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A C^1 ARNOL′D-LIOUVILLE THEOREM

by

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Dedicated to Jean-Christophe Yoccoz
A generous man

Abstract. — In this paper, we prove a version of Arnol′d-Liouville theorem for C^2 Hamiltonians having enough C^1 commuting Hamiltonians. We show that the Lipschitz regularity of the foliation by invariant Lagrangian tori is crucial to determine the Dynamics on each Lagrangian torus and that the C^1 regularity of the foliation by invariant Lagrangian tori is crucial to prove the continuity of Arnol′d-Liouville coordinates. We also explore various notions of C^0 and Lipschitz integrability.

***Résumé* (Un théorème C^1 d'Arnol′d-Liouville).** — Dans cet article, nous montrons une version du théorème d'Arnol′d-Liouville pour des hamiltoniens de classe C^2 qui ont assez de hamiltoniens de classe C^1 commutant avec eux. Nous montrons que le caractère Lipschitz du feuilletage en tores lagrangiens invariants est crucial pour déterminer la dynamique sur chaque tore invariant et que la régularité C^1 du feuilletage est cruciale pour montrer la continuité des coordonnées d'Arnol′d-Liouville. Nous explorons aussi différentes notions d'intégrabilité au sens C^0 ou Lipschitz.

1. Introduction and Main Results

This article elaborates on the following question.

Question. — *If a C^2 Hamiltonian system has enough commuting integrals* [(1)]*, can we precisely describe the Hamiltonian Dynamics, even in the case of non C^2 integrals?*

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(1) These notions are precisely described in the rest of the introduction.

When the integrals are C^2, Arnol′d-Liouville Theorem (see [4]) implies that the compact level sets are tori and gives a precise description of the Dynamics. The proof in [4] relies on the Abelian group action generated by the commuting Hamiltonian flows. Unfortunately, the result is not valid for C^1 integrals : in this case, there is *a priori* no Hamiltonian flow that we can associate to the C^1 integrals, the Abelian group action does not exist and so the proof in [4] does not work.

Note that in some case, a C^0-integrability can be shown without knowing if the integrals can be chosen smooth: the case of Tonelli Hamiltonians (2)with no conjugate points on $\mathbb{T}^n \times \mathbb{R}^n$ (see Theorem B.1). C^0-integrability for such Hamiltonians is proved in [1] and some partial results concerning the Dynamics on the invariant graphs are given, but no result similar to Arnol′d-Liouville Theorem is proved. The only case where a more accurate result is obtained is when the Tonelli Hamiltonian gives rise to a Riemannian metric after Legendre transform. Burago & Ivanov proved in [5] that a Riemannian metric with no conjugate points is smoothly integrable, but this is specific to the Riemannian case and cannot be adapted to the general Tonelli case.

In these statements and all our results, the topology of the invariant leaves is prescribed: they are tori. Indeed, we cannot use the Abelian group action defined by the commuting Hamiltonian flows as in classical Arnol′d-Liouville Theorem to recover this topology. And our use of Herman's theory requires the level set to be $\mathbb{T}^n$, so we work with $T^*\mathbb{T}^n$ with the topology prescribed.

In this article, we consider the case of Lipschitz and C^1-integrability, that are intermediary between C^0 and C^2 integrability. For a C^2 Hamiltonian that satisfies a certain non-degeneracy condition called A-non-degeneracy condition (3) and that is C^1-integrable, we will prove

— we can define global continuous Arnol′d-Liouville coordinates, which are defined by using a symplectic homeomorphism;
— the Dynamics restricted to every invariant torus is C^1-conjugate to a rotation;
— we can even define a flow for the continuous Hamiltonian vectorfields that are associated to the C^1 integrals (see Proposition 4.2).

For a Tonelli Hamiltonian that is Lipschitz integrable, we will prove that the Dynamics restricted to every invariant torus is Lipschitz conjugate to a rotation, but we will obtain no information concerning the transverse dependence to the conjugacy.
Let us add that the wide class of Hamiltonians that are defined on the cotangent bundle of the n-dimensional torus and strictly convex in the fiber direction is a part of the set of A-non-degenerate Hamiltonians. (4) In particular, Tonelli Hamiltonian are A-non-degenerate.

In order to state our results, let us now introduce some definitions. The definitions can be divided into three classes: the integrability conditions (Definition 1.2, 1.3,

(2) See Definition 1.1.
(3) See Definition 1.4.
(4) This will be proved in Proposition 3.2.

1.6, 1.7, 1.8), the nondegeneracy conditions (Definition 1.1, 1.4) and the symplectic coordinates (Definition 1.5).

Definition 1.1. — *A C^2 Hamiltonian $H : T^*N \to \mathbb{R}$ for a compact Riemannian manifold N is* Tonelli *if the following two assumptions are satisfied.*

— *H has super-linear growth, i.e., $\frac{H(q,p)}{\|p\|} \to \infty$ as $\|p\| \to \infty$.*

— *H is convex in the fiber, i.e., $\frac{\partial^2 H}{\partial p^2}(q,p)$ is positive definite for all q, p.*

For example, a mechanical Hamiltonian $H : T^*\mathbb{T}^n \to \mathbb{R}$ given by $H(q,p) = \frac{1}{2}\|p\|^2 + V(q)$ is Tonelli.

Notations. — If H is a C^1 Hamiltonian defined on a symplectic manifold $(M^{(2n)}, \omega)$, we denote by X_H *the Hamiltonian vectorfield*, that is defined by

$$\forall\, x \in M,\ \forall\, v \in T_xM, \quad \omega(X_H(x), v) = dH(x)\cdot v.$$

If moreover H is C^2, the Hamiltonian flow associated to H, that is the flow of X_H, is denoted by (φ_t^H).

— If H and K are two C^1 Hamiltonians that are defined on M, their *Poisson bracket* is

$$\{H, K\}(x) = DH(x)\cdot X_K(x) = \omega(X_H(x), X_K(x)).$$

Definition 1.2. — *Let $k \geq 1$ be an integer. Let $\mathscr{U} \subset M^{(2n)}$ be an open subset and let $H : M \to \mathbb{R}$ be a $C^{\sup\{2,k\}}$ Hamiltonian. Then H is C^k* completely integrable *in $\mathscr{U}$ if*

— *$\mathscr{U}$ is invariant by the Hamiltonian flow of H;*

— *there exist n C^k functions $H_1, H_2, \dots, H_n : \mathscr{U} \to \mathbb{R}$ so that*

  — *at every $x \in \mathscr{U}$, the family $dH_1(x), \dots, dH_n(x)$ is independent;*

  — *for every i, j, we have $\{H_i, H_j\} = 0$ and $\{H_i, H\} = 0$.*

Remarks. — 1. We cannot always take $H_1 = H$. At the critical points of H, $dH(x) = 0$ and a Tonelli Hamiltonian has always critical points. However, if we consider only the part of phase space without critical points, we can indeed take $H = H_1$.

2. Observe that when $k = 1$, the C^1 Hamiltonians $H_1, \dots, H_n$ don't necessarily define a flow because the corresponding vector field is just continuous. Hence the proof of Arnol′d-Liouville theorem (see for example [**4**]) cannot be used to determine what the Dynamics is on the invariant Lagrangian submanifold $\{H_1 = c_1, \dots, H_n = c_n\}$. That is why the results we give in Theorem 1.1 and 1.2 below are non-trivial.

   In fact, in the setting of next definition for $k = 1$ and when the Hamiltonian is Tonelli, we will prove in Proposition 4.2 a posteriori that each H_i surprisingly defines a flow.

We will sometimes need the following narrower definition of C^k integrability.

Definition 1.3. — *A C^2 Hamiltonian $H : \mathcal{V} \to \mathbb{R}$ that is defined on some open subset $\mathcal{V} \subset T^*N$ is called G-C^k* completely integrable *on some open subset $\mathcal{U} \subset \mathcal{V}$ if*
- *it is C^k completely integrable and*
- *with the same notations as in the definition of C^k complete integrability, every Lagrangian submanifold $\{H_1 = c_1, \dots, H_n = c_n\}$ is the graph of a C^k map.*

Remarks. — 1. Observe that for any $k \geq 1$, G-C^k integrability is equivalent to the existence of an invariant C^k foliation into Lagrangian graphs. The direct implication is a consequence of the fact that the H_i are commuting in the Poisson sense (i.e., $\{H_i, H_j\} = 0$). For the reverse implication, denote the invariant C^k foliation by $(\eta_a)_{a\in U}$ where a is in some open subset of $\mathbb{R}^n$. Then we define a C^k map $A = (A_1, \dots, A_n)$ by

$$A(q,p) = a \Longleftrightarrow p = \eta_a(q).$$

Observe that each Lagrangian graph $\mathcal{T}_a$ of η_a is in the energy level $\{A_i = a_i\}$. Hence $X_{A_i}(q,p) \in T_{(q,p)}\mathcal{T}_{A(q,p)}$ and thus

$$\{A_i, A_j\}(q,p) = \omega(X_{A_i}(q,p), X_{A_j}(q,p)) = 0$$

because all the $\mathcal{T}_a$ are Lagrangian. In a similar way, $\{H, A_i\} = 0$.

2. When $k \geq 2$, $\mathcal{U} = T^*N$ and H is a Tonelli Hamiltonian, C^k-integrability implies G-C^k integrability and that $N = \mathbb{T}^n$. Let us give briefly the arguments: in this case the set of fixed points of the Hamiltonian flow $\mathcal{N} = \{\frac{\partial H}{\partial p} = 0\}$ is an invariant submanifold that is a graph. Hence $\mathcal{N}$ is one of the invariant tori given by Arnol′d-Liouville theorem. Therefore $N = \mathbb{T}^n$ and all the invariant tori of the foliation are also graphs : this is true for those that are close to $\mathcal{N}$, and in this case there is a uniform Lipschitz constant because the Hamiltonian is Tonelli. Using this Lipschitz constant, we can extend the neighborhood where they are graphs to the whole $T^*\mathbb{T}^n$.

We next introduce a non-degeneracy condition called A-non-degeneracy. We will prove the following proposition in Section 4.2.

Proposition 1.1. — *Assume that $(q,a) \in \mathbb{T}^n \times U \mapsto (q, \eta_a(q)) \in \mathcal{U}$ is a C^1 foliation of an open subset $\mathcal{U}$ of $\mathbb{T}^n \times \mathbb{R}^n$ into Lagrangian graphs. Then the map $\mathsf{c} : U \to \mathbb{R}^n$ defined by*

$$\mathsf{c}(a) = \int_{\mathbb{T}^n} \eta_a(q)dq$$

is a C^1-diffeomorphism from U onto its image.

Notations. — Let $H : \mathcal{U} \subset T^*\mathbb{T}^n \to \mathbb{R}$ be a C^1 integrable Hamiltonian.
- We will denote by

$$(\mathcal{T}_a)_{a\in U} = (\{(q, \eta_a(q)) : q \in \mathbb{T}^n\})_{a\in U}$$

  the invariant foliation and define the function $\mathsf{c} : U \to \mathbb{R}^n$ by $\mathsf{c}(a) = \int_{\mathbb{T}^n} \eta_a(q)dq$.
- The *A-function* $A_H : \mathsf{c}(U) \to \mathbb{R}$ is defined by $A_H(c) = H(0, \eta_{\mathsf{c}^{-1}(c)}(0))$.