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ON THE EQUIVARIANT BLOW-NASH CLASSIFICATION OF SIMPLE INVARIANT NASH GERMS

BY FABIEN PRIZIAC

ABSTRACT. — We make progress towards the classification of simple Nash germs invariant under the involution changing the sign of the first coordinate, with respect to equivariant blow-Nash equivalence, which is an equivariant Nash version of blow-analytic equivalence, taking advantage of invariants for this relation, the equivariant zeta functions.

RÉSUMÉ (*Sur l'équivalence Nash après éclatements équivariante des germes Nash invariants simples*). — On effectue des avancées en direction de la classification des germes Nash simples invariants sous l'involution changeant le signe de la première coordonnée, par rapport à l'équivalence Nash après éclatements équivariante, qui est une version Nash équivariante de l'équivalence analytique après éclatements, en tirant parti d'invariants pour cette relation, les fonctions zêta équivariantes.

1. Introduction

The classification of real analytic germs requires us to carefully choose the equivalence relation used. One may think about the (right) C^1-equivalence. However, this is too strong, as illustrated by the example of the Whitney

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family $f_t(x,y) = xy(y-x)(y-tx)$, $t > 1$ (f_t and $f_{t'}$ are C^1-equivalent if and only if $t = t'$), while the topological equivalence is too rough. In [19], T.-C. Kuo suggested an equivalence relation for which the Whitney family has only one equivalence class: the blow-analytic equivalence. More generally, any analytically parameterized family of isolated singularities has a locally finite classification with respect to blow-analytic equivalence.

Two real analytic germs are said to be blow-analytically equivalent if, roughly speaking, they become analytically equivalent after compositions with real modifications, e.g. compositions of blowings-up along smooth centers. From the definition of this equivalence relation, further studies on real analytic germs were stimulated. In particular, invariants have been constructed for blow-analytic equivalence, like the Fukui invariants ([15]), as well as the zeta functions of S. Koike and A. Parusiński ([18]), inspired by the motivic zeta functions of J. Denef and F. Loeser ([8]), using the Euler characteristic with compact supports as a motivic measure.

A refinement of blow-analytic equivalence was defined for Nash germs, that is, germs of real analytic functions with a semialgebraic graph, by G. Fichou in [10]: the blow-Nash equivalence, that is, the Nash equivalence after compositions with Nash modifications. The algebraicity involved allowed Fichou to use the virtual Poincaré polynomial ([22] and [9]), which is an additive and multiplicative invariant on $\mathcal{AS}$ sets ([20] and [21]) encoding more information than the Euler characteristic with compact supports, in order to define new zeta functions, invariant for the blow-Nash equivalence of Nash germs. Recently, in [6] J.-B. Campesato gave an equivalent alternative definition of blow-Nash equivalence as arc-analytic equivalence, proving that the blow-Nash equivalence of [10] is, indeed, an equivalence relation; he defined a more general invariant for it, the motivic local zeta function.

In [13], G. Fichou used his zeta functions of [10] to classify simple Nash germs (a germ is called simple if sufficiently small perturbations provide only finitely many analytic classes) with respect to blow-Nash equivalence. He showed that this classification actually coincides with the real analytic one, that is, the ADE-classification of [2]. An analog result for blow-analytic equivalence is not known.

In this paper, we are interested in real analytic germs invariant under right composition with the action of the group $G = \mathbb{Z}/2\mathbb{Z}$ only changing the sign of the first coordinate (which we will simply call invariant germs). In [24], we defined the equivariant blow-Nash equivalence for invariant Nash germs, which is, roughly speaking, an equivariant Nash equivalence after compositions with equivariant Nash modifications. Using the equivariant virtual Poincaré series ([14]), which is an additive invariant on G-$\mathcal{AS}$ sets, as a motivic measure, we constructed "equivariant" zeta functions, which are invariants for the equivariant blow-Nash equivalence.

Similarly to the nonequivariant frame, we ask if the equivariant blow-Nash classification of invariant Nash germs could coincide with the equivariant Nash classification for sufficiently "tame" invariant singularities. The equivariant analytic classification of simple invariant real analytic germs was established by V.I. Arnold in [1] (see also [16]). The representatives for this classification are the invariant singularities A_k, B_k, C_k, D_k, E_6, E_7, E_8 and F_4 (see Theorem 2.1 below). We will first show that a simple invariant Nash germ is G-blow-Nash equivalent (and even G-Nash equivalent) to one of these germs. The largest part of our study will then consist in trying to distinguish, with respect to G-blow-Nash equivalence, the invariant $ABCDEF$-singularities, notably using the equivariant zeta functions.

For some cases, we will be faced with either the equality of the respective equivariant zeta functions of a couple of germs or the fact that they are equal if and only if the respective equivariant virtual Poincaré series of specific sets are equal. The former situation is, in particular, due to the fact that the equivariant virtual Poincaré series can not distinguish two different algebraic actions on the same sphere as soon as there is at least one fixed point. As for the latter situation, we do not know if the invariance of the virtual Poincaré polynomial under bijection with the $\mathcal{AS}$ graph (see [23]) "generalizes" to an invariance of the equivariant virtual Poincaré series under equivariant bijection with the $\mathcal{AS}$ graph. If this is proven to be true, it should allow us to compute all the coefficients of the equivariant zeta functions considered.

The next section is devoted to the equivariant Nash classification of simple invariant Nash germs; we prove that it coincides with the equivariant real analytic classification of [1] and [16]. Indeed, two invariant Nash germs are equivariantly Nash equivalent if and only if they are equivariantly analytically equivalent (Proposition 2.3). This can be deduced from an equivariant Nash approximation theorem of E. Bierstone and P. Milman in [4].

In Section 3, we justify the fact that a germ G-Nash equivalent to a germ of the list $ABCDEF$ is, in particular, G-blow-Nash equivalent to it. On the other hand, one can notice that, forgetting the G-action, the invariant singularities A_k and B_k, or C_k and D_k, E_6 and F_4, are both A-, or D-, E-singularities. Since equivariant blow-Nash equivalence is a particular case of blow-Nash equivalence, and because the ADE-singularities are not blow-Nash equivalent to one another ([13]), we are reduced to comparing, with respect to G-blow-Nash equivalence, the invariant germs of the families A_k and B_k, or C_k and D_k, E_6 and F_4.

Section 4 recalls the definition of the tools we are going to use to do so: the equivariant zeta functions. Sections 6, 7 and 8 are devoted to the comparison of the invariant germs of a specific couple of families (A_k and B_k, C_k and D_k, and finally E_6 and F_4). We proceed as follows. We begin by computing the first coefficients of the equivariant zeta functions (that is, the coefficients of degree strictly smaller than the degree of the germs) in order to extract first cases of non-G-blow-Nash equivalence. Reducing our study to the remaining cases, we then compute the coefficient of degree equal to the degree of the germs.

Finally, for the cases for which it is not sufficient, we compare the coefficients of higher degrees of the respective equivariant zeta functions.

These comparisons lead to interesting examples of computations of equivariant virtual Poincaré series. The first one, to which Section 5 is devoted, is the computation of the equivariant virtual Poincaré series of the fibers over 0, -1 and $+1$ of the quadratic forms $Q_{p,q}(y) := \sum_{i=1}^{p} y_i^2 - \sum_{j=1}^{q} y_{p+j}^2$, equipped with four different actions of G.

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2. Equivariant Nash classification of invariant simple Nash germs

Consider the affine space $\mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$. We denote by s the involution of $\mathbb{R}^n$ changing the sign of the first coordinate x_1:

$$s : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1, x_2, \ldots, x_n) & \mapsto & (-x_1, x_2, \ldots, x_n) \end{array}$$

This equips $\mathbb{R}^n$ with a linear action of the group $G = \{id_{\mathbb{R}^n}, s\}$.

In this paper, a function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ will be said to be invariant if f is invariant under the right composition with s, that is, if f is the germ of an equivariant function (we equip $\mathbb{R}$ with the trivial action of G).

In [1] and [16], the classification of invariant, simple, real analytic germs $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with respect to equivariant analytic equivalence, that is, right equivalence via an equivariant analytic diffeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$:

Theorem 2.1 ([1], [16]). — *An invariant simple real analytic function germ $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is equivariantly analytically equivalent to one and only one invariant germ of the following list:*

$$\begin{array}{llll} A_k,\ k \geq 0\colon & \pm x_1^2 \pm x_2^{k+1} + Q, & E_6\colon & \pm x_1^2 + x_2^3 \pm x_3^4 + Q, \\ B_k,\ k \geq 2\colon & \pm x_1^{2k} \pm x_2^2 + Q, & E_7\colon & \pm x_1^2 + x_2^3 + x_2 x_3^3 + Q, \\ C_k,\ k \geq 3\colon & x_1^2 x_2 \pm x_2^k + Q, & E_8\colon & \pm x_1^2 + x_2^3 + x_3^5 + Q, \\ D_k,\ k \geq 4\colon & \pm x_1^2 + x_2^2 x_3 \pm x_3^{k-1} + Q, & F_4\colon & \pm x_1^4 + x_2^3 + Q, \end{array}$$

where $Q = \pm x_s^2 \pm \cdots \pm x_n^2$, with $s = 4$ for singularities D_k and E_k, and $s = 3$ in the other cases.

Remark 2.2. — If we forget the action of the involution s on $\mathbb{R}^n$, we notice that the families A_k and B_k, and C_k and D_k, E_6 and F_4, E_7, E_8, of Theorem 2.1 are singularities A, and D, E_6, E_7, E_8.

In this paper, we are interested in the classification of invariant Nash germs $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, that is, germs of equivariant analytic functions with semialgebraic graphs. Recall (see, for instance, [5] Corollary 8.1.6) that a Nash germ can be considered as an algebraic power series, via its Taylor series. The above classification is also valid for invariant simple Nash germs with respect to equivariant Nash equivalence, that is, right equivalence via an equivariant Nash diffeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, according to the following proposition:

Proposition 2.3. — *Let $f, h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two invariant Nash germs. Then f and h are equivariantly Nash equivalent if and only if they are equivariantly analytically equivalent.*

This property is a particular case of the following result:

Theorem 2.4. — *Let G be a reductive algebraic group acting linearly on $\mathbb{R}^n$ and $\mathbb{R}^p$. Consider two equivariant Nash germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ and $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. If f and h are equivariantly analytically equivalent, then they are equivariantly Nash equivalent.*

Remark 2.5. —

- Since a Nash diffeomorphism is, in particular, analytic, the converse is obviously true.
- Any finite group is reductive.

Proof (of Theorem 2.4). — Suppose there exists an equivariant analytic diffeomorphism $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f \circ \phi = h$. Denote $F(x, y) := f(y) - h(x)$ for $x, y \in \mathbb{R}^n$. Then $F : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^p, 0)$ is a Nash germ and can be considered as an algebraic power series in $\mathbb{R}_{alg}[[x, y]]^p$, and $\phi(x)$ as an equivariant convergent power series in $\mathbb{R}\{x, y\}$ such that $F(x, \phi(x)) = 0$.

Therefore, by Theorem A of [4] and Example 11.3 of [26], we can approximate $\phi(x)$ by an equivariant algebraic power series $\widetilde{\phi}(x)$ such that $F(x, \widetilde{\phi}(x)) = 0$, and we do the approximation closely enough so that $\widetilde{\phi}(x)$ remains a diffeomorphism. As a consequence, $\widetilde{\phi} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is an equivariant Nash diffeomorphism such that $f \circ \widetilde{\phi} = h$.

Notice that, actually, Theorem A of [4] is about approximation of equivariant formal solutions of polynomial equations by equivariant algebraic power series, but it is also true for algebraic power series equations. Indeed, following G. Rond's ideas, it is possible to reduce it to the case of polynomial equations, as in [3] Lemma 5.2 and [7] Reduction (2) of the proof of Theorem 1.1, using arguments of the proof of Lemma 8.1 in [25], along with the fact that the morphism $\mathbb{R}[x, y]_{(x,y)} \to \mathbb{R}_{alg}[[x, y]]$ is faithfully flat by [5] Corollary 8.7.16. $\square$

3. Equivariant blow-Nash equivalence

Now, we want to study the classification of invariant simple Nash germs with respect to G-blow-Nash equivalence via an equivariant blow-Nash isomorphism