ISSN 0012-9593

quatrième série - tome 53 fascicule 4 juillet-août 2020

ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE

Daniel LOUGHRAN & Alexei N. SKOROBOGATOV & Arne SMEETS

Pseudo-split fibers and arithmetic surjectivity

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 428 euros.
Abonnement avec supplément papier :
Europe : 576 €. Hors Europe : 657 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 53, 2020, p. 1037 à 1070

PSEUDO-SPLIT FIBERS AND ARITHMETIC SURJECTIVITY

BY Daniel LOUGHRAN, Alexei N. SKOROBOGATOV AND Arne SMEETS

Abstract. – Let $f : X \to Y$ be a dominant morphism of smooth, proper and geometrically integral varieties over a number field k, with geometrically integral generic fiber. We give a necessary and sufficient geometric criterion for the induced map $X(k_v) \to Y(k_v)$ to be surjective for almost all places v of k. This generalizes a result of Denef which had previously been conjectured by Colliot-Thélène, and can be seen as an optimal geometric version of the celebrated Ax-Kochen theorem.

Résumé. – Soit $f : X \to Y$ un morphisme dominant de variétés lisses, propres et géométriquement intègres définies sur un corps de nombres k, dont la fibre générique est géométriquement intègre. Nous donnons un critère géometrique, à la fois nécessaire et suffisant, pour que l'application induite $X(k_v) \to Y(k_v)$ soit surjective pour presque toute place v de k. Ceci généralise un résultat de Denef précédemment conjecturé par Colliot-Thélène. Notre résultat peut être vu comme une version géométrique optimale du célèbre théorème de Ax-Kochen.

1. Introduction

1.1. – A famous theorem of Ax-Kochen [6] states that any homogeneous polynomial over $\mathbf{Q}_p$ of degree d in at least d^2+1 variables has a non-trivial zero, provided that p avoids a certain finite exceptional set of primes depending only on d. This was originally proved using model theory. Denef recently found purely algebro-geometric proofs [12, 13]. In [13], he did so by proving a more general conjecture of Colliot-Thélène [8, §3, Conjecture].

The essential notion (first introduced by the second author in [32, Definition 0.1]) appearing in this conjecture is that of a *split scheme*:

Definition 1.1. – Let k be a perfect field. A scheme X of finite type over k is called *split* if X contains an irreducible component of multiplicity 1 which is geometrically irreducible.

Here the *multiplicity* of an irreducible component Z of X is the length of the local ring of X at the generic point of Z. In particular, it has multiplicity 1 if and only if it is generically reduced. Denef's result [13, Theorem 1.2] is the following.

doi:10.24033/asens.2439

THEOREM 1.2 (Denef). – *Let $f : X \to Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over a number field k, with geometrically integral generic fiber. Assume that for every modification $f' : X' \to Y'$ of f with X' and Y' smooth such that the generic fibers of f and f' are isomorphic, the fiber $(f')^{-1}(D)$ is a split $\kappa(D)$-variety for every $D \in (Y')^{(1)}$.*

Then $Y(k_v) = f(X(k_v))$ for all but finitely many places v of k.

Here k_v denotes the completion of k at the place v, $(Y')^{(1)}$ denotes the set of points of codimension 1 in Y', and $\kappa(D)$ is the residue field of D. A *modification* of f is a commutative diagram

(1.1)
$$\begin{array}{ccc} X' & \xrightarrow{\alpha_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\alpha_Y} & Y, \end{array}$$

where $f' : X' \to Y'$ is a dominant morphism of proper and geometrically integral varieties over k, and $\alpha_X : X' \to X$ and $\alpha_Y : Y' \to Y$ are birational morphisms.

One obtains the Ax-Kochen theorem by applying Theorem 1.2 to the universal family of all hypersurfaces of degree d in $\mathbf{P}^n$ with $n \geq d^2$; that the hypotheses of the theorem are satisfied in this case was shown by Colliot-Thélène (see [8, Remarque 4]).

1.2. – In this paper we strengthen Denef's result, by determining conditions which are both *necessary and sufficient* for the map $f : X(k_v) \to Y(k_v)$ to be surjective for almost all places v. Our result uses the following weakening of Definition 1.1 (in §2.2 we also give a more general definition over arbitrary ground fields).

DEFINITION 1.3. – Let k be a perfect field with algebraic closure $\bar{k}$. A scheme X of finite type over k is called *pseudo-split* if every element of $\mathrm{Gal}(\bar{k}/k)$ fixes some irreducible component of $X \times_k \bar{k}$ of multiplicity 1.

It is clear that pseudo-splitness is weaker than splitness, the latter meaning that a *single* irreducible component of $X \times_k \bar{k}$ of multiplicity 1 is fixed by *all* of $\mathrm{Gal}(\bar{k}/k)$. With this terminology, we can state our generalization of Denef's result as follows:

THEOREM 1.4. – *Let k be a number field. Let $f : X \to Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over k with geometrically integral generic fiber. Then $Y(k_v) = f(X(k_v))$ for all but finitely many places v of k if and only if for every modification $f' : X' \to Y'$ of f, with X' and Y' smooth, and for every point $D \in (Y')^{(1)}$, the fiber $(f')^{-1}(D)$ is a pseudo-split $\kappa(D)$-variety.*

In the notation introduced by the first and third named authors in their recent work [25, §3], the morphisms $f : X \to Y$ satisfying the conclusion of the theorem are exactly the morphisms such that $\Delta(f') = 0$ for every modification f' of f.

1.3. – Theorem 1.4 will be deduced from finer results. With $f : X \to Y$ as in Theorem 1.2, Colliot-Thélène asked in [9, §13.1] how the geometry of f relates to the surjectivity of the map $X(k_v) \to Y(k_v)$, for a possibly infinite collection of places v. He called this phenomenon "surjectivité arithmétique" (note that this is different from the notion of arithmetic surjectivity studied in [16]). We develop general criteria which allow one to decide whether, for an *individual* (but large) place v, the map $X(k_v) \to Y(k_v)$ is surjective. They involve certain invariants which we call "s-invariants," defined in §3—local versions of the δ-invariants introduced in [25, §3]; their definition is given in terms of the geometry of f and does not involve model theory.

The following result is proved in §6 using tools from logarithmic geometry, in particular, a logarithmic version of Hensel's lemma and "weak toroidalisation". It should be viewed as the main theorem of the paper and is a geometric criterion, in the style of Colliot-Thélène's conjecture, for surjectivity of the map $X(k_v) \to Y(k_v)$.

THEOREM 1.5. – *Let k be a number field. Let $f : X \to Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over k, with geometrically integral generic fiber. Then there exist a modification $f' : X' \to Y'$ of f with X' and Y' smooth, and a finite set of places S of k such that for all $v \notin S$ the following are equivalent:*

(1) *the map $X(k_v) \to Y(k_v)$ is surjective;*

(2) *for every codimension 1 point $D' \in (Y')^{(1)}$, we have $s_{f',D'}(v) = 1$.*

The invariants $s_{f',D'}(v)$ appearing in the statement will be defined in §3. They are defined in terms of the Galois action on the irreducible components of the fiber of f' over D'. One benefit of our approach is that it yields a single model for f which can be used to test arithmetic surjectivity using a finite list of criteria.

A simple consequence of Theorem 1.5 is the following:

THEOREM 1.6. – *Let $f : X \to Y$ be a dominant morphism of smooth, proper and geometrically integral varieties over a number field k, with geometrically integral generic fiber. The set of places v such that $Y(k_v) = f(X(k_v))$ is Frobenian.*

Here we use the term "Frobenian" in the sense of Serre [31, §3.3] (see §3.1). Frobenian sets of places have a density, but being Frobenian is much stronger than just having a density; for example, an infinite Frobenian set has positive density. It is also possible to prove Theorem 1.6 using model-theoretic results and techniques such as quantifier elimination [5, 28]; our method avoids these and is completely algebro-geometric. However, we know of no model-theoretic proof of the finer Theorems 1.4 and 1.5. (From a model-theoretic perspective, one may view Theorem 1.5 as an explicit instance of quantifier elimination).

1.4. – Some of the ingredients of our proof are already present in the work of Denef [12, 13], e.g., the use of the weak toroidalisation theorem [4, 3]. We need more ingredients from logarithmic geometry, cf. §5—essentially a few basic properties of log smooth morphisms and log blow-ups. The choice of a log smooth model for the morphism makes some of its arithmetic properties more transparent, and can be seen as a convenient way to come up with a Galois stratification, in the sense of Fried and Sacerdote [14]. On the other hand, we also use work of Serre [31] on Frobenian functions, expanding upon what was done in [25].