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Xuecheng WANG

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

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GLOBAL REGULARITY FOR THE 3D FINITE DEPTH CAPILLARY WATER WAVES

BY XUECHENG WANG

ABSTRACT. – In this paper, we prove global regularity, scattering, and the non-existence of small traveling waves for the $3D$ capillary waves system in the flat bottom setting for smooth localized small initial data.

To construct global solutions, we highly exploit the symmetric structures inside the capillary waves system and control both a low order weighted norm and a high order weighted norm of the profile of a good substitution variable over time to show that, although the nonlinear solution itself doesn't decay sharply at rate $1/(1+t)$ over time, the "$1+\alpha$" derivatives of the nonlinear solution indeed decay sharply, where α is some fixed positive number.

RÉSUMÉ. – Dans cet article, on démontre la régularité globale, la dispersion des solutions et la non-existence des petites ondes progressives pour un système d'equations des ondes capillaires en dimension 3 avec des petites données initiales régulières et localisées, dans le cas des fonds plats.

Pour construire des solutions globales, on exploite les structures symétriques du système d'ondes capillaires et contrôle à la fois les évolutions des deux normes avec poids du profil d'une bonne variable substitutive, l'une d'ordre petit et l'autre d'ordre grand. En conséquence, on montre que les dérivées d'ordre $1 + \alpha$ de la solution non-linéaire décroissent rapidement au taux de $1/(1 + t)$, bien que la solution elle-même ne décroisse pas aussi rapidement, où α est un nombre positif fixé.

1. Introduction

1.1. The set-up of problem and previous results

We study the evolution of a constant density irrotational inviscid fluid, e.g., water, inside a time dependent domain $\Omega(t) \subset \mathbb{R}^3$, which has a fixed flat bottom Σ and a moving interface $\Gamma(t)$. Above the water region $\Omega(t)$ is vacuum. We neglect the gravity effect and only consider the surface tension effect in this paper. The problem under consideration is also known as the capillary waves system.

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After normalizing the depth of $\Omega(t)$ to be "1," we can represent $\Omega(t)$, $\Gamma(t)$, and Σ in the Eulerian coordinates as follows,

$$\Omega(t) := \{(x, y) : x \in \mathbb{R}^2, -1 \leq y \leq h(t, x)\},$$
$$\Gamma(t) := \{(x, h(t, x) : x \in \mathbb{R}^2\}, \quad \Sigma := \{(x, -1) : x \in \mathbb{R}^2\},$$

where $h(t, x)$ represents the height of interface, which will be a small perturbation of zero.

Let "u" and "p" denote the velocity and the pressure of the fluid respectively. Then the evolution of fluid can be described by the free boundary Euler equation as follows,

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \nabla \cdot u = 0, \quad \nabla \times u = 0, \quad \text{in } \Omega(t). \tag{1.1}$$

The free surface $\Gamma(t)$ moves with the normal component of the velocity according to the kinematic boundary condition as follows,

$$\partial_t + u \cdot \nabla \text{ is tangent to } \cup_t \Gamma(t).$$

The pressure p satisfies the Young-Laplace equation as follows,

$$p = \sigma H(h), \quad \text{on } \Gamma(t),$$

where "σ" denotes the surface tension coefficient, which will be normalized to be one, and $H(h)$ represents the mean curvature of the interface, which is given as follows,

$$H(h) = \nabla \cdot \Big(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\Big).$$

Lastly, the following Neumann type boundary condition holds on the bottom Σ,

$$u \cdot \vec{\mathbf{n}} = 0, \quad \text{on } \Sigma.$$

Because the bottom is assumed to be fixed, the fluid cannot go through the bottom. This explains why the above boundary condition holds.

Since the velocity field is irrotational, we can represent it in terms of a velocity potential ϕ. Let ψ be the restriction of the velocity potential on the boundary $\Gamma(t)$, more precisely, $\psi(t, x) := \phi(t, x, h(t, x))$. From the divergence free condition and the boundary conditions, we can derive the Laplace equation with two boundary conditions as follows,

$$(\partial_y^2 + \Delta_x)\phi = 0, \quad \frac{\partial \phi}{\partial \vec{\mathbf{n}}}\Big|_\Sigma = 0, \quad \phi|_{\Gamma(t)} = \psi. \tag{1.2}$$

Hence, we can reduce the study of the motion of fluid in $\Omega(t)$ to the study of the evolution of the height function "$h(t, x)$" and the restricted velocity potential "$\psi(t, x)$" as follows,

$$\begin{cases} \partial_t h = G(h)\psi, \\ \partial_t \psi = H(h) - \frac{1}{2}|\nabla \psi|^2 + \dfrac{(G(h)\psi + \nabla h \cdot \nabla \psi)^2}{2(1 + |\nabla h|^2)}, \end{cases} \tag{1.3}$$

where $G(h)\psi = \sqrt{1 + |\nabla h|^2}\,\mathcal{N}(h)\psi$ and $\mathcal{N}(h)\psi$ is the Dirichlet-Neumann operator at the interface $\Gamma(t)$. See e.g., [42] for the derivation of the system (1.3).

The capillary waves system (1.3) has the conserved energy and the conserved momentum as follows, see e.g., [7],

(1.4)
$$\mathscr{H}(h(t), \psi(t)) := \Big[\int_{\mathbb{R}^2} \frac{1}{2}\psi(t) G(h(t))\psi(t) + \frac{|\nabla h(t)|^2}{1 + \sqrt{1 + |\nabla h(t)|^2}} dx\Big] = \mathscr{H}(h(0), \psi(0)),$$

$$\int_{\mathbb{R}^2} h(t,x)dx = \int_{\mathbb{R}^2} h(0,x)dx. \tag{1.5}$$

From [34, Lemma 3.4], we know that

(1.6) (Flat bottom setting) :

$$\Lambda_{\leq 2}[G(h)\psi] = |\nabla| \tanh |\nabla|\psi - \nabla \cdot (h\nabla\psi) - |\nabla| \tanh |\nabla|(h|\nabla| \tanh |\nabla|\psi),$$

$$\text{(Flat bottom setting)}: \quad \Lambda_{\leq 2}[\partial_t \psi] = \Delta h - \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}(|\nabla| \tanh |\nabla|\psi)^2, \tag{1.7}$$

where $\Lambda_{\leq 2}[\mathcal{N}]$ denotes the linear terms and the quadratic terms of the nonlinearity $\mathcal{N}$.

From the above Taylor expansions, in the small solution regime, the conserved Hamiltonian in (1.4) tells us that the L^2-norm of $(\nabla_x h, |\nabla|\sqrt{\tanh |\nabla|}\psi)$ doesn't change much over time. More precisely, the following approximation holds,

$$\frac{1}{4}\big(\|\nabla h(t)\|_{L^2}^2 + \||\nabla| P_{\leq 1}[\psi(t)]\|_{L^2}^2 + \||\nabla|^{1/2} P_{\geq 1}[\psi(t)]\|_{L^2}^2\big) \leq \mathcal{H}(h(t), \psi(t))$$

$$= \mathcal{H}(h(0), \psi(0)) \leq 4\big(\|\nabla h(t)\|_{L^2}^2 + \||\nabla| P_{\leq 1}[\psi(t)]\|_{L^2}^2 + \||\nabla|^{1/2} P_{\geq 1}[\psi(t)]\|_{L^2}^2\big). \tag{1.8}$$

There is an extensive literature on the study of the water waves system. Without being exhaustive, we only discuss some previous works here and refer readers to the references therein.

Previous results on the local existence of the water waves system. – Due to the quasilinear nature of the water waves systems, to obtain the local existence, it is very important to get around the losing derivatives issue. Early works of Nalimov [32] and Yosihara [41] considered the local well-posedness of the small perturbation of a flat interface such that the Rayleigh-Taylor sign condition holds. It was first discovered by Wu [37, 38] that the Rayleigh-Taylor sign condition holds without the smallness assumptions in the infinite depth setting. She showed the local existence for arbitrary size of initial data in Sobolev spaces. After the breakthrough of Wu's work, there are many important works devoted to improve the understanding of local well-posedness of the full water waves system and the free boundary Euler equations. Christodoulou-Lindblad [10] and Lindblad [31] considered the gravity waves with vorticity. Beyer-Gunter [8] considered the effect of surface tension. Lannes [30] considered the finite depth setting. See also Shatah-Zeng [33], and Coutand-Shkoller [11]. It turns out that local well-posedness also holds even if the curvature of the interface is unbounded and the bottom is very rough even without regularity assumption, only a finite separation condition is required, see the works of Alazard-Burq-Zuily [1, 2] for more detailed and precise description of this result.

Previous results on the long time behavior of the water waves system. – The long time behavior of the water waves system is more difficult and challenging. To study the long time behavior, the low frequency part of the solution plays an essential role. It is very interesting to see that the water waves systems in different settings have very different behavior at the low frequency part. Even for a small perturbation of static solution and flat interface, we only have few results so far. Note that it is possible to develop the so-called "splash-singularity" for a large perturbation, see [9] and references therein for more details.

We first discuss previous results in the infinite depth setting. The first long-time result for the water waves system is due to the work of Wu [39], where she proved the almost global