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CHAOS FOR CONVOLUTION OPERATORS ON THE SPACE OF ENTIRE FUNCTIONS OF INFINITELY MANY COMPLEX VARIABLES

by Blas M. Caraballo & Vinícius V. Fávaro

Abstract. — In sharp contrast to a classical result of Godefroy and Shapiro, Mujica and the second author showed that no translation operator on the space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ of entire functions of infinitely many complex variables is hypercyclic. In an attempt to better understand the dynamics of such operators, in this work we show, firstly, that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is cyclic or n-supercyclic for any positive integer n. In the opposite direction, we show that every non-trivial convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is mixing. Particularizing Arai's concept of Li-Yorke chaos to non-metrizable topological vector spaces, we show that non-trivial convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ are also Li-Yorke chaotic.

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Résumé (*Chaos pour les opérateurs de convolution sur l'espace des fonctions entières en une infinité de variables complexes*). — Contrastant fortement avec un résultat classique de Godefroy et Shapiro, Mujica et le deuxième auteur ont montré qu'aucun opérateur de translation sur l'espace $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ des fonctions entières en une infinité de variables complexes est hyper cyclique. Pour mieux comprendre la dynamique de tels opérateurs, dans ce travail, nous montrons premièrement qu'aucun opérateur de convolution sur $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ n'est cyclique ni n-supercyclique, quelque que soit l'entier positif n. Dans le sens opposé, nous montrons que tous les opérateurs de convolution non triviaux sur $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ sont mélangeant. En appliquant le concept, défini par Arai, de chaos de Li-Yorke sur des espaces vectoriels topologiques non métrisables, nous montrons que les opérateurs de convolution non triviaux sur $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ sont également Li-Yorke chaotiques.

Dedicated to the memory of Professor Jorge Mujica (1946–2017)

1. Introduction

In the last 30 years, the study of the dynamics of continuous linear operators (in short, operators) on topological vector spaces has been intensively explored. References [6, 25] provide deep and detailed surveys of the theory. In this paper, we are mainly interested in the linear dynamics of convolution operators on spaces of entire functions of infinitely many complex variables. We remark that several results about the linear dynamics of (not necessarily convolution) operators on spaces of entire functions of infinitely many complex variables have appeared in the last few decades, see, for instance, [3, 7, 9, 10, 11, 15, 18, 19, 20, 22, 24, 29, 31].

A classical result due to Godefroy and Shapiro [23] states that every nontrivial convolution operator on the space $\mathcal{H}(\mathbb{C}^n)$ of entire functions of several complex variables is hypercyclic (the definition will be given in Section 2). Moreover, Bonilla and Grosse-Erdmann [13] showed that these convolution operators are even frequently hypercyclic, which is a stronger notion than hypercyclicity. In sharp contrast with these results, Mujica and the second author [18] proved that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ can be hypercyclic. At first sight this result seems surprising, since it is well known that every $f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ depends only on finitely many variables (see [17, p. 162]). Based on these facts, the following question arises:

> Which other dynamical concepts are satisfied by convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$?

The purpose of this paper is to answer this question, either positively or negatively, for the following concepts (the definitions will follow in Section 2): n-supercyclicity, cyclicity, Li-Yorke chaos (notions that are weaker than hypercyclicity), and mixing. In contrast with the aforementioned result of Godefroy and Shapiro, we will show that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ can be either cyclic or n-supercyclic for any positive integer n (Theorem 3.1), but every non-trivial convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is mixing (Theorem 3.3).

Recall that the Birkhoff transitivity theorem gives the equivalence between hypercyclicity and topological transitivity for operators on Fréchet spaces. In particular, mixing implies hypercyclicity in this case. The Baire category theorem (the metrizability and completeness of the space) is the key to proving the Birkhoff transitivity theorem. For operators on a Fréchet space E, it is well known that hypercyclicity does not imply mixing. So, it is natural to ask if mixing implies hypercyclicity when E is not a Fréchet space. In [11, p. 254, Example 1], Bonet constructed the following example, which is mixing but not hypercyclic. Let $\mathcal{T} = \lambda B, \lambda > 1$, where B is the backward shift on ℓ_2. The operator $\mathcal{T}$ is mixing (hence, hypercyclic), and it has a dense set P of periodic points. Let e_i be the i-th canonical unit vector of ℓ_2. We denote by $\mathcal{E}$ the dense topological subspace of ℓ_2, which is the linear span of $\{e_i : i \in \mathbb{N}\} \cup P$. Clearly, $\mathcal{T}$ acts continuously from $\mathcal{E}$ into itself, it has a dense set of periodic points on $\mathcal{E}$, and it is mixing. Since every element of $\mathcal{E}$ has a finite orbit, $\mathcal{T}$ cannot be hypercyclic.

Note that the space in Bonet's example is metrizable (in fact, it is normed) but not complete. So, it is natural to ask if, in a non-metrizable complete, separable locally convex space, every mixing operator is hypercyclic. The answer is no, and the same example given by Grosse-Erdmann to answer [11, Open question 13.(2)] is a mixing non-hypercyclic operator. The details of this example will appear in Example 3.2. New examples of mixing non-hypercyclic operators on a non-metrizable complete, separable locally convex space will be provided by Theorem 3.3.

The notion of chaos in linear dynamics was introduced by Godefroy and Shapiro [23] in 1991. They adopted Devaney's definition of chaos. Recall that an operator on a Fréchet space is *chaotic* if it is hypercyclic and it has a dense set of periodic points. The notion of chaos for operators on an arbitrary topological vector space was given by Bonet [11]. He adopted Devaney's definition of chaos replacing the condition "hypercyclicity" with "topological transitivity", but both concepts coincide in Fréchet spaces (note that the Bonet and Grosse-Erdamnn examples above are chaotic). In addition to these notions explored in linear dynamics, we mention the first mathematical definition of chaos given in 1975 by Li and Yorke in [28], which is currently known as Li-Yorke chaos. The classical notion of Li-Yorke chaos was introduced for continuous maps defined on metric spaces. By [8, Theorem 9], hypercyclic operators on Fréchet spaces are Li-Yorke chaotic.

Since $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is a complete non-metrizable locally convex space, the classical notion of Li-Yorke chaos does not make sense for convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$. Recently, Arai [2] introduced the notion of Li-Yorke chaos for an action of a group on a uniform space. Since every topological vector space is a uniform space, we will adopt Arai's definition of Li-Yorke chaos (the definition is given in Section 2). Using this definition we will prove that every hypercyclic operator on a topological vector space is Li-Yorke chaotic (Corollary 3.5), and every non-trivial convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is Li-Yorke chaotic (Theorem 3.6). We will also observe in Remark 3.8 that Grosse-Erdmann's example is Li-Yorke chaotic, whereas Bonet's example is not.

It is worth mentioning that the criteria that appear in the literature to prove that an operator does or does not satisfy some notion of linear dynamics are, in general, for operators defined on F-spaces. Since $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is not a metric space, the known criteria are not useful to prove Theorems 3.1 and 3.3. However, to show Theorem 3.6, we will adapt a criterion obtained by Bernardes *et al* [8] for operators on Fréchet spaces to operators on Hausdorff topological vector spaces. This criterion is the key of the proof.

The following diagram summarizes the relation between the notions of linear dynamics raised in this paper for operators on an arbitrary topological vector space:

$$\begin{array}{ccccccc} \text{Li-Yorke chaos} & \Longleftarrow & \text{hypercyclicity} & \Longrightarrow & \text{supercyclicity} & \Longrightarrow & \text{cyclicity} \\ & & \Downarrow & & \Downarrow & & \\ \text{mixing} & \Longrightarrow & \text{topological transitivity} & & n\text{-supercyclicity} & & \end{array}$$

2. Preliminaries

Let V be a subset of a Hausdorff topological complex vector space E and let $T\colon E \to E$ be a continuous linear operator (from now on, we just write operator). The *orbit of V under T*, denoted by $\mathrm{orb}_T(V)$, is the subset of E given by

$$\mathrm{orb}_T(V) = \bigcup_{k=0}^{\infty} T^k(V).$$

If $V = \{x\}$ is a singleton, and $\mathrm{orb}_T(V) = \{T^k x : k \in \mathbb{N}_0\}$ is dense in E, where $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, then T is said to be *hypercyclic* and x a *hypercyclic vector* for T. If the linear span of $\mathrm{orb}_T(V)$ is dense in E, then T is said to be *cyclic* and x a *cyclic vector* for T. If $V = \mathrm{span}\{x\}$ and $\mathrm{orb}_T(V) = \mathbb{C} \cdot \{T^k x : k \in \mathbb{N}_0\}$ is dense in E, then T is said to be *supercyclic* and x a *supercyclic vector* for T. If V is a vector subspace of dimension n and $\mathrm{orb}_T(V)$ is dense in E, then T is said to be *n-supercyclic* and V a *supercyclic subspace* for T. Note that the notions of 1-supercyclicity and supercyclicity are equivalent. Also, an n-supercyclic operator, for $n = 2, 3, \ldots$, need not be cyclic (for an infinite dimensional example, see [14]). Hilden and Wallen [27] proved that no operator on $\mathbb{C}^n$, $n = 2, 3 \ldots$, can be supercyclic. So, n-supercyclicity does not imply supercyclicity in general. For properties and results about supercyclicity and n-supercyclicity, we refer the reader to [14, 21, 26, 27]. We say that T is *mixing* (respectively *topologically transitive*) if for any two non-empty open sets $U_1, U_2 \subset E$, there is $n_0 \in \mathbb{N}$ such that $T^n(U_1) \cap U_2 \neq \emptyset$, for all $n \geq n_0$ (respectively, $T^{n_0}(U_1) \cap U_2 \neq \emptyset$).

As was stated in the Introduction, the classical notion of Li-Yorke chaos was introduced for maps defined on metric spaces. The definition is the following: given a metric space (M, d) and a continuous map $f\colon M \to M$, we recall that a pair $(x, y) \in M \times M$ is called a *Li-Yorke pair* for f if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

A *scrambled set* for f is a subset S of M such that (x, y) is a Li–Yorke pair for f whenever x and y are distinct points in S. The map f is said to be *Li–Yorke chaotic* if there exists an uncountable scrambled set for f.

As we also mentioned in the Introduction, Arai [2] introduced the notion of Li-Yorke chaos for an action of a group on a uniform space. Adopting Arai's definition of Li-Yorke chaos in the particular case for operators on a Hausdorff topological vector space E, we have the following:

- A pair $(x, y) \in E \times E$ is said to be *asymptotic* for T if for any neighborhood of zero U, there exists $k \in \mathbb{N}$ such that $T^n(x - y) \in U$ for every $n \geq k$, that is, if $T^n(x - y) \to 0$. A pair $(x, y) \in E \times E$ is said to be *proximal* for T if for any neighborhood of zero U, there exists $n \in \mathbb{N}$ such that $T^n(x - y) \in U$, that is, if the sequence $\{T^n(x - y)\}_{n=1}^{\infty}$ has a subnet converging to zero.
- A pair $(x, y) \in E \times E$ is said to be a *Li-Yorke pair* for T if it is proximal but not asymptotic. In other words, (x, y) is a Li-Yorke pair for T if and only if the sequence $\{T^n(x - y)\}_{n=1}^{\infty}$ does not converge to zero but has a subnet converging to zero.

Using this definition of Li-Yorke pair, *scrambled set* and *Li-Yorke chaos* for operators on a Hausdorff topological vector space are defined as in the case of maps on a metric space.

It is easy to check that if E is metrizable, and we consider a translation-invariant metric (this metric exists by definition of metrizability), then both definitions of Li-Yorke chaos coincide.

2.1. Convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$. — In this section, we prove some technical results about convolution operators that we need to show the main results of this work. First, we present some preliminary results about the space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$.

Given the topological product $\mathbb{C}^{\mathbb{N}} = \prod_{n=1}^{\infty} \mathbb{C}$, we consider the complex vector space of all entire functions $f\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}$, which is denoted by $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$. It is well known that there are only two usual locally convex topologies on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$: the compact open topology τ_0 and its bornological associated topology τ_δ (see