes. Plusieurs résultats qualitatifs, établis pour les groupes finis par Quillen et d'autres, leurs sont généralisés. On rappelle les méthodes de calcul de l'algèbre homologique, puis on explique l'intervention déterminante des foncteurs polynomiaux stricts dans la démonstration qui établit que la cohomologie d'un schéma en groupes fini est de type fini.

Ce texte donne une introduction aux propriétés algébriques de l'algèbre de Steenrod et de la catégorie des modules instables. Les relations avec la catégorie des foncteurs polynomiaux sont établies.

On rappelle dans cette note la construction de l'algèbre de Steenrod en théorie de l'homotopie comme algèbre d'opérations naturelles stables. On donne ses principales propriétés.

Pour beaucoup d'anneaux R, l'homologie du groupe linéaire infini avec coefficients s'obtient en effectuant le produit tensoriel de son homologie avec coefficients triviaux par un autre terme, qui n'est autre que la K-théorie stable de l'anneau. Le théorème de Scorichenko exprime la K-théorie stable comme homologie des foncteurs.

INTRODUCTION

by

Vincent Franjou

This book is a sequel to a series of lectures given in Nantes, December 12–15, 2001, for Société Mathématique de France's "État de la recherche" session. The lectures presented an overview of a few recent applications of polynomial functors to homotopy and representation theory.

The organizers' interest in polynomial functors stems from a purely topological problem: Sullivan's fixed points conjecture. The proof of the conjecture in the mid 1980's [**10, 9**] required a detailed study of the category of unstable modules over the Steenrod algebra. It revealed [**8**] that this category is closely related to the category of polynomial functors between vector spaces over a finite field. This opened the way for a productive interaction between representation theory in positive characteristic and homotopy theory. This was a timely development, because Green's 1980 book [**7**] had just shown that many ideas developed by Schur for complex numbers [**14**] extend in any characteristic. Nevertheless, functors were not yet used as such: topologists would use the Steenrod algebra, representation theorists would use Green's Schur algebras, and both would use functors only as a conceptual framework.

In the 1990's, techniques developed in the category of polynomial functors allowed elegant cohomological calculations for finite fields [**3**]. These techniques apply to a wide scope of situations where functors appear. They were used by Suslin and Friedlander to prove [**6**] the finite generation of the cohomology of finite group schemes. This in turn allowed new and impressive homological calculations to be carried out [**2**], resulting in new insights into the cohomology of the general linear groups GL_n in positive characteristic, and its relations to the cohomology of the finite groups $\mathrm{GL}_n(\mathbb{F}_q)$. Finally, extending functors techniques to the (twice as good) bifunctors allowed Scorichenko to elegantly describe [**17**] Waldhausen's Stable K-theory.

This book grew out of the success of the Nantes meeting. Its purpose is not to present a detailed account of fifteen years of research, especially in a topic which claims to cross the barriers of specialization. Its purpose is to give the reader a glimpse of a few aspects of polynomial functors, and encourage her to complete her knowledge through the bibliography.

Before we embark on presenting the different chapters, we would like to give the reader a concrete feeling for the kind of objects this book deals with. Through examining an easy example of representation of the general linear group, the next few pages present a modern reading of Schur's ideas, and how they lead to functors. On the way, we explain that the same representation is often meant to be quite different objects, a fact which has been a source of confusion for many of us in the past. And because most of this book deals with polynomials over finite fields, we included a section on those, again in the hope that we can spare the reader unnecessary worries.

Representations, polynomial representations and functors: an example

Let k be a field, and let us consider homogeneous polynomials in two variables x and y. Let the group $\mathrm{GL}_2(k)$ act on the left on $k[x,y]$ by linear substitution. Precisely, we let a 2×2 matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ act on x to $ax+by$ and on y to $cx+dy$. This defines an action on the two-dimensional k-vector space of homogeneous polynomials of degree 1. Let us call V this vector space. Let us denote by S^2V the three-dimensional k-vector space of homogeneous polynomials of degree two. The action of the same matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, seen in the monomial basis (x^2, y^2, xy) of S^2V, is given by the matrix:

$$\begin{pmatrix} a^2 & c^2 & ac \\ b^2 & d^2 & bd \\ 2ab & 2cd & ad+bc \end{pmatrix}. \tag{1}$$

This formula shows us that the action extends immediately to all (even singular) matrices. This is because the coefficients above are polynomials, and no division occurs. Note as well that the coefficients can be seen as formal polynomials, that is: elements in the polynomial algebra $k[a,b,c,d]$, where the matrix coordinates a, b, c, d are now indeterminates. The representation S^2V is then called a polynomial representation of the general linear group GL_2 over k. Because all the coefficients are homogeneous polynomials of degree two, one says that S^2V itself is homogeneous of degree 2. This is the point of view explained in Friedlander's lectures.

Even such a simple example might discourage the reader to look in S^2V for an invariant vector, or proper invariant subspace. Rightly so, for S^2V is indeed a simple representation of $\mathrm{GL}_2(k)$, at least when the characteristic of k is 0. But things are different when the characteristic of the field k is 2: the subspace generated in S^2V by x^2 and y^2 is invariant by linear substitution (so the representation S^2V is not simple). This subspace looks very much like V itself, except there is a square taken whenever there is a chance. We denote the subspace spanned by x^2 and y^2 by $V^{(1)}$, and call it the Frobenius twist on V. The squaring map $V \to S^2(V)$ is additive but not k-linear (except when the characteristic two field is the prime field $\mathbb{F}_2$), and the k-linear map $V^{(1)} \to S^2(V)$ fixes this problem. It is called the Frobenius map.

The quotient of S^2V by this subspace is easily identified through the last diagonal term of the matrix (1): it is the determinant representation, and we get a short exact sequence of equivariant maps:

$$0 \longrightarrow V^{(1)} \longrightarrow S^2V \longrightarrow \det \longrightarrow 0.$$

The next question is whether S^2V splits as a direct sum of $V^{(1)}$ and of the determinant.

Let us look briefly at the case when k is the prime field $\mathbb{F}_2$. In this case, the invariant polynomial $x^2 + y^2 + xy$ provides a section which splits the determinant representation off S^2V. However this is exceptional, and the general situation is that S^2V does not split:

(i) First, the splitting does not extend when one lets the singular matrices act as well.

(ii) Second, there is no such splitting when the field k is larger (*e.g.* $k = \mathbb{F}_4$).

(iii) Third, there is no such splitting of S^2V as a polynomial representation of GL_2.

(iv) Fourth, there is no such splitting for homogeneous polynomials of degree 2 in more than two variables.

In order to explain what is meant in the fourth statement, we need to explain how we allow V to vary. Let V now denote any vector space over k. We view V, as above, as the k-vector space of homogeneous polynomial of degree one in $\dim(V)$ variables. The left action of $\mathrm{GL}(V)$ on V so corresponds to the above linear substitution action (the indeterminates x, y can be seen as coordinates for a chosen basis of V). Homogeneous polynomials of degree d form the d-th symmetric power $S^d(V)$. Again, the simplest case is for $d = 1$, and $S^1(V) = V$. The next step is easy: describe the d-th symmetric power of a k-linear map. The correspondance thus obtained:

$$\mathrm{Hom}_k(V, W) \longrightarrow \mathrm{Hom}_k(S^d(V), S^d(W))$$

is compatible with composition of maps. This is the property that makes S^d a functor. Because S^d is a covariant functor of V, endomorphisms of V act on the left on $S^d(V)$. In this way, the action on $S^d(V)$ is described in a very concise way, and in a way that makes it independent of the choice of a basis in V. In doing so, we gain control on the changes we may like to perform on $S^d(V)$: change of V primarily, or change of fields.

Let us go back to our example and express it in terms of functors. We are looking at the *functor* S^2 for a field of characteristic 2. This means that instead of studying the representation S^2V for a specific vector space V, we now look at the compatible family of the $\mathrm{End}(V)$-representations $S^2(V)$ for all finite dimensional vector spaces V. The quotient $S^2(V)/V^{(1)}$ defines the second exterior power $\Lambda^2(V)$, which equals the determinant representation when V is two-dimensional. All in all, we obtain a short exact sequence of natural transformations of functors:

$$0 \longrightarrow V^{(1)} \longrightarrow S^2(V) \longrightarrow \Lambda^2(V) \longrightarrow 0.$$

We retrieve the above example when V is two-dimensional.

This raises new questions, for example as when such a short exact sequence of natural transformations admits a natural splitting. We have already said that this