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MIXED HODGE STRUCTURES AND FORMALITY OF SYMMETRIC MONOIDAL FUNCTORS

BY Joana CIRICI AND Geoffroy HOREL

Abstract. – We use mixed Hodge theory to show that the functor of singular chains with rational coefficients is formal as a lax symmetric monoidal functor, when restricted to complex varieties whose weight filtration in cohomology satisfies a certain purity property. This has direct applications to the formality of operads or, more generally, of algebraic structures encoded by a colored operad. We also prove a dual statement, with applications to formality in the context of rational homotopy theory. In the general case of complex varieties with non-pure weight filtration, we relate the singular chains functor to a functor defined via the first term of the weight spectral sequence.

Résumé. – Nous utilisons la théorie de Hodge mixte pour montrer que le foncteur des chaînes singulières à coefficients rationnels est formel, comme foncteur symétrique monoïdal lax, lorsqu'on le restreint aux variétés complexes dont la filtration par le poids en cohomologie satisfait une certaine propriété de pureté. Ce résultat a des applications directes à la formalité d'opérades ou plus généralement à des structures algébriques encodées par une opérade colorée. Nous prouvons aussi le résultat dual, avec des applications à la formalité dans le contexte de la théorie de l'homotopie rationnelle. Dans le cas général d'une variété dont la filtration par le poids n'est pas pure, nous relions le foncteur des chaînes singulières à un foncteur défini par la première page de la suite spectrale des poids.

1. Introduction

There is a long tradition of using Hodge theory as a tool for proving formality results. The first instance of this idea can be found in [18] where the authors prove that compact Kähler manifolds are formal (i.e., the commutative differential graded algebra of differential forms is quasi-isomorphic to its cohomology). In the introduction of that paper, the authors explain that their intuition came from the theory of étale cohomology and the fact that the degree n étale cohomology group of a smooth projective variety over a finite field is pure

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of weight n. This purity is what heuristically prevents the existence of non-trivial Massey products. In the setting of complex algebraic geometry, Deligne introduced in [16, 17] a filtration on the rational cohomology of every complex algebraic variety X, called the *weight filtration*, with the property that each of the successive quotients of this filtration behaves as the cohomology of a smooth projective variety, in the sense that it has a Hodge-type decomposition. Deligne's mixed Hodge theory was subsequently promoted to the rational homotopy of complex algebraic varieties (see [35], [27], [36]). This can then be used to make the intuition of the introduction of [18] precise. In [22] and [12], it is shown that purity of the weight filtration in cohomology implies formality, in the sense of rational homotopy, of the underlying topological space. However, the treatment of the theory in these references lacks functoriality and is restricted to smooth varieties in the first paper and to projective varieties in the second.

In another direction, in the paper [26], the authors elaborate on the method of [18] and prove that operads (as well as cyclic operads, modular operads, etc.) internal to the category of compact Kähler manifolds are formal. Their strategy is to introduce the functor of de Rham currents which is a functor from compact Kähler manifolds to chain complexes that is lax symmetric monoidal and quasi-isomorphic to the singular chain functor as a lax symmetric monoidal functor. Then they show that this functor is formal as a lax symmetric monoidal functor. Recall that, if $\mathcal{C}$ is a symmetric monoidal category and $\mathscr{A}$ is an abelian symmetric monoidal category, a lax symmetric monoidal functor $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathscr{A})$ is said to be formal if it is weakly equivalent to $H \circ F$ in the category of lax symmetric monoidal functors. It is then straightforward to see that such functors send operads in $\mathcal{C}$ to formal operads in $\mathrm{Ch}_*(\mathscr{A})$. The functoriality also immediately gives us that a map of operads in $\mathcal{C}$ is sent to a formal map of operads or that an operad with an action of a group G is sent to a formal operad with a G-action. Of course, there is nothing specific about operads in these statements and they would be equally true for monoids, cyclic operads, modular operads, or more generally any algebraic structure that can be encoded by a colored operad.

The purpose of this paper is to push this idea of formality of symmetric monoidal functors from complex algebraic varieties in several directions in order to prove the most general possible theorem of the form "purity implies formality". Before explaining our results more precisely, we need to introduce a bit of terminology.

Let X be a complex algebraic variety. Deligne's weight filtration on the rational n-th cohomology vector space of X is bounded by

$$0 = W_{-1}H^n(X,\mathbb{Q}) \subseteq W_0H^n(X,\mathbb{Q}) \subseteq \cdots \subseteq W_{2n}H^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q}).$$

If X is smooth then $W_{n-1}H^n(X,\mathbb{Q}) = 0$, while if X is projective $W_nH^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q})$. In particular, if X is smooth and projective then we have

$$0 = W_{n-1}H^n(X,\mathbb{Q}) \subseteq W_nH^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q}).$$

In this case, the weight filtration on $H^n(X,\mathbb{Q})$ is said to be *pure of weight n*. More generally, for α a rational number and X a complex algebraic variety, we say that the weight filtration on $H^*(X,\mathbb{Q})$ is α-*pure* if, for all $n \geq 0$, we have

$$\mathrm{Gr}^W_p H^n(X,\mathbb{Q}) := \frac{W_pH^n(X,\mathbb{Q})}{W_{p-1}H^n(X,\mathbb{Q})} = 0 \text{ for all } p \neq \alpha n.$$

The bounds on the weight filtration tell us that this makes sense only when $0 \leq \alpha \leq 2$. Note as well that if we write $\alpha = a/b$ with $(a, b) = 1$, α-purity implies that the cohomology is concentrated in degrees that are divisible by b, and that $H^{bn}(X, \mathbb{Q})$ is pure of weight an.

Aside from smooth projective varieties, some well-known examples of varieties with 1-pure weight filtration are: projective varieties whose underlying topological space is a $\mathbb{Q}$-homology manifold ([17, Theorem 8.2.4]) and the moduli spaces $\mathcal{M}_{Dol}$ and $\mathcal{M}_{dR}$ appearing in the non-abelian Hodge correspondence ([28]). Some examples of varieties with 2-pure weight filtration are: complements of hyperplane arrangements ([33]), which include the moduli spaces $\mathcal{M}_{0,n}$ of smooth projective curves of genus 0 with n marked points, and complements of toric arrangements ([22]). As we shall see in Section 8, complements of codimension d subspace arrangements are examples of smooth varieties whose weight filtration in cohomology is $2d/(2d-1)$-pure. For instance, this includes configuration spaces of points in $\mathbb{C}^d$.

Our main result is Theorem 7.3. We show that, for a non-zero rational number α, the singular chains functor

$$S_*(-, \mathbb{Q}) : \mathrm{Var}_{\mathbb{C}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to complex varieties whose weight filtration in cohomology is α-pure. Here $\mathrm{Var}_{\mathbb{C}}$ denotes the category of complex algebraic varieties (i.e the category of schemes over $\mathbb{C}$ that are reduced, separated and of finite type). This generalizes the main result of [26] on the formality of $S_*(X, \mathbb{Q})$ for any operad X in smooth projective varieties, to the case of operads in possibly singular and/or non-compact varieties with pure weight filtration in cohomology.

As direct applications of the above result, we prove formality of the operad of singular chains of some operads in complex varieties, such as the noncommutative analog of the (framed) little 2-discs operad introduced in [19] and the monoid of self-maps of the complex projective line studied by Cazanave in [11] (see Theorems 7.4 and 7.7). We also reinterpret in the language of mixed Hodge theory the proofs of the formality of the little disks operad and Getzler's gravity operad appearing in [38] and [23]. These last two results do not fit directly in our framework, since the little disks operad and the gravity operad do not quite come from operads in algebraic varieties. However, the action of the Grothendieck-Teichmüller group provides a bridge to mixed Hodge theory.

In Theorem 8.1 we prove a dual statement of our main result, showing that Sullivan's functor of piecewise linear forms

$$\mathcal{A}^*_{PL} : \mathrm{Var}_{\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to varieties whose weight filtration in cohomology is α-pure, where α is a non-zero rational number.

This gives functorial formality in the sense of rational homotopy for such varieties, generalizing both "purity implies formality" statements appearing in [22] for smooth varieties and in [12] for singular projective varieties. Our generalization is threefold: we allow α-purity (instead of just 1-and 2-purity), we obtain functoriality and we study possibly singular and open varieties simultaneously.

Theorems 7.3 and 8.1 deal with situations in which the weight filtration is pure. In the general context with mixed weights, it was shown by Morgan [35] for smooth varieties and