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A HENSTOCK-KURZWEIL TYPE INTEGRAL ON ONE-DIMENSIONAL INTEGRAL CURRENTS

by Antoine Julia

Abstract. — We define a non-absolutely convergent integration method on integral currents of dimension 1 in Euclidean space. This integral is closely related to the Henstock-Kurzweil and Pfeffer integrals. Using it, we prove a generalized fundamental theorem of calculus on these currents. A detailed presentation of Henstock-Kurzweil integration is given in order to make the paper accessible to non-specialists.

Résumé (*Une intégrale à la Henstock-Kurzweil sur les courants entiers de dimension* 1). — On définit une intégrale non absolument convergente sur les courants entiers euclidiens de dimension 1. Cette intégrale est inspirée des intégrales de Henstock-Kurzweil et de Pfeffer. Dans ce contexte, on démontre un théorème fondamental généralisé sur ces courants. On donne aussi une présentation détaillée de l'intégrale de Henstock et Kurzweil, pour les non-spécialistes.

1. Introduction

The goal of this paper is to present an integration method for functions defined on the support of an integral current of dimension 1 in Euclidean space. This method is inspired from the Henstock-Kurzweil (HK) and Pfeffer integrals

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[23, 20, 32], and, like them, tailored to the study of the fundamental theorem of calculus. The HK integral is a variant of the Riemann integral; it is more general than the Lebesgue integral, in the sense that all Lebesgue integrable functions are HK integrable, but *non-absolutely convergent*; there exist functions that are HK integrable, while their absolute value is not; in the same way that the series $\sum_k (-1)^k k^{-1}$ converges, while $\sum_k k^{-1}$ does not. A classical example of such functions is the derivative of $x \mapsto x^2 \sin(x^{-2})$ for $x \in (0, 1\,]$.

An integral that solves this problem was defined in two different ways by A. Denjoy and O. Perron in the early twentieth century, see [36]. R. Henstock and J. Kurzweil independently found a simpler, equivalent definition of this integral; we will use the HK formalism, but refer the reader to [18] for a comparison of these three approaches. For functions defined on a bounded interval $[\,a, b\,]$, the fundamental theorem of calculus of the integral of HK is the following:

Theorem A ([33, Theorem 6.1.2]). — *Let $f : [\,a, b\,] \to \mathbb{R}$ be a continuous function that is differentiable everywhere, then its derivative f' is HK integrable on $[\,a, b\,]$ and there holds:*

$$(HK) \int_a^b f' = f(b) - f(a). \tag{1}$$

Note that several other integration methods have been defined for which Theorem A holds, see, in particular, [2, 3, 7] and a “minimal” theory in [6]. The (Denjoy-Perron-)Henstock-Kurzweil integral is now well understood and its integrable functions and their primitives have been completely characterized [8]. It is also interesting to note that a small variation in the definition of the HK integral yields the McShane integral [28], which is equivalent to the Lebesgue integral. Lastly, although the present paper focuses on scalar valued functions, we mention that HK-like integration of Banach space valued functions has raised considerable interest [15, 37, 5], in particular with an application to the Cauchy problem in [10].

The Riemann-like formulation of the HK integral makes it straightforward to allow for singularities in the above theorem; if f is only differentiable at all but countably many points of $[\,a, b\,]$, the result still holds. This statement is, in some sense, optimal. Indeed, as shown by Z. Zahorsky in [41], the set of non-differentiability points of a continuous function is a countable union of G_δ sets. In particular, if it is uncountable, it must contain a Cantor subset by [30, Lemma 5.1], and to any Cantor subset of an interval having zero Lebesgue measure, one can associate a “Devil’s Staircase,” which has derivative equal to 0 outside of the set and is non-constant.

However, the differentiability condition can be relaxed and replaced by a pointwise Lipschitz condition. Thus, a more general statement is

THEOREM B ([33, Theorem 6.6.9]). — *Let $f : [a, b] \to \mathbb{R}$ be a continuous function that is pointwise Lipschitz at all but countably many points, then, it is differentiable almost everywhere in $[a, b]$, its derivative f' is HK integrable on $[a, b]$, and identity* (1) *holds.*

Natural extensions of the fundamental theorem of calculus include the Gauss-Green (or divergence) theorem and Stokes' theorem. For the former in bounded sets of finite perimeter an integral was developed by W.F. Pfeffer in [32], following, in particular, J. Mawhin [26] and J. Mařìk [25]. The results naturally extend to Stokes' theorem on smooth oriented manifolds. For singular varieties, an integral adapted to Stokes' theorem was defined by the author on certain types of integral currents in Euclidean spaces [21, 22]. Let us also mention works on integration on more fractal objects with different methods [40, 19, 42].

The present paper corresponds to the second chapter of the author's thesis [21], which focuses on one-dimensional integral currents. These are treated separately from the higher-dimensional ones, as they can be decomposed into a countable family of curves. We, thus, define an integral closer to the Henstock-Kurzweil one, which we call the $\mathcal{R}_1$ integral.

Given an integral current T of dimension 1 in $\mathbb{R}^n$, define $\operatorname{Indec}(T)$ to be the subset of $\operatorname{spt} T$ containing the points in the support of an indecomposable piece of T (see Section 3 for the notations on currents). Denote by $\|T\|$ the carrying measure of T, by $\overrightarrow{T}$ its tangent vector field and by $\operatorname{spt} T$ its support. The main result of this paper is the following:

THEOREM 1.1 (Fundamental theorem of calculus for 1-currents). — *Let T be a fixed integral current of dimension 1 in $\mathbb{R}^n$ and u be a continuous function on $\operatorname{spt} T$. Suppose that u is pointwise Lipschitz at all but countably many points in $\operatorname{Indec}(T)$ and that u is differentiable $\|T\|$ almost everywhere, then $x \mapsto \langle \mathrm{D}u(x), \overrightarrow{T}(x) \rangle$ is $\mathcal{R}_1$ integrable on T and*

$$(\partial T)(u) = (\mathcal{R}_1) \int_T \langle \mathrm{D}u, \overrightarrow{T} \rangle.$$

This theorem is equivalent to Theorem B when T represents a bounded interval.

Summary of the paper. — In Section 2, we define the integral of Henstock and Kurzweil and its main properties along with schemes of proofs of the main theorems. We also give an equivalent definition of integrability — inspired from the Pfeffer integral — which will be useful in the sequel. It is important to note that the Pfeffer integral is not equivalent to the HK integral.

In Section 3, we recall the definition of integral currents of dimension 1 in Euclidean spaces and define the main ingredients of $\mathcal{R}_1$ integration: pieces of a current and functions on the space of pieces of a current; we also study the derivation of these functions, following H. Federer [16, Section 2.9] and W.F. Pfeffer [33, Section 9.3] and [34]. Section 4 contains the definition of $\mathcal{R}_1$ integration and the proof of its main properties, as well as the proof of Theorem 1.1.

Possible generalizations. — First, one can ask if u could be allowed to be discontinuous outside of the set of positive lower-density points of $\|T\|$, yet remain bounded. Proposition 3.5 and Example 2.10 show that this is not straightforward.

A natural question would be whether Theorem 1.1 could be generalized to normal currents in Euclidean spaces. Indeed, by a Theorem of S.K. Smirnov [38], normal currents of dimension 1 also admit a decomposition into Lipschitz curves. More precisely, given a current T of dimension 1, with finite mass and finite boundary mass in $\mathbb{R}^n$, there exists a finite measure μ on the space of finite length Lipschitz curves in $\mathbb{R}^n$ such that

$$T = \int [\![\gamma]\!] \,\mathrm{d}\mu(\gamma), \tag{2}$$

where $[\![\gamma]\!]$ is the integral current of dimension 1 associated to the Lipschitz curve γ with multiplicity 1 and orientation given by the parameterization. However, there is no a-priori constraint on the measure μ; it can be somewhat diffuse, as the carrying measure of a normal current can be absolutely continuous with respect to the Lebesgue measure. It is, therefore, impossible to work with countable sums of pieces, and one would probably need another notion of piece of a normal current to define suitable Riemann sums. Recall that Fubini-type arguments do not work well with non-absolutely convergent integrals, as shown in [33, Section 11.1]. Note also that the space of curves, on which we would have to integrate is far from Euclidean.

Another natural idea would be to consider integral currents of dimension 1 in Banach spaces or complete metric spaces, following [1] or [13].

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Finally, I wish to mention that the notions of pieces and subcurrents studied in this paper and in [21, 22] are very close to the subcurrents defined by E. Paolini and E. Stepanov in [31] for normal currents in metric spaces.

2. The integral of Kurzweil and Henstock

Here, we give a short presentation of the Henstock-Kurzweil integral. The proofs of these results can be found in any treaty on the subject, such as [27, 33, 18, 17]. Let us also mention the very detailed recent book [29] — in French, and Appendix H to [11].

2.1. Definition and classical properties. — A non-negative function defined on a set $E \subseteq \mathbb{R}$ is called a **gauge** if its zero set is countable. In the classical definition of the Henstock-Kurzweil integral, gauges are always positive, but for our purposes it makes sense to allow the gauge to take on the value zero in a countable set. A **tagged family** in an interval $[\,a, b\,]$ is a finite collection of pairs $([\,a_j, b_j\,], x_j)_{j=1,2,\ldots,p}$, where one has $a \le a_1 < b_1 \le a_2 < \cdots \le a_p < b_p \le b$ and for all j, $x_j \in [\,a_{j-1}, a_j\,]$. The **body** of a family $\mathcal{P}$ is the union denoted by $[\mathcal{P}]$ of all the intervals in $\mathcal{P}$. A **tagged partition** in $[\,a, b\,]$ is a tagged family whose body is $[\,a, b\,]$. If δ is a gauge on $[\,a, b\,]$, we say that a tagged family (or a tagged partition) is δ-fine, when for all j, $b_j - a_j < \delta(x_j)$. In particular, it holds that $\delta(x_j) > 0$, for all j.

Definition 2.1. — A function f defined on a compact interval $[\,a, b\,]$ is **Henstock-Kurzweil integrable on** $[\,a, b\,]$ if there exists a real number α, such that for all $\epsilon > 0$, there exists a positive gauge δ on $[\,a, b\,]$, such that for each δ-fine tagged partition $\mathcal{P} = \{([\,a_{j-1}, a_j\,], x_j)\}_{j=1,\ldots,p}$, it holds that:

$$\left| \sum_{j=1}^{p} f(x_j)(a_j - a_{j-1}) - \alpha \right| < \epsilon.$$

In the following, we will write $\sigma(f, \mathcal{P})$ for the sum on the left-hand side, whenever $\mathcal{P}$ is a tagged family. If α exists as above, we denote it by $(HK)\int_a^b f$. This definition is well posed as a consequence of the following key result.

Lemma 2.2 (Cousin's lemma [18, Lemma 9.2]). — *If I is a closed bounded interval, and δ is a positive gauge on I, then a δ-fine tagged partition of I exists.*

To characterize integrability, the following proposition is useful:

Proposition 2.3 (Cauchy criterion for integrability [18, Theorem 9.7]). — *A function f is HK integrable on the interval $[\,a, b\,]$, if and only if for each $\epsilon > 0$ there exists a positive gauge δ on $[\,a, b\,]$, such that whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are δ-fine tagged partitions of $[\,a, b\,]$, it holds that*

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \epsilon.$$

Let us also list some fundamental properties of HK integrable functions.