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COHOMOLOGICAL RANK FUNCTIONS ON ABELIAN VARIETIES

BY Zhi JIANG AND Giuseppe PARESCHI

ABSTRACT. – Generalizing the *continuous rank function* of Barja-Pardini-Stoppino, in this paper we consider *cohomological rank functions* of $\mathbb{Q}$-twisted (complexes of) coherent sheaves on abelian varieties. They satisfy a natural transformation formula with respect to the Fourier-Mukai-Poincaré transform, which has several consequences. In many concrete geometric contexts these functions provide useful invariants. We illustrate this with two different applications, the first one to GV-subschemes and the second one to multiplication maps of global sections of ample line bundles on abelian varieties.

RÉSUMÉ. – En généralisant la fonction rang continu de Barja-Pardini-Stoppino, nous considérons dans cet article les fonctions rang cohomologiques des (complexes de) faisceaux cohérents sur les variétés abéliennes. Ils répondent à une formule de transformation naturelle par rapport à la transformée de Fourier-Mukai-Poincaré, ce qui a plusieurs conséquences. Dans de nombreux contextes géométriques concrets, ces fonctions fournissent des invariants utiles. Nous illustrons ceci avec deux applications différentes, la première pour les sous-schémas GV et la seconde pour la multiplication de sections globales des fibrés en droites amples sur les variétés abéliennes.

Introduction

In their paper [3] M.A. Barja, R. Pardini and L. Stoppino introduce and study the *continuous rank function* associated to a line bundle M on a variety X equipped with a morphism $X \xrightarrow{f} A$ to a polarized abelian variety. Motivated by their work, we consider more generally *cohomological rank functions*—defined in a similar way—of a bounded complex $\mathcal{F}$ of coherent sheaves on a polarized abelian variety $(A, \underline{l})$ defined over an algebraically closed field of characteristic zero. As it turns out, these functions often encode interesting geometric

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information. The purpose of this paper is to establish some general structure results about them and show some examples of application.

Let L be an ample line bundle on an abelian variety A, let $\underline{l} = c_1(L)$ and let $\varphi_{\underline{l}} : A \to \widehat{A}$ be the corresponding isogeny. The cohomological rank functions of $\mathcal{F} \in \mathrm{D}^b(A)$ with respect to the polarization $\underline{l}$ are initially defined (see Definition 2.1 below) as certain continuous rational-valued functions

$$h^i_{\mathcal{F},\underline{l}} : \mathbb{Q} \to \mathbb{Q}^{\geq 0}.\ ^{(1)}$$

The definition of these functions is peculiar to abelian varieties (and more generally to irregular varieties), as it uses the isogenies $\mu_b : A \to A$, $z \mapsto bz$. For $x \in \mathbb{Z}$, $h^i_{\mathcal{F},\underline{l}}(x) := h^i_{\mathcal{F}}(x\underline{l})$ coincides with the generic value of $h^i(A, \mathcal{F} \otimes L^x)$, for L varying among all line bundles representing $\underline{l}$. This is extended to all $x \in \mathbb{Q}$ using the isogenies μ_b. In fact the rational numbers $h^i_{\mathcal{F}}(x\underline{l})$ can be interpreted as generic cohomology ranks of the $\mathbb{Q}$-twisted coherent sheaf (or, more generally, $\mathbb{Q}$-twisted complex of coherent sheaves) $\mathcal{F}\langle x\underline{l}\rangle$ (in the sense of Lazarsfeld [15, §6.2A]).

The above functions are closely related to the Fourier-Mukai transform $\Phi_{\mathcal{P}} : \mathrm{D}^b(A) \to \mathrm{D}^b(\widehat{A})$ associated to the Poincaré line bundle and our first point consists in exploiting systematically such a relation. We prove the following transformation formula (Proposition 2.3 below):

$$h^i_{\mathcal{F}}(x\underline{l}) = \frac{(-x)^g}{\chi(\underline{l})} h^i_{\varphi_{\underline{l}}^* \Phi_{\mathcal{P}}(\mathcal{F})}(-\frac{1}{x}\underline{l}) \qquad \text{for } x \in \mathbb{Q}^-, \tag{0.1}$$

$$h^i_{\mathcal{F}}(x\underline{l}) = \frac{x^g}{\chi(\underline{l})} h^{g-i}_{\varphi_{\underline{l}}^* \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)}(\frac{1}{x}\underline{l}) \qquad \text{for } x \in \mathbb{Q}^+. \tag{0.2}$$

This has several consequences, summarized in the following theorem. The proof and discussion of the various items are found in Sections 2,3 and 4.

Theorem A. – *Let* $\mathcal{F} \in \mathrm{D}^b(A)$ *and* $i \in \mathbb{Z}$. *Let* $g = \dim A$.
(1) (Corollaries 2.4, 2.6, 2.7.) *For each* $x_0 \in \mathbb{Q}$ *there are* $\epsilon^-, \epsilon^+ > 0$ *and two* (explicit, see below) *polynomials* $P^+_{i,\mathcal{F},x_0}, P^-_{i,\mathcal{F},x_0} \in \mathbb{Q}[x]$ *of degree* $\leq g$ *such that* $P^+_{i,\mathcal{F},x_0}(x_0) = P^-_{i,\mathcal{F},x_0}(x_0)$ *and*

$$h^i_{\mathcal{F}}(x\underline{l}) = P^-_{i,\mathcal{F},x_0}(x) \quad \text{for } x \in (x_0 - \epsilon^-, x_0] \cap \mathbb{Q},$$
$$h^i_{\mathcal{F}}(x\underline{l}) = P^+_{i,\mathcal{F},x_0}(x) \quad \text{for } x \in [x_0, x_0 + \epsilon^+) \cap \mathbb{Q}.$$

(2) (Proposition 4.4) *Let* $k < g$ *and* $x_0 \in \mathbb{Q}$. *If the function* $h^i_{\mathcal{F},\underline{l}}$ *is strictly of class* $\mathcal{C}^k$ *at* x_0 *then the jump locus* $J^{i+}(\mathcal{F}\langle x_0\rangle)$ (see §4 for the definition) *has codimension* $\leq k+1$.
(3) (Theorem 3.2) *The function* $h^i_{\mathcal{F},\underline{l}}$ *extends to a continuous function* $h^i_{\mathcal{F},\underline{l}} : \mathbb{R} \to \mathbb{R}^{\geq 0}$. (2)

(1) In the present context the above-mentioned continuous rank function of Barja-Pardini-Stoppino is recovered as $h^0_{f_*M,\underline{l}}$ (see the notation above).
(2) This theorem provides partial answers to some questions raised, in the specific case of the above mentioned continuous rank functions $h^0_{f_*M,\underline{l}}$, in [3], e.g., Question 8.11. We also point out that for such functions item (3) of the present theorem, as well as some additional properties, were already proved in loc. cit. via different methods.

It follows from (1) that for $x_0 \in \mathbb{Q}$ the function $h^i_{\mathcal{F},\underline{l}}$ is smooth at x_0 if and only if the two polynomials $P^-_{i,\mathcal{F},x_0}$ and $P^+_{i,\mathcal{F},x_0}$ coincide. If this is not the case x_0 is called a *critical point*.

It turns out (Corollary 2.4) that for $x_0 \in \mathbb{Z}$ the two polynomials $P^-_{i,\mathcal{F}\,x_0}(x)$ and $P^+_{i,\mathcal{F},x_0}(x)$ are obtained from the Hilbert polynomials (with respect to the polarization $\underline{l}$) of the two coherent sheaves $\mathcal{G}^{i,+}_{x_0} := \varphi^*_{\underline{l}} R^i\Phi_{\mathcal{P}}(\mathcal{F} \otimes L^{x_0})$ and $\mathcal{G}^{i,-}_{x_0} := \varphi^*_{\underline{l}} R^{g-i}\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee \otimes L^{-x_0})$ in the following way:

$$P^-_{i,x_0,\mathcal{F}}(x) = \frac{(-x)^g}{\chi(\underline{l})}\chi_{\mathcal{G}^{i,+}_{x_0}}(-\frac{1}{x}\underline{l}), \tag{0.3}$$

$$P^+_{i,x_0,\mathcal{F}}(x) = \frac{x^g}{\chi(\underline{l})}\chi_{\mathcal{G}^{i,-}_{x_0}}(\frac{1}{x}\underline{l}). \tag{0.4}$$

For non-integer $x_0 \in \mathbb{Q}$ the two polynomials $P^-_{i,\mathcal{F}\,x_0}(x)$ and $P^+_{i,\mathcal{F},x_0}(x)$ have a similar description after reducing to the integer case (Corollary 2.6). Thus item (2) of Theorem A tells that, for $x_0 \in \mathbb{Q}$, the first k coefficients of the polynomials $P^-_{i,x_0,\mathcal{F}}(x)$ and $P^+_{i,x_0,\mathcal{F}}(x)$ coincide as soon as the rank function $\operatorname{Pic}^0 A \to \mathbb{Z}^{\geq 0}$ defined by $\alpha \mapsto h^i(\mathcal{F} \otimes L^{x_0} \otimes P_\alpha)$ has jump locus of codimension $\geq k+1$. In this last formulation we are implicitly assuming that x_0 is integer but for rational x_0 the situation is completely similar. However there might be irrational critical points (see e.g., Example 4.1), and at present we lack any similar interpretation for them.

In Section 5 we relate cohomological rank functions with the notions of GV, M-regular and IT(0)-sheaves, which are extended here to the $\mathbb{Q}$-twisted setting. We provide formulations of Hacon's results ([9]), and some related ones, which are simpler and more convenient even for usual sheaves. Finally, in Section 6 we point out some integral properties of cohomological rank functions.

It seems that the critical points of the function and the polynomials $P^-_{i,\mathcal{F},x_0}$ and $P^+_{i,\mathcal{F},x_0}$ are interesting and sometimes novel invariants in many concrete geometric situations. We exemplify this in the following two applications.

Application to GV-subschemes. – Our first example concerns GV-subschemes of principally polarized abelian varieties (here we will assume that the ground field is $\mathbb{C}$). This notion (we refer to Section 7 below for the definition and basic properties) was introduced in [24] in the attempt of providing a Fourier-Mukai approach to the minimal class conjecture ([6]), predicting that the only effective algebraic cycles representing the minimal classes $\frac{\theta^{g-d}}{(g-d)!} \in H^{2(g-d)}(A,\mathbb{Z})$ are (translates of) the subvarieties $\pm W_d(C)$ of Jacobians $J(C)$, and $\pm F$, the Fano surface in the intermediate Jacobian of a cubic threefold.

It is known that the subvarieties $W_d(C)$ of Jacobians, as well as the Fano surface ([10]) are GV-subschemes and that, on the other hand, geometrically non-degenerate GV-subschemes have minimal classes ([24]). Therefore it was conjectured in loc. cit. that geometrically non-degenerate GV-subschemes are either (translates of) $\pm W_d(C)$ or $\pm F$ as above. Denoting g the dimension of the p.p.a.v. and d the dimension of the subscheme, this is known only in a few cases: (i) for $d = 1$ and $d = g-2$ (loc. cit.); (ii) for $g = 5$, settled in the recent work [5], (iii) for Jacobians and intermediate Jacobians of generic cubic threefolds, as consequences of the main results of respectively [6] and [11]. In the recent work [28] it is proved that geometrically non-degenerate GV-subschemes are reduced and irreducible and that the geometric genus of their desingularizations is the expected one, namely $\binom{g}{d}$.

As an application of cohomological rank functions we prove that the Hilbert polynomial as well as all $h^i(\mathcal{O}_X)$'s are the expected ones:

Theorem B. – *Let X be geometrically non-degenerate GV-subvariety of dimension d of a principally polarized complex abelian variety (A, θ). Then:*
(1) *(Theorem 7.5)* $\chi_{\mathcal{O}_X}(x\underline{\theta}) = \sum_{i=0}^{d} \binom{g}{i}(x-1)^i$.
(2) *(Theorem 7.7)* $h^i(\mathcal{O}_X) = \binom{g}{i}$ *for all* $i = 1, \ldots, d$.

The proof of (1) is based on the study of the function $h^0_{\mathcal{O}_X}(x\underline{\theta})$ at the highest critical point (which turns out to be $x = 1$). (2) follows from (1) via another argument involving the Fourier-Mukai transform.

As a corollary of (2), combining with the results of [28] and [5], we have

Proposition C (Corollary 7.9). – *A 2-dimensional geometrically non-degenerate GV-subscheme is normal with rational singularities.*

Application to multiplication maps of global sections of line bundles and normal generation of abelian varieties. – Finally we illustrate the interest of cohomological rank functions in another example: the ideal sheaf of one (closed) point $p \in A$. The functions $h^i_{\mathcal{I}_p}(x\underline{l})$ seem to be highly interesting ones, especially in the perspective of basepoint-freeness criteria for primitive line bundles on abelian varieties. While we defer this to a subsequent paper, here we content ourselves to point out an elementary—but surprising—relation with multiplication maps of global sections of powers of line bundles. We consider the critical point

$$\beta(\underline{l}) = \inf\{x \in \mathbb{Q} \mid h^1_{\mathcal{I}_p}(x\underline{l}) = 0\}$$

(as the notation suggests, such notion does not depend on $p \in A$). A standard argument shows that in any case $\beta(\underline{l}) \leq 1$ and $\beta(\underline{l}) = 1$ if and only if the polarization $\underline{l}$ has base points, i.e., a line bundle L representing $\underline{l}$ (or, equivalently, all of them) has base points. Therefore, given a rational number $x = \frac{a}{b}$, it is suggestive to think that the inequality $\beta(\underline{l}) < x$ holds if and only if "the rational polarization $x\underline{l}$ is basepoint-free". Explicitly, this means the following: let $\mu_b : A \to A$ be the multiplication-by-b isogeny. Then, as it follows from the definition of cohomological rank functions, $\beta(\underline{l}) < \frac{a}{b}$ means that the finite scheme $\mu_b^{-1}(p)$ imposes independent conditions to all translates of a given line bundle L^{ab} with $c_1(L) = \underline{l}$. In turn $\frac{a}{b} = \beta(\underline{l})$ means that $\mu_b^{-1}(p)$ imposes dependent conditions to a proper closed subset of translates of the line bundle L^{ab} as above. [3] At present we don't know how to compute, or at least bound efficiently, the invariant $\beta(\underline{l})$ of a *primitive* polarization $\underline{l}$ (except for principal polarizations of course).

Here is one of the reasons why one is led to consider the number $\beta(\underline{l})$. Let $\underline{n}$ be another polarization on A. We assume that $\underline{n}$ is basepoint-free. Let N be a line bundle representing $\underline{n}$ and let M_N be the kernel of the evaluation map of global sections of N. We consider the critical point

$$s(\underline{n}) = \inf\{x \in \mathbb{Q} \mid h^1_{M_N}(x\underline{n}) = 0\}$$

[3] Writing $1 = \frac{b}{b}$ one recovers the usual notions of basepoint-freeness and base locus.

(again this invariant does not depend on the line bundle N representing $\underline{n}$). Well known facts about the vector bundles M_N yield that, given $x \in \mathbb{Z}^+$, $s(\underline{n}) \leq x$ if and only if the multiplication maps of global sections

$$H^0(N) \otimes H^0(N^x \otimes P_\alpha) \to H^0(N^{x+1} \otimes P_\alpha) \tag{0.5}$$

are surjective for general $\alpha \in \widehat{A}$ and, furthermore, $s(\underline{n}) < x$ if and only if the surjectivity holds for all $\alpha \in \widehat{A}$. Now the cohomological rank function leads to consider a "fractional" version of the maps (0.5). Writing $x = \frac{a}{b}$, these are the multiplication maps of global sections

$$H^0(N) \otimes H^0(N^{\mathrm{ab}} \otimes P_\alpha) \to H^0(\mu_b^*(N) \otimes N^{\mathrm{ab}} \otimes P_\alpha) \tag{0.6}$$

obtained by composing with the natural inclusion $H^0(N) \hookrightarrow H^0(\mu_b^* N)$. It follows that $s(\underline{n}) \leq \frac{a}{b}$ if and only if the maps (0.6) are surjective for general $\alpha \in \widehat{A}$. The strict inequality holds if the surjectivity holds for all $\alpha \in \widehat{A}$.(4) As a simple consequence of the formula (0.1) applied to $\underline{n} = h\underline{l}$ we have

Theorem D. – *Let h be an integer such that the polarization $h\underline{l}$ is basepoint-free* (hence $h \geq 1$ if $\underline{l}$ is basepoint-free, $h \geq 2$ otherwise). *Then*

$$s(h\underline{l}) = \frac{\beta(\underline{l})}{h - \beta(\underline{l})}. \tag{0.7}$$

Since $\beta(\underline{l}) \leq 1$ it follows that

$$s(h\underline{l}) \leq \frac{1}{h-1}$$

and equality holds if and only if $\beta(\underline{l}) = 1$, i.e., $\underline{l}$ has base points.

Surprisingly, this apparently unexpressive result summarizes, generalizes and improves what is known about the surjectivity of multiplication maps of global sections and projective normality of line bundles on abelian varieties. For example, the case $h = 2$ alone tells that $s(2\underline{l}) \leq 1$, with equality if and only if $\beta(\underline{l}) = 1$, i.e., $\underline{l}$ has base points. In view of the above, this means that the multiplication maps (0.5) for a second power $N = L^2$ and $x = 1$ are in any case surjective for general $\alpha \in \widehat{A}$, and in fact for all $\alpha \in \widehat{A}$ as soon as $\underline{l}$ is basepoint free. This is a classical result which implies all classical results on projective normality of abelian varieties proven via theta-groups by Mumford, Koizumi, Sekiguchi, Kempf, Ohbuchi and others (see [13] §6.1-2, [4] §7.1-2 and references therein, see also [20] and [23] for a theta-group-free treatment). We refer to Section 8 below for more on this.

Finally if $\underline{l}$ is basepoint-free and $h = 1$ the above theorem tells that $\beta(\underline{l}) < \frac{1}{2}$ if and only if the multiplication maps (0.5) for $N = L$ and $x = 1$ are surjective for all $\alpha \in \widehat{A}$. Using a well known argument, this implies

Corollary E. – *Assume that $\underline{l}$ is basepoint-free and $\beta(\underline{l}) < \frac{1}{2}$. Then $\underline{l}$ is projectively normal* (this means that all line bundles $L \otimes P_\alpha$ are projective normal).

(4) Again a simple computation shows that when x is an integer, writing $x = \frac{xb}{b}$ one recovers the usual notions of surjectivity of the maps (0.5) for every (resp. for general) $\alpha \in \widehat{A}$.

This is at the same time an explanation and a generalization of Ohbuchi's theorem ([19]) asserting that, given a polarization $\underline{n}$, $2\underline{n}$ is projectively normal as soon as $\underline{n}$ is basepoint-free.

Finally, we remark that, although the applications presented in this paper concern abelian varieties and their subvarieties, the study of cohomological rank functions can be applied to the wider context of *irregular varieties*, namely varieties having non-constant morphisms to abelian varieties, say $f : X \to A$ (as mentioned above this is indeed the point of view of the paper [3]). Given an element $\mathcal{F} \in \mathrm{D}^b(X)$, this can be done by considering the cohomological rank functions of the complex $Rf_* \mathcal{F}$.

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1. Notation and background material

We work on an algebraically closed ground field of characteristic zero.

A polarization $\underline{l}$ on an abelian variety is the class of an ample line bundle L in $\mathrm{Pic} A/\mathrm{Pic}^0 A$. The corresponding isogeny is denoted

$$\varphi_{\underline{l}} : A \to \widehat{A}$$

where $\widehat{A} := \mathrm{Pic}^0 A$. For $b \in \mathbb{Z}$

$$\mu_b : A \to A \qquad z \mapsto bz$$

denotes the multiplication-by-b homomorphism.

Let A be a g-dimensional abelian variety. We denote $\mathcal{P}$ the Poincaré line bundle on $A \times \widehat{A}$. For $\alpha \in \widehat{A}$ the corresponding line bundle in A is denoted by P_α, i.e., $P_\alpha = \mathcal{P}_{|A \times \{\alpha\}}$. We always denote $\hat{e}$ the origin of $\hat{A}$.

Let $\mathrm{D}^b(A)$ be the bounded derived category of coherent sheaves on A and denote by

$$\Phi_{\mathcal{P}}^{A \to \widehat{A}} : \mathrm{D}^b(A) \to \mathrm{D}^b(\widehat{A})$$

the Fourier-Mukai functor associated to $\mathcal{P}$. It is an equivalence ([17]), whose quasi-inverse is

$$\Phi_{\mathcal{P}^\vee[g]}^{\widehat{A} \to A} : \mathrm{D}^b(\widehat{A}) \to \mathrm{D}^b(A). \tag{1.1}$$

When possible we will suppress the direction of the functor from the notation, writing simply $\Phi_{\mathcal{P}}$. Since $\mathcal{P}^\vee = (-1_A, 1_{\widehat{A}})^* \mathcal{P} = (1_A, -1_{\widehat{A}})^* \mathcal{P}$ it follows that $\Phi_{\mathcal{P}^\vee} = (-1)^* \Phi_{\mathcal{P}}$. Finally, we will denote $R^i \Phi_{\mathcal{P}}$ the induced i-th cohomology functors.

For the reader's convenience we list some useful facts, in use throughout the paper, concerning the above Fourier-Mukai equivalence.

Exchange of direct and inverse image of isogenies ([17] (3.4)). – Let $\varphi : A \to B$ be an isogeny of abelian varieties and let $\hat{\varphi} : \widehat{B} \to \widehat{A}$ be the dual isogeny. Then

$$\hat{\varphi}^* \Phi_{\mathcal{P}_A}(\mathcal{F}) = \Phi_{\mathcal{P}_B} \varphi_*(\mathcal{F}), \qquad \hat{\varphi}_* \Phi_{\mathcal{P}_B}(\mathcal{G}) = \Phi_{\mathcal{P}_A} \varphi^*(\mathcal{G}). \tag{1.2}$$

Exchange of derived tensor product and derived Pontryagin product ([17] (3.7)). – We have

$$\Phi_{\mathcal{P}}(\mathcal{F} * \mathcal{G}) = (\Phi_{\mathcal{P}}\mathcal{F}) \otimes (\Phi_{\mathcal{P}}\mathcal{G}) \qquad \Phi_{\mathcal{P}}(\mathcal{F} \otimes \mathcal{G}) = (\Phi_{\mathcal{P}}\mathcal{F}) * (\Phi_{\mathcal{P}}\mathcal{G})[g]. \tag{1.3}$$

Serre-Grothendieck duality ([17] (3.8). See also [26] Lemma 2.2). – As customary, for a given projective variety X (in what follows X will be either A or $\widehat{A}$) and $\mathcal{F} \in \mathrm{D}^b(X)$, we denote $\mathcal{F}^\vee := \mathcal{R}Hom(\mathcal{F}, \mathcal{O}_X) \in \mathrm{D}^b(X)$. Then

$$(\Phi_{\mathcal{P}}\mathcal{F})^\vee = \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)[g]. \tag{1.4}$$

The transform of a non-degenerate line bundle ([17] Prop. 3.11(1)). – Given an ample line bundle on A, the Fourier-Mukai transform $\Phi_{\mathcal{P}}(L)$ is a locally free sheaf (concentrated in degree 0) on $\widehat{A}$, denoted by $\widehat{L}$, of rank equal to $h^0(L)$. Moreover

$$\varphi_{\underline{l}}^*\widehat{L} \simeq H^0(L) \otimes L^{-1} = (L^{-1})^{\oplus h^0(L)}. \tag{1.5}$$

The Pontryagin product with a non-degenerate line bundle ([17] (3.10)). – Given a non-degenerate line bundle N on A, we denote $\underline{n} = c_1(N)$. Let $\mathcal{F} \in \mathrm{D}^b(A)$. Then

$$\mathcal{F} * N = N \otimes \varphi_{\underline{n}}^*\big(\Phi_{\mathcal{P}}((-1)^*\mathcal{F}) \otimes N)\big). \tag{1.6}$$

(Hyper)cohomology and derived tensor product ([26] Lemma 2.1). – Let $\mathcal{F} \in \mathrm{D}^b(A)$ and $\mathcal{G} \in \mathrm{D}^b(\widehat{A})$. We have

$$H^i(A, \mathcal{F} \otimes \Phi_{\mathcal{P}}^{\widehat{A}\to A}(\mathcal{G})) = H^i(\widehat{A}, \Phi_{\mathcal{P}}^{A\to\widehat{A}}(\mathcal{F}) \otimes \mathcal{G}). \tag{1.7}$$

2. Cohomological rank functions on abelian varieties

In this section we define a certain non-negative rational number as the rank of the cohomology of a coherent sheaf (or, more generally, of the hypercohomology of a complex of coherent sheaves) twisted with a rational power of a polarization. This definition is already found in [1] and, somewhat implicitly, a notion like that was already in use in [14] (proof of Thm 17.12) and [27] (proof of Thm 4.1). This provides rational cohomological rank functions satisfying certain transformation formulas under Fourier-Mukai transform (Prop. 2.4 below). It follows that these functions are polynomial almost everywhere and extend to continuous functions on an open neighborhood of $\mathbb{Q}$ in $\mathbb{R}$ (Corollaries 2.6 and 2.7).

Definition 2.1. – (1) Given $\mathcal{F} \in \mathrm{D}^b(A)$ and $i \in \mathbb{Z}$, define

$$h^i_{\mathrm{gen}}(A, \mathcal{F})$$

as the dimension of hypercohomology $H^i(A, \mathcal{F} \otimes P_\alpha)$, for α general in $\widehat{A}$.[(5)]
(2) Given $\mathcal{F} \in \mathrm{D}^b(A)$, a polarization $\underline{l}$ on A and $x = \frac{a}{b} \in \mathbb{Q}$, $b > 0$, we define

$$h^i_{\mathcal{F}}(x\underline{l}) = b^{-2g}\, h^i_{\mathrm{gen}}(A, \mu_b^*(\mathcal{F}) \otimes L^{ab}).$$

(5) It is well known that hypercohomology groups as the above satisfy the usual base-change and semicontinuity properties, see e.g., [26] proof of Lemma 3.6 and [8] 7.7.4 and Remark 7.7.12(ii).

The definition is dictated from the fact that the degree of $\mu_b : A \to A$ is b^{2g} (see the previous section for the notation) and $\mu_b^*(\underline{l}) = b^2\underline{l}$. Therefore the pullback via μ_b of the class $\frac{a}{b}\underline{l}$ is $ab\underline{l}$. It is easy to check that the definition does not depend on the representation $x = \frac{a}{b}$. For example, if $n \in \mathbb{Z}$, writing $n = \frac{nb}{b}$ one gets

$$\begin{aligned} b^{-2g} h^i_{\mathrm{gen}}((\mu_b^* \mathcal{F}) \otimes L^{b^2 n}) &= b^{-2g} h^i_{\mathrm{gen}}(\mu_b^*(\mathcal{F} \otimes L^n)) \\ &= b^{-2g} \sum_{\alpha \in \hat{\mu}_b^{-1}(\hat{e})} h^i_{\mathrm{gen}}(\mathcal{F} \otimes L^n \otimes P_\alpha) = h^i_{\mathrm{gen}}(\mathcal{F} \otimes L^n), \end{aligned}$$

where $\hat{e}$ is the identity point of $\widehat{A}$ and $\hat{\mu}_b : \widehat{A} \to \widehat{A}$ is the dual isogeny.

REMARK 2.2. – [Coherent sheaves $\mathbb{Q}$-twisted by a polarization] Let $\underline{l}$ be a polarization on our abelian variety A. Following Lazarsfeld ([15]), but somewhat more restrictively, we will define *coherent sheaves $\mathbb{Q}$-twisted by $\underline{l}$* as equivalence classes of pairs $(\mathcal{F}, x\underline{l})$ where $\mathcal{F}$ is a coherent sheaf on A and $x \in \mathbb{Q}$, with respect to the equivalence relation generated by $(\mathcal{F} \otimes L^h, x\underline{l}) \sim (\mathcal{F}, (h + x)\underline{l})$, for L a line bundle representing $\underline{l}$ and $h \in \mathbb{Z}$. Such thing is denoted $\mathcal{F}\langle x\underline{l}\rangle$ (note that $\mathcal{F} \otimes P_\alpha\langle x\underline{l}\rangle = \mathcal{F}\langle x\underline{l}\rangle$ for $\alpha \in \widehat{A}$). Similarly, one can define complexes of coherent sheaves $\mathbb{Q}$-twisted by the polarization $\underline{l}$. Now the quantity $h^i_{\mathcal{F}}(x\underline{l})$ depends only on the $\mathbb{Q}$-twisted complex $\mathcal{F}\langle x\underline{l}\rangle$ and one may think of it as the (generic) cohomology rank $h^i(A, \mathcal{F}\langle x\underline{l}\rangle)$.

Some immediate basic properties of generic cohomology ranks defined above are:

(a) $\chi_{\mathcal{F}}(x\underline{l}) = \sum_i (-1)^i h^i_{\mathcal{F}}(x\underline{l})$, where $\chi_{\mathcal{F}}(x\underline{l})$ is the Hilbert polynomial, i.e., the Euler characteristic.

(b) Serre duality: $h^i_{\mathcal{F}}(x\underline{l}) = h^{g-i}_{\mathcal{F}^\vee}(-x\underline{l})$.

(c) Serre vanishing: *given a coherent sheaf $\mathcal{F}$ there is a $x_0 \in \mathbb{Q}$ such that $h^i_{\mathcal{F}}(x\underline{l}) = 0$ for all $i > 0$ and for all rational $x \geq x_0$.*

Proof of (c). – It is well known that there is $n_0 \in \mathbb{Z}$ such that $h^i(A, \mathcal{F} \otimes L^{n_0} \otimes P_\alpha) = 0$ for all $i > 0$ and for all $\alpha \in \mathrm{Pic}^0 X$. Following the terminology of Mukai, this condition is referred to as follows: *$\mathcal{F} \otimes L^n$ satisfies IT(0)* (the Index Theorem with index 0, see also §5 below). Therefore, for all $b \in \mathbb{Z}^+$, $\mu_b^*(\mathcal{F}) \otimes L^{b^2 n_0}$ satisfies IT(0). The tensor product of a coherent IT(0) sheaf with a locally free IT(0) sheaf is IT(0) (see e.g., [27, Prop. 3.1] for a stronger result). Therefore $\mu_b^*(\mathcal{F}) \otimes L^m$ satisfies IT(0) for all $b \in \mathbb{Z}^+$ and $m \geq b^2 n_0$. This is more than enough to ensure that $h^i_{\mathcal{F}}(x\underline{l}) = 0$ for all rational numbers $x \geq n_0$.[6] □

The following proposition describes the behavior of the generic cohomology ranks with respect to the Fourier-Mukai transform.

PROPOSITION 2.3. – *Let $\mathcal{F} \in \mathrm{D}^b(A)$ and let $\underline{l}$ be a polarization on A. Then, for $x \in \mathbb{Q}^+$*

$$h^i_{\mathcal{F}}(x\underline{l}) = \frac{x^g}{\chi(\underline{l})} h^{g-i}_{\varphi_{\underline{l}}^* \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)}\left(\frac{1}{x}\underline{l}\right)$$

[6] More precisely this proves, in the terminology of Section 5 below, that the $\mathbb{Q}$-twisted coherent sheaves $\mathcal{F}\langle x\underline{l}\rangle$ satisfy IT(0) for all $x \in \mathbb{Q}^{\geq n_0}$.

and, for $x \in \mathbb{Q}^-$,

$$h^i_{\mathcal{F}}(x\underline{l}) = \frac{(-x)^g}{\chi(\underline{l})} h^i_{\varphi_{\underline{l}}^* \Phi_{\mathcal{P}}(\mathcal{F})}(-\frac{1}{x}\underline{l}).$$

Proof. – Let us start with the case $x = \frac{a}{b} \in \mathbb{Q}^+$. Then,

$$\begin{aligned} h^i_{\mathcal{F}}(x\underline{l}) = \frac{1}{b^{2g}} h^i_{\mathrm{gen}}(A, \mu_b^* \mathcal{F} \otimes L^{\mathrm{ab}}) &= \frac{1}{b^{2g}} \dim \mathrm{Ext}^i_A(\mu_b^* \mathcal{F}^\vee, L^{\mathrm{ab}}_\alpha) \\ &= \frac{1}{b^{2g}} \dim \mathrm{Ext}^i_{\hat{A}}(\Phi_{\mathcal{P}}(\mu_b^* \mathcal{F}^\vee), \Phi_{\mathcal{P}}(L^{\mathrm{ab}}_\alpha)), \end{aligned}$$

where $\alpha \in \hat{A}$ is general, $L^{\mathrm{ab}}_\alpha := L^{\mathrm{ab}} \otimes P_\alpha$, and the last equality holds by Mukai's equivalence [17].

Note that, by (1.2), $\Phi_{\mathcal{P}}(\mu_b^* \mathcal{F}^\vee) \simeq \hat{\mu}_{b*} \Phi_{\mathcal{P}}(\mathcal{F}^\vee)$ where $\hat{\mu}_b : \widehat{A} \to \widehat{A}$ is the multiplication by b on $\widehat{A}$. By (1.5) $R\Phi_{\mathcal{P}}(L^{\mathrm{ab}}_\alpha) := \widehat{L^{\mathrm{ab}}_\alpha}$ is a vector bundle on $\widehat{A}$ and

$$\mu_{\mathrm{ab}}^* \varphi_{\underline{l}}^* \widehat{L^{\mathrm{ab}}_\alpha} = \varphi_{ab\underline{l}}^* \widehat{L^{\mathrm{ab}}_\alpha} \simeq ((L^{\mathrm{ab}}_\alpha)^{-1})^{\oplus h^0(L^{\mathrm{ab}})}. \tag{2.1}$$

Hence, for general $\alpha \in \widehat{A}$,

$$\begin{aligned} h^i_{\mathcal{F}}(x\underline{l}) &= \frac{1}{b^{2g}} \dim \mathrm{Ext}^i_{\widehat{A}}(\hat{\mu}_{b*} \Phi_{\mathcal{P}}(\mathcal{F}^\vee), \widehat{L^{\mathrm{ab}}_\alpha}) = \frac{1}{b^{2g}} \dim \mathrm{Ext}^i_{\widehat{A}}(\Phi_{\mathcal{P}}(\mathcal{F}^\vee), \hat{\mu}_b^* \widehat{L^{\mathrm{ab}}_\alpha}) \\ &= \frac{1}{b^{2g}} \dim \mathrm{Ext}^{g-i}_{\widehat{A}}(\hat{\mu}_b^* \widehat{L^{\mathrm{ab}}_\alpha}, \Phi_{\mathcal{P}}(\mathcal{F}^\vee)) \\ &= \frac{1}{\deg \hat{\mu}_a \deg \varphi_{\underline{l}}} \frac{1}{b^{2g}} \dim \mathrm{Ext}^{g-i}_A(\varphi_{\underline{l}}^* \hat{\mu}_a^* \hat{\mu}_b^* \widehat{L^{\mathrm{ab}}_\alpha}, \varphi_{\underline{l}}^* \hat{\mu}_a^* \Phi_{\mathcal{P}}(\mathcal{F}^\vee)) \\ &= \frac{1}{\chi(\underline{l})^2} \frac{1}{a^{2g} b^{2g}} \dim \mathrm{Ext}^{g-i}_A(\varphi_{ab\underline{l}}^* \widehat{L^{\mathrm{ab}}_\alpha}, \mu_a^* \varphi_{\underline{l}}^* \Phi_{\mathcal{P}}(\mathcal{F}^\vee)) \\ &\overset{(2.1)}{=} \frac{1}{\chi(\underline{l})} \frac{1}{a^g b^g} h^{g-i}(A, \mu_a^* \varphi_{\underline{l}}^* \Phi_{\mathcal{P}}(\mathcal{F}^\vee) \otimes L^{\mathrm{ab}}_\alpha). \end{aligned}$$

We also note that $(-1)^*_{\widehat{A}} \Phi_{\mathcal{P}}(\mathcal{F}^\vee) = \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)$ and $(-1)^*_A \underline{l} = \underline{l}$. Hence, applying $(-1)^*_A$, we get

$$h^i_{\mathcal{F}}(x\underline{l}) = \frac{1}{\chi(\underline{l})} \frac{1}{a^g b^g} h^{g-i}_{\mathrm{gen}}(\mu_a^* \varphi_{\underline{l}}^* \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \otimes L^{\mathrm{ab}}) = \frac{1}{\chi(\underline{l})} \frac{a^g}{b^g} h^{g-i}_{\varphi_{\underline{l}}^* \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)}(\frac{b}{a}\underline{l}).$$

By similar argument (or by Serre duality) we get the equalities when $x \in \mathbb{Q}^-$. □

COROLLARY 2.4. – *Under the same hypothesis and notation of the previous proposition, for each $i \in \mathbb{Z}$ there are $\epsilon^-, \epsilon^+ > 0$ and two polynomials $P^-_{i,\mathcal{F}}, P^+_{i,\mathcal{F}} \in \mathbb{Q}[x]$ of degree $\leq \dim A$ such that, for $x \in (-\epsilon^-, 0) \cap \mathbb{Q}$,*

$$h^i_{\mathcal{F}}(x\underline{l}) = P^-_{i,\mathcal{F}}(x)$$

and, for $x \in (0, \epsilon^+) \cap \mathbb{Q}$

$$h^i_{\mathcal{F}}(x\underline{l}) = P^+_{i,\mathcal{F}}(x).$$

More precisely

$$\begin{aligned} h^i_{\mathcal{F}}(x\underline{l}) &= \frac{(-x)^g}{\chi(\underline{l})} \chi_{\varphi_{\underline{l}}^* R^i \Phi_{\mathcal{P}}(\mathcal{F})}(-\frac{1}{x}\underline{l}) \quad \text{for } x \in (-\epsilon^-, 0) \cap \mathbb{Q} \\ h^i_{\mathcal{F}}(x\underline{l}) &= \frac{x^g}{\chi(\underline{l})} \chi_{\varphi_{\underline{l}}^* R^{g-i} \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)}(\frac{1}{x}\underline{l}) \quad \text{for } x \in (0, \epsilon^+) \cap \mathbb{Q}. \end{aligned}$$

Proof. – The statement follows from Proposition 2.4 via Serre vanishing (see (c) above in this section). Indeed for a sufficiently small $x \in \mathbb{Q}^+$ we have that $h^k_{\varphi_{\underline{l}}^* R^j \Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)}(\frac{1}{x}\underline{l})) = 0$ for all $k \neq 0$ and all j. Therefore the hypercohomology spectral sequence computing $h^{g-i}_{\varphi_{\underline{l}}^* \Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)}(\frac{1}{x}\underline{l}))$ [7] collapses so that

$$h^{g-i}_{\varphi_{\underline{l}}^* \Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)}(\frac{1}{x}\underline{l})) = h^0_{\varphi_{\underline{l}}^* R^{g-i} \Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)}(\frac{1}{x}\underline{l})) = \chi_{\varphi_{\underline{l}}^* R^{g-i} \Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)}(\frac{1}{x}\underline{l})).$$

This proves the statement for $x > 0$. The proof for the case $x < 0$ is the same. □

REMARK 2.5. – It follows from the proof that one can take as ϵ^- the minimum, for all i, of $\frac{1}{x_i}$, where x_i is a bound ensuring Serre vanishing for twists with powers of L of the sheaf $\varphi_{\underline{l}}^* R^i \Phi_{\mathscr{P}}(\mathscr{F})$. Similarly for ϵ^+.

The next corollary shows that the statement of the previous corollary holds more generally in $\mathbb{Q}$-twisted setting.

COROLLARY 2.6. – *Same hypothesis and notation of the previous proposition. Let $x_0 \in \mathbb{Q}$. For each $i \in \mathbb{Z}$ there are $\epsilon^-, \epsilon^+ > 0$ and two polynomials $P^-_{i,\mathscr{F},x_0}, P^+_{i,\mathscr{F},x_0} \in \mathbb{Q}[x]$ of degree $\leq \dim A$ such that, for $x \in (x_0 - \epsilon^-, x_0) \cap \mathbb{Q}$,*

$$h^i_{\mathscr{F}}(x\underline{l}) = P^-_{i,\mathscr{F},x_0}(x)$$

and, for $x \in (x_0, x_0 + \epsilon^+) \cap \mathbb{Q}$

$$h^i_{\mathscr{F}}(x\underline{l}) = P^+_{i,\mathscr{F},x_0}(x).$$

Proof. – This follows by reducing to the previous corollary via the formula

$$h^i_{\mathscr{F}}((x_0 + y)\underline{l}) = b^{-2g} h^i_{\mu_b^*(\mathscr{F}) \otimes L^{ab}}(b^2 y\underline{l})$$

for $x_0 = \frac{a}{b}$, $b > 0$. □

As a consequence we have

COROLLARY 2.7. – *The functions $h^i_{\mathscr{F},\underline{l}} : \mathbb{Q} \to \mathbb{Q}^{\geq 0}$ extend to continuous functions $h^i_{\mathscr{F},L} : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}$ containing $\mathbb{Q}$, satisfying the condition of Corollary 2.6 above, namely for each $x_0 \in U$ there exist $\epsilon^-, \epsilon^+ > 0$ and two polynomials $P^-_{i,\mathscr{F},x_0}, P^+_{i,\mathscr{F},x_0} \in \mathbb{Q}[x]$ of degree $\leq \dim A$, having the same value at x_0, such that*

$$h^i_{\mathscr{F}}(x_{\underline{l}}) = \begin{cases} P^-_{i,\mathscr{F},x_0}(x) & \textit{for } x \in (x_0 - \epsilon^-, x_0] \\ P^+_{i,\mathscr{F},x_0}(x) & \textit{for } x \in [x_0, x_0 + \epsilon^+). \end{cases}$$

[7] This expression stands for the spectral sequence computing the hypercohomology groups $H^{g-i}(A, \mu_a^*(\varphi_{\underline{l}}^* \Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)) \otimes L^{ab} \otimes P_\alpha)$ for $\frac{b}{a} = \frac{1}{x}$ and a general $\alpha \in \mathrm{Pic}^0 X$.

Proof. – Let $x_0 \in \mathbb{Q}$. The assertion follows from Corollary 2.6 because, for $x_0 \in \mathbb{Q}$, $P^-_{i,\mathcal{F},x_0}(x_0) = P^+_{i,\mathcal{F},x_0}(x_0) = h^i_{\mathcal{F}}(x_0\underline{l})$. To prove this we can assume, using Corollary 2.6, that $x_0 = 0$. By Corollary 2.4 we have that $P^-_{i,\mathcal{F},x_0}(0)$ (resp. $P^+_{i,\mathcal{F},x_0}(0)$) coincide, up the same multiplicative constant, with the coefficients of degree g of the Hilbert polynomial of the sheaves $\varphi^*_{\underline{l}} R^i\Phi_{\mathcal{P}}(\mathcal{F})$, resp. $\varphi^*_{\underline{l}} R^{g-i}\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)$. Hence they coincide, up to the same multiplicative constant, with the generic ranks of the above sheaves. By cohomology and base change, and Serre duality, such generic ranks coincide with $h^i_{\mathcal{F},\underline{l}}(0)$. □

REMARK 2.8. – It seems likely that $U = \mathbb{R}$, hence the cohomological rank functions would be piecewise-polynomial (compare [3, Question 8.11]). This would follow from the absence of accumulation points in $\mathbb{R} \smallsetminus U$, but at present we don't know how to prove that. In any case, in the next section we prove that the cohomological rank functions extend to continuous functions on the whole $\mathbb{R}$.

Given two objects $\mathcal{F}$ and $\mathcal{G}$ in $\mathrm{D}^b(A)$ and $f \in \mathrm{Hom}_{\mathrm{D}^b(A)}(\mathcal{F},\mathcal{G})$ one can define similarly the i-th cohomological rank, nullity and corank of the maps twisted with a rational multiple of a polarization $\underline{l}$ as the generic rank, nullity and corank of the maps

$$H^i(A,\mu_b^*(\mathcal{F})\otimes L^{\mathrm{ab}}\otimes P_\alpha)\to H^i(A,\mu_b^*(\mathcal{G})\otimes L^{\mathrm{ab}}\otimes P_\alpha).$$

This gives rise to functions $\mathbb{Q}\to\mathbb{Q}^{\geq}0$ satisfying the same properties. Let us consider, for example, the rank, (the kernel and the corank have completely similar description) and let us denote it $rk\big(h^i_f(x\underline{l})\big)$.

PROPOSITION 2.9. – *Let $x_0\in\mathbb{Q}$. For each $i\in\mathbb{Z}$ there are $\epsilon^-,\epsilon^+>0$ and two polynomials $P^-_{i,f,x_0}, P^+_{i,f,x_0}\in\mathbb{Q}[x]$ of degree $\leq \dim A$ such that, for $x\in(x_0-\epsilon^-,x_0)\cap\mathbb{Q}$,*

$$rk\big(h^i_f(x\underline{l})\big) = P^-_{i,f,x_0}(x)$$

and, for $x\in(x_0,x_0+\epsilon^+)\cap\mathbb{Q}$

$$rk\big(h^i_f(x\underline{l})\big) = P^+_{i,f,x_0}(x).$$

Proof. – As above, we can assume that $x_0=0$. By Corollary 2.4 and its proof there is a $\epsilon^->0$ such that for $x=\frac{a}{b}\in(-\epsilon^-,0)$ (with $a<0$ and $b>0$), $rk\big(h^i_f(x\underline{l})\big)$ coincides with $\frac{(-x)^g}{\chi(\underline{l})}rk(F_{-\frac{1}{x}})$ where $F_{-\frac{1}{x}}$ is the natural map

$$F_{-\frac{1}{x}}: H^0\big(\mu^*_{-a}(\varphi^*_{\underline{l}}R^i\Phi_{\mathcal{P}}\mathcal{F})\otimes L^{-ab}\big)\to H^0\big(\mu^*_{-a}(\varphi^*_{\underline{l}}R^i\Phi_{\mathcal{P}}\mathcal{G})\otimes L^{-ab}\big).$$

By an easy calculation with Serre vanishing (see (c) in this Section), up to taking a smaller ϵ^- the image of the map $F_{-\frac{1}{x}}$ is H^0 of the image of the map coherent sheaves

$$\mu^*_{-a}(\varphi^*_{\underline{l}}R^i\Phi_{\mathcal{P}}\mathcal{F})\otimes L^{-ab}\to\mu^*_{-a}(\varphi^*_{\underline{l}}R^i\Phi_{\mathcal{P}}\mathcal{G})\otimes L^{-ab}$$

and its dimension is

$$\chi_{Im(\varphi^*_{\underline{l}}R^i\Phi_{\mathcal{P}}(f))}(-\frac{1}{x}\underline{l}).$$

In conclusion, for $x\in(-\epsilon^-,0)$

$$rk\big(h^i_f(x\underline{l})\big)=\frac{(-x)^g}{\chi(\underline{l})}\chi_{Im(\varphi^*_{\underline{l}}R^i\Phi_{\mathcal{P}}(f))}(-\frac{1}{x}\underline{l}) := P^-_{i,f,0}(x).$$

Similarly, for $x \in (0, \epsilon^+)$

$$rk\big(h^i_f(x\underline{l})\big) = \frac{(x)^g}{\chi(\underline{l})}\chi_{Im(\varphi^*_{\underline{l}} R^{g-i}\Phi_{\mathscr{P}^\vee}(f))}(\frac{1}{x}\underline{l}) := P^+_{i,f,0}(x). \qquad \square$$

3. Continuity as real functions

The aim of this section is to prove Theorem 3.2 below, asserting that the cohomological rank functions extend to continuous functions on the whole $\mathbb{R}$ (see Remark 2.8). We start with a version of Serre's vanishing needed in the proof.

LEMMA 3.1. – *Let A be an abelian variety and let L be a very ample line bundle on A. Let $\mathscr{F}$ be a coherent sheaf of dimension n on A. There exist two integers M^- and M^+ such that for all integers $m \in \mathbb{Z}^+$, for all $k = 0, \dots n$ and all sufficiently general complete intersections $Z_k = D_1 \cap D_2 \cap \cdots \cap D_k$ of k divisors $D_i \in |m^2\rho_i L|$ with $0 < \rho_i < 1$ rational with $m^2\rho_i \in \mathbb{Z}$* (here we understand $Z_0 = A$)*, and for all $\alpha \in \widehat{A}$, the following conditions hold:*

$$\begin{cases} h^i(\mu_m^*\mathscr{F}|_{Z_k} \otimes L^{m^2t} \otimes P_\alpha) = 0 \\ \qquad \textit{for all} \quad i \geq 1 \textit{ and } t \in \mathbb{Z}^{\geq M^+}, \\ h^i(\mu_m^*\mathscr{F}|_{Z_k} \otimes L^{m^2t} \otimes P_\alpha) = \chi(\mathscr{E}xt^{g-i}(\mu_m^*\mathscr{F}|_{Z_k}, \mathcal{O}_A) \otimes L^{-m^2t}) \\ \qquad \textit{for all } i \leq n-1-s \textit{ and } t \in \mathbb{Z}^{\leq M^-}. \end{cases}$$

The pair (M^-, M^+) will be referred to as an effective cohomological bound for $\mathscr{F}$.

Proof. – Note that the statement makes sense since L is assumed to be very ample and $\tau_i := m^2\rho_i \in \mathbb{Z}^+$. Since Z_k is a general complete intersection, the Koszul resolution of $\mathcal{O}_{Z_k}$, tensored with $\mu_m^*\mathscr{F}$

$$0 \to \mu_m^* F \otimes L^{-\sum_i \tau_i} \to \cdots \to \mu_m^*\mathscr{F} \otimes (\bigoplus_i L^{-\tau_i}) \to \mu_m^*\mathscr{F} \to \mu_m^* F|_{Z_k} \to 0 \tag{3.1}$$

is exact. Therefore the bound of the upper line is a variant of Serre vanishing, in the version of the previous section, via a standard diagram-chase.

Concerning the lower line, we first prove it for $k = 0$. By Serre duality

$$\begin{aligned} H^i(\mu_m^*\mathscr{F} \otimes L^{m^2t} \otimes P_\alpha) &= H^{g-i}((\mu_m^*\mathscr{F})^\vee \otimes L^{-m^2t} \otimes P_\alpha^\vee) \\ &= H^{g-i}(\mu_m^*(\mathscr{F}^\vee) \otimes L^{-m^2t} \otimes P_\alpha^\vee). \end{aligned}$$

Since $\mathscr{F}$ is a coherent sheaf of dimension n, $\mathscr{E}xt^j(F, \mathcal{O}_A)$ vanishes for $j < g-n$ while it has codimension $\geq j$ with support contained in the support of $\mathscr{F}$ for $j \geq g-n$ (see for instance [12, Proposition 1.1.6]). We apply Serre vanishing to find an integer N such that such that

$$H^j(\mu_m^*\mathscr{E}xt^i(\mathscr{F}, \mathcal{O}_A) \otimes L^{-m^2t} \otimes P_\alpha^\vee) = 0 \quad \text{for all } t \in \mathbb{Z} \text{ such that } -t \geq N \text{ and } j \geq 1.$$

The statement of the bottom line for $k = 0$ follows via the spectral sequence

$$H^h(\mu_m^*\mathscr{E}xt^{g-i-h}(\mathscr{F}, \mathcal{O}_A) \otimes L^{-m^2t} \otimes P_\alpha^\vee) \Rightarrow H^{g-i}((\mu_m^*\mathscr{F})^\vee \otimes L^{-m^2t} \otimes P_\alpha^\vee).$$

At this point the statement of the lower line for all $k \leq g - n$ follows as above from the case $k = 0$ and the fact that for a general choice of a very ample divisor D we have a short exact sequence for all $j \geq 0$

$$0 \to \mu_m^* \mathcal{E}xt^j(\mathcal{F}, \mathcal{O}_A) \to \mu_m^* \mathcal{E}xt^j(\mathcal{F}, \mathcal{O}_A) \otimes \mathcal{O}_A(D) \to \mathcal{E}xt^{j+1}(\mu_m^* \mathcal{F}|_D, \mathcal{O}_A) \to 0$$

(see [12, Lemma 1.1.13]). □

THEOREM 3.2. – *Let A be an abelian variety, let $\underline{l}$ be a polarization on A and $\mathcal{F} \in \mathrm{D}^b(A)$. The functions $x \mapsto h^i_{\mathcal{F}}(x\underline{l})$ extend to continuous functions on $\mathbb{R}$. Such functions are bounded above by a polynomial function of degree at most $n = \dim \mathcal{F}$, whose coefficients involve only the intersection numbers of the support of $\mathcal{F}$ with powers of L, the ranks of the cohomology sheaves of $\mathcal{F}$ on the generic points of their support, and an effective bound (N^-, N^+) of generic cohomology of $\mathcal{F}$.*

Proof. – We can assume that $\underline{l}$ is very ample. The proof will be in some steps. To begin with, we prove the statement under the assumption that $\mathcal{F}$ is a pure sheaf. Let $V_1, \dots, V_s$ be the irreducible components of the support of $\mathcal{F}$ with reduced scheme structures. Hence each V_j is an integral variety. Let t_j be the length of $\mathcal{F}$ at the generic point of V_j and define

$$u(\mathcal{F}) := \sum_j t_j (V_j \cdot L^n)_A.$$

We have seen in the previous lemma that $h^i_{\mathcal{F}}(x\underline{l})$ are natural polynomial functions for $x \leq M^-$ and $x \geq M^+$. We now deal with the case when $M^- \leq x \leq M^+$. More precisely we will prove, by induction on $n = \dim \mathcal{F}$, the following statements

(a) $h^i_{\mathcal{F},\underline{l}}$ extends to a continuous function on $\mathbb{R}$;

(b) $h^0_{\mathcal{F},\underline{l}}(x) \leq \frac{u(\mathcal{F})}{n!}(x - M^-)^n$, for $x \geq M^-$;
$h^i_{\mathcal{F},\underline{l}}(x) \leq 2^{n-1} u(\mathcal{F})(M^+ - M^-)^{n-1}$, for $M^- \leq x \leq M^+$,
and $h^n_{\mathcal{F},\underline{l}}(x) \leq \frac{u(\mathcal{F})}{n!}(M^+ - x)^n$, for $x \leq M^+$.

These assertions are clear if $\dim \mathcal{F} = 0$. Assume that they hold for all pure sheaves of dimension $\leq n - 1$. We will prove that they imply the following assertions:

For all pure sheaves $\mathcal{F}$ with $\dim \mathcal{F} = n$ *and for all rational numbers x and* $0 < \epsilon < 1$

$$h^0_{\mathcal{F}}((x+\epsilon)\underline{l}) - h^0_{\mathcal{F}}(x\underline{l}) \leq \epsilon \frac{u(\mathcal{F})}{(n-1)!}(x + \epsilon - M^-)^{n-1} \text{ for } x \geq M^-; \tag{3.2}$$

$$|h^i_{\mathcal{F}}((x+\epsilon)\underline{l}) - h^i_{\mathcal{F}}(x\underline{l})| \leq \epsilon 2^{n-1} u(\mathcal{F})(M^+ - M^-)^{n-1} \text{ for } M^- \leq x < x + \epsilon \leq M^+; \tag{3.3}$$

$$h^n_{\mathcal{F}}(x\underline{l}) - h^n_{\mathcal{F}}((x+\epsilon)\underline{l}) \leq \epsilon \frac{u(\mathcal{F})}{(n-1)!}(M^+ - x - \epsilon)^{n-1} \text{ for } x + \epsilon \leq M^+. \tag{3.4}$$

Take M sufficiently large and divisible such that Mx and $M\epsilon$ are integers, take a general divisor $D \in |M^2 \epsilon L|$ and consider the short exact sequence:

$$0 \to \mu_M^* \mathcal{F} \otimes L^{M^2 x} \xrightarrow{\cdot D} \mu_M^* \mathcal{F} \otimes L^{M^2(x+\epsilon)} \to \mu_M^* \mathcal{F} \otimes L^{M^2(x+\epsilon)}|_D \to 0.$$

Taking the long exact sequence of cohomology of the above sequence tensored with a general $P_\alpha \in \widehat{A}$, we see that

(3.5)
$$h^0_{\mathscr{F}}((x+\epsilon)\underline{l}) - h^0_{\mathscr{F}}(x\underline{l}) \le \frac{1}{M^{2g}} h^0_{\mathrm{gen}}(\mu_M^*\mathscr{F}\otimes L^{M^2(x+\epsilon)}|_D) = \frac{1}{M^{2g}} h^0_{\mu_M^*\mathscr{F}|_D}(M^2(x+\epsilon)\underline{l}).$$

Note that $\mu_M^*\mathscr{F}|_D$ is a pure sheaf on A of dimension $n-1$. It is also easy to see that an effective cohomological bound of $\mu_M^*\mathscr{F}|_D$ is (M^2M^-, M^2M^+). Hence condition (b)$_{n-1}$ above yields that

$$h^0_{\mu_M^*\mathscr{F}|_D}(M^2(x+\epsilon)\underline{l}) \le \frac{u(\mu_M^*\mathscr{F}|_D)}{(n-1)!} M^{2n-2}(x+\epsilon-M^-)^{n-1}.$$

The components of the support of $\mu_M^*\mathscr{F}|_D$ are $\mu^{-1}V_1\cap D, \dots, \mu^{-1}V_s\cap D$. Hence

$$u(\mu_M^*\mathscr{F}|_D) = \sum_j t_j(\mu_M^{-1}V_j\cdot D\cdot L^{n-1})_A = \frac{\epsilon}{M^{2n-2}}\sum_j t_j(\mu_M^*V_j\cdot\mu_M^*L^n)_A = \epsilon M^{2g-2n+2}r(\mathscr{F}).$$

It follows that

$$h^0_{\mathscr{F}}((x+\epsilon)\underline{l}) - h^0_{\mathscr{F}}(x\underline{l}) \le \epsilon\frac{u(\mathscr{F})}{(n-1)!}(x+\epsilon-M_0)^{n-1}$$

i.e., (3.2)$_n$. The estimate (3.4)$_n$ is proved exactly in the same way as the $h^0_{\mathscr{F},L}$ case. Concerning (3.3)$_n$ note that for $M^-\le x < x+\epsilon \le M^+$,

$$\begin{aligned}|h^i_{\mathscr{F}}((x+\epsilon)\underline{l}) - h^i_{\mathscr{F}}(x\underline{l})| &\le \frac{1}{M^{2g}}\big(h^{i-1}_{\mathrm{gen}}(\mu_M^*\mathscr{F}|_D\otimes L^{M^2(x+\epsilon)}) + h^i_{\mathrm{gen}}(\mu_M^*\mathscr{F}|_D\otimes L^{M^2(x+\epsilon)})\big)\\ &\le \epsilon 2^{n-1}u(\mathscr{F})(M^+-M^-)^{n-1}.\end{aligned}$$

This concludes the proof of the estimates (3.2) (3.3) (3.4) under the assumption that $(b)_{n-1}$ holds.

Turning to $(a)_n$ and $(b)_n$, note that the functions $h^i_{\mathscr{F}}(x\underline{l})$ satisfy the statement of Corollaries 2.6 and 2.7. Therefore the left derivative $D^-h^i_{\mathscr{F}}(x\underline{l})$ and the right derivative $D^+h^i_{\mathscr{F}}(x\underline{l})$ exist on all $x\in\mathbb{Q}$ (in fact on all $x\in U$ of Cor. 2.7), and they coincide away of a discrete subset. The inequalities (3.2)$_n$, (3.3)$_n$ and (3.4)$_n$ show that both derivatives are bounded above by the corresponding polynomials of degree $n-1$. Note that by Lemma 3.1 and the assumption that $\mathscr{F}$ is pure, we have $h^0_{\mathscr{F}}(M^-\underline{l}) = 0$ and $h^i_{\mathscr{F}}(M^+\underline{l}) = 0$ for $i\ge 1$, and hence by integration, the above bounds for derivatives imply $(a)_n$ and $(b)_n$ for all $x\in U$ as above and hence, by continuity, for all $x\in\mathbb{R}$. This concludes the proof of the theorem for pure sheaves.

Next, we prove the theorem for all coherent sheaves $\mathscr{F}$ on A. Assume that $\dim\mathscr{F} = n$. We consider the torsion filtration of $\mathscr{F}$:

$$T_0(\mathscr{F})\subset T_1(\mathscr{F})\subset\cdots\subset T_{n-1}(\mathscr{F})\subset T_n(\mathscr{F}) = \mathscr{F},$$

where $T_i(\mathscr{F})$ is the maximal subsheaf of $\mathscr{F}$ of dimension i and hence $\mathscr{Q}_i := T_i(\mathscr{F})/T_{i-1}(\mathscr{F})$ is a pure sheaf of dimension i. We see that $h^0_{\mathscr{F}}(x\underline{l}) \le h^0_{T_{n-1}(\mathscr{F})}(x\underline{l}) + h^0_{\mathscr{Q}_n}(x\underline{l})$ and we also have, adopting the previous notation,

$$\begin{aligned} h^0_{\mathscr{F}}((x+\epsilon)\underline{l}) - h^0_{\mathscr{F}}(x\underline{l}) &\leq \frac{1}{M^{2g}} h^0_{\mu_M^*\mathscr{F}|_D}(M^2(x+\epsilon)\underline{l}) \\ &\leq \frac{1}{M^{2g}}\big(h^0_{\mu_M^* T_{n-1}(\mathscr{F})|_D}(M^2(x+\epsilon)\underline{l}) + h^0_{\mu_M^* \mathscr{Q}_n|_D}(M^2(x+\epsilon)\underline{l})\big). \end{aligned}$$

We then proceed by induction on $\dim \mathscr{F}$ and the results on the pure sheaf case to prove the continuity of the function $h^0_{\mathscr{F}}(x\underline{l})$ and its boundedness. The proof of continuity for other cohomology rank functions of $\mathscr{F}$ is similar.

Similarly the proof of the statement of the theorem for objects of the bounded derived category follows the same lines, using the functorial hypercohomology spectral sequences

$$E_2^{h,k}(\mu_m^*(\mathscr{F})\otimes L^r\otimes P_\alpha) := H^h(A, \mathscr{H}^k(\mu_m^*\mathscr{F}\otimes L^r\otimes P_\alpha)) \Rightarrow H^{h+k}(A, \mu_m^*\mathscr{F}\otimes L^r\otimes P_\alpha).$$

This time, for $x\in\mathbb{Q}$, $x=\frac{a}{b}$ with $b>0$ one defines the cohomological rank functions for the groups appearing at each page: $e^{h,k}_{r,\mathscr{F}}(x\underline{l}) := b^{-2g}\dim E_r^{h,k}(\mu_b^*(\mathscr{F})\otimes L^a\otimes P_\alpha)$ for general $\alpha\in\widehat{A}$. Using Proposition 2.9 these functions are already defined in $\mathbb{R}$ minus a discrete set satisfying the property stated in Corollary 2.6. By induction on r and on the dimension of the cohomology sheaves one proves that these functions can be extended to continuous functions satisfying the same property. They are bounded as above. From this and the convergence one gets the same statements for the functions $h^i_{\mathscr{F}}(x\underline{l})$. We leave the details to the reader. □

4. Critical points and jump loci

A *critical point for the function* $x\mapsto h^i_{\mathscr{F}}(x\underline{l})$ is a $x_0\in\mathbb{R}$ where the function is not smooth. We denote $S^i_{\mathscr{F},\underline{l}}$ the set of critical points of $h^i_{\mathscr{F}}(x\underline{l})$ and let $S_{\mathscr{F},\underline{l}}=\bigcup_i S^i_{\mathscr{F},\underline{l}}$. This is the subject of this section. In all examples we know, the critical points of a cohomological rank function are finitely many, and satisfy the conclusion of Corollary 2.7. We expect this to be true in general.

It follows from the results of Section 2 that for $x_0\in\mathbb{Q}$, or more generally for x_0 in the open set U of Corollary 2.7, x_0 is a critical point if and only if the polynomials $P^-_{i,\mathscr{F},x_0}$ and $P^+_{i,\mathscr{F},x_0}$ do not coincide. As we will see below it is easy to produce examples of rational critical points. However they can be irrational—even for line bundles on abelian varieties—as shown by the following example.

Example 4.1. – Let $(A,\underline{l})$ be a polarized abelian variety and let M be a non-degenerate line bundle on A. Consider the polynomial $P(x)=\chi_M(x\underline{l})$. By Mumford, all roots of $P(x)$ are real numbers. Denote them by $\lambda_1>\lambda_2>\cdots>\lambda_k$, let m_i be the multiplicity of λ_i and finally denote by $\lambda_0=\infty$, $m_0=0$, and $\lambda_{k+1}=-\infty$. We know by [18, Page 155] that for any rational number $x=\frac{a}{b}\in(\lambda_{i+1},\lambda_i)$ with $b>0$, the line bundle $M^b\otimes L^a$ is non-degenerate and has index $a_i:=\sum_{0\leq k\leq i} m_k$. Hence

$$h^k_M(x\underline{l}) = \begin{cases} (-1)^k P(x) & \text{if } k=a_i \text{ for some } 0\leq i\leq k, \text{ and } x\in(\lambda_{i+1},\lambda_i),\\ 0 & \text{otherwise.}\end{cases}$$

Hence the set of critical points of M is $\{\lambda_1,\ldots,\lambda_k\}$. If A is a simple abelian variety and M and L are linearly independent in NS(A) (such abelian varieties exist and they are called

Shimura-Hilbert-Blumenthal varieties, see for instance [7]), then all roots of $P(x)$ are irrational. Actually, if some root $\lambda_i = \frac{a}{b}$ is rational, then $M^b \otimes L^a$ is a non-trivial degenerate line bundle and hence its kernel is a non-trivial abelian subvariety of A, which is a contradiction.

A critical point $x_0 \in \mathbb{R}$ is said to be *of index* k if the function $h^i_{\mathscr{F}}(x\underline{l})$ is of class $\mathcal{C}^k$ but not $\mathcal{C}^{k+1}$ at x_0. By Corollaries 2.6 and 2.7 if $x_0 \in \mathbb{Q}$ (or, more generally, $x_0 \in U$ as above) this can be equivalently stated as follows

$$P^+_{i,\mathscr{F},x_0} - P^-_{i,\mathscr{F},x_0} = (x - x_0)^{k+1} Q(x),$$

with $Q(x) \in \mathbb{Q}[x]$ of degree $\leq g-k-1$ such that $Q(x_0) \neq 0$ (in particular, it follows that the index is at most $g-1$).The main result of this section is Proposition 4.4, relating the index of a rational critical point with the dimension of the jump locus.

It is not difficult to exhibit cohomological rank functions with critical points even of index zero. i.e., the function is non-differentiable at such a point. The simple examples below serve also as illustration of the method of calculation provided by the results of Section 2.

EXAMPLE 4.2. – Let $A = B \times E$, a principally polarized product of a principally polarized $(g-1)$-dimensional abelian variety B and an elliptic curve E. Let Θ_B be a principal polarization on B and p a closed point of E. Let $\mathscr{F} = \mathcal{O}_B(\Theta_B) \boxtimes \mathcal{O}_E$. It is well known that:

(a) the FM transform—on E—of the sheaf $\mathcal{O}_E$ is $k(\hat{e})[-1]$, the one-dimensional skyscraper sheaf at the origin, in cohomological degree 1.

(b) The FM transform—on B—of the sheaf $\mathcal{O}_B(-\Theta_B)$ is equal to $\mathcal{O}_{\widehat{B}}(\Theta_{\widehat{B}})[-(g-1)]$.

By Künneth formula it follows from (a) that $R^0\Phi_{\mathscr{P}}(\mathscr{F}) = 0$, hence $h^0_{\mathscr{F}}(x\underline{l}) = 0$ for $x < 0$ (of course this was obvious from the beginning). On the other hand, again from Künneth formula together with (a) and (b) it follows that

$$R\Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee) = R^g\Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)[-g] = i_{\widehat{B}*}(\mathcal{O}_{\widehat{B}}(\Theta_{\widehat{B}}))[-g],$$

where $i_{\widehat{B}} : \widehat{B} \to \widehat{A}$ is the natural inclusion $\hat{b} \mapsto (\hat{b}, \hat{e})$. Hence, for $x > 0$

$$h^0_{\mathscr{F}}(x\underline{l}) = (x)^g h^0_{R^g\Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)}(\frac{1}{x}) = x^g(1 + \frac{1}{x})^{g-1} = x(1+x)^{g-1}.$$

In conclusion

$$h^0_{\mathscr{F}}(x\underline{l}) = \begin{cases} 0 & \text{for } x \leq 0, \\ x(1+x)^{g-1} & \text{for } x \geq 0. \end{cases}$$

(Of course the same calculation could have been worked out in a completely elementary way.) Hence $x_0 = 0$ is critical point of index zero.

EXAMPLE 4.3. – Let A be the Jacobian of a smooth curve of genus g, equipped with the natural principal polarization and let $i : C \hookrightarrow A$ be an Abel-Jacobi embedding. Let $p \in C$ and let $\mathscr{F} = i_*\mathcal{O}_C((g-1)p)$. We claim that $x_0 = 0$ is a critical point of index zero for the function $h^0_{\mathscr{F}}(x\underline{l})$. Notice that $\mathscr{F}^\vee = i_*\omega_C(-(g-1)p)[1-g]$ and $\deg_C(\omega_C(-(g-1)p)) = g-1$. Hence $R^0\Phi_{\mathscr{P}}(\mathscr{F}) = 0$, while

$$R\Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee) = R^g\Phi_{\mathscr{P}^\vee}(\mathscr{F}^\vee)[-g] = R^1\Phi_{\mathscr{P}^\vee}\big(i_*\omega_C(-(g-1)p)\big)[-g] := \mathscr{H}[-g]$$

is a torsion sheaf in cohomological degree g (supported at a translate of a theta-divisor, where it is of generic rank equal to 1). From Proposition 2.3 it follows that $h^0_{\mathcal{F}}(x\underline{l}) = 0$ for $x \leq 0$ and $h^1_{\mathcal{F}}(x\underline{l}) = 0$ for $x \geq 0$. Hence, by (a) of §1,

$$h^0_{\mathcal{F}}(x\underline{l}) = \begin{cases} 0 & \text{for } x \leq 0, \\ \chi_{\mathcal{F}}(x\underline{l}) = gx & \text{for } x \geq 0. \end{cases}$$

This proves what claimed.

One can show that $x_0 = 0$ is a critical point of index $g - d - 1$ of the h^0-function of the sheaf $i_* \mathcal{O}_C(dp)$, with $0 \leq d \leq g - 1$.

As it will be clear in the sequel, the previous examples are explained by the presence of a jump locus of codimension one.

Jump loci. – We introduce some terminology. Let $(A, \underline{l})$ be a polarized abelian variety. Let $\mathcal{F} \in \mathrm{D}^b(A)$ and $x_0 \in \mathbb{Q}$. The *jump locus* of the i-the cohomology of $\mathcal{F}$ at $x_0 = \frac{a}{b}$ is the closed subscheme of $\widehat{A}$ consisting of the points α such that $h^i(A, (\mu_b^* \mathcal{F}) \otimes L^{ab} \otimes P_\alpha))$ is strictly greater than the generic value, where L is a line bundle representing $\underline{l}$. A different choice of the line bundle L changes the jump locus in a translate of it while a different fractional representation of x_0, say $x_0 = \frac{ah}{bh}$ changes the jump locus in its inverse image via the isogeny $\mu_h : \widehat{A} \to \widehat{A}$. Therefore, strictly speaking, for us the jump locus at x_0 of a cohomological rank function $h^i_{\mathcal{F}}(x\underline{l})$ will be an equivalence class of (reduced) subschemes with respect to the equivalence relation generated by translations and multiplication isogenies. In this paper we will be only concerned with the dimension of these loci. We will denote it by $\dim J^{i+}(\mathcal{F}\langle x_0 \underline{l}\rangle)$.

PROPOSITION 4.4. – *Let $\mathcal{F} \in \mathrm{D}^b(A)$. If $x_0 \in \mathbb{Q}$ is a critical point of index k for $h^i_{\mathcal{F},\underline{l}}$, then* $\operatorname{codim}_{\widehat{A}} J^{i+}(\mathcal{F}\langle x_0\rangle) \leq k + 1$.

Proof. – We may assume that $x_0 = 0$. By Corollary 2.4 we know that in a left neighborhood of 0, $h^i_{\mathcal{F}}(x\underline{l}) = \frac{(-x)^g}{\chi(\underline{l})} \chi_{\varphi_{\underline{l}}^* R^i \Phi_{\mathcal{P}}(\mathcal{F})}(-\frac{1}{x}\underline{l})$ and in a right neighborhood of 0, $h^i_{\mathcal{F}}(x\underline{l}) = \frac{x^g}{\chi(\underline{l})} \chi_{\varphi_{\underline{l}}^* R^{g-i} \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)}(\frac{1}{x}\underline{l})$. We denote

$$P_1(x) := \chi_{\varphi_{\underline{l}}^* R^i \Phi_{\mathcal{P}}(\mathcal{F})}(x\underline{l}) = a_g x^g + a_{g-1} x^{g-1} + \cdots + a_1 x + a_0$$

and

$$P_2(x) := \chi_{\varphi_{\underline{l}}^* R^{g-i} \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)}(x\underline{l}) = b_g x^g + b_{g-1} x^{g-1} + \cdots + b_1 x + b_0.$$

It follows $h^i_{\mathcal{F}}(x\underline{l})$ is strictly of class $\mathcal{C}^k$ at 0 if and only if

$$(-1)^j a_{g-j} = b_{g-j} \quad \text{for } j = 0, \ldots, k \quad \text{and} \quad (-1)^{k+1} a_{g-k-1} \neq b_{g-k-1}. \tag{4.1}$$

We also note that for a coherent sheaf $\mathcal{Q}$,

$$\chi_{\mathcal{Q}}(x\underline{l}) = \int_A \mathrm{ch}(\mathcal{Q}) e^{x\underline{l}} = \sum_{j \geq 0} \frac{1}{(g-j)!} (\mathrm{ch}_j(\mathcal{Q}) \cdot \underline{l}^{g-j})_A x^{g-j}.$$

On the other hand, by Grothendieck duality (1.4) we have $\Phi_{\mathcal{P}}(\mathcal{F})^\vee = \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)[g]$. Thus we have a natural homomorphism $R^{g-i}\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \to \mathcal{H}om(R^i\Phi_{\mathcal{P}}(\mathcal{F}), \mathcal{O}_{\widehat{A}}) := \mathcal{H}$

by the Grothendieck spectral sequence. By base change and Serre duality such homomorphism is an isomorphism of vector bundles on the open set V whose closed points are the $\alpha \in \widehat{A}$ such that $h^i(\mathcal{F} \otimes P_\alpha)$ takes the generic value. Now assume that the complement of V, i.e., a representative of $J^{i+}(\mathcal{F})$, has codimension $> k+1$. Hence $\mathrm{ch}(R^{g-i}\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)) - \mathrm{ch}(\mathcal{H}) \in \mathrm{CH}^{>k+1}(\widehat{A})$. Since $R^i\Phi_{\mathcal{P}}(\mathcal{F})$ is a vector bundle on V, thus $\mathrm{ch}_j(\mathcal{H}) = (-1)^j \mathrm{ch}_j(R^i\Phi_{\mathcal{P}})(\mathcal{F})$ for $j \le k+1$. This implies that

$$\begin{aligned}(-1)^j a_{g-j} &= \frac{(-1)^j}{(g-j)!}(\varphi_{\underline{l}}^* \mathrm{ch}_j\big(R^i\Phi_{\mathcal{P}})(\mathcal{F}) \cdot \underline{l}^{g-j}\big)_A \\ &= \frac{1}{(g-j)!}(\varphi_{\underline{l}}^* \mathrm{ch}_j\big(R^{g-i}\Phi_{\mathcal{P}^\vee})(\mathcal{F}^\vee) \cdot \underline{l}^{g-j}\big)_A = b_{g-j},\end{aligned}$$

for $j = 0, \ldots, k+1$, which contradicts (4.1). This concludes the proof. □

5. Generic vanishing, M-regularity and IT(0) of $\mathbb{Q}$-twisted sheaves on abelian varieties

The notions of GV, M-regular and IT(0)-sheaves (and other related ones) are useful in the study of the geometry of abelian and irregular varieties via the Fourier-Mukai transform associated to the Poincaré line bundle. In this section we extend such notions to the $\mathbb{Q}$-twisted setting. In doing that we don't claim any originality, as this point of view was already taken, at least implicitly, in the work [27], Proof of Thm 4.1, and goes back to work of Hacon ([9]). It turns out that the $\mathbb{Q}$-twisted formulation of Hacon's criterion for being GV, and related results, is simpler and more expressive even for usual (non-$\mathbb{Q}$-twisted) coherent sheaves or, more generally, objects of $\mathrm{D}^b(A)$. In the last part of the section we go back to cohomological rank functions. First we show how they can be used to provide a characterization of M-regularity and related notions. Finally we show the maximal critical points relates to the notion of $\mathbb{Q}$-twisted GV sheaves.

As for jump loci (see the previous Section), one can define the *cohomological support locus* of the i-th cohomology of the $\mathbb{Q}$-twisted object of $\mathrm{D}^b(A)$, say $\mathcal{F}\langle x_0\underline{l}\rangle$, as the equivalence class (with respect to the equivalence relation generated by translations and inverse images by multiplication-isogenies) of the loci

$$\{\alpha \in \widehat{A} | h^i(A, (\mu_b^*\mathcal{F}) \otimes L^{\mathrm{ab}} \otimes P_\alpha)) > 0\}.$$

If $h^i_{\mathcal{F}}(x_0\underline{l}) = 0$ it coincides with the jump locus while it is simply $\widehat{A}$ if $h^i_{\mathcal{F}}(x\underline{l}) > 0$. Its dimension is well-defined, and we will denote it $\dim V^i(\mathcal{F}\langle x_0\underline{l}\rangle)$.

The $\mathbb{Q}$-twisted object $\mathcal{F}\langle x_0\underline{l}\rangle$ is said to be *GV* if $\mathrm{codim}_{\widehat{A}} V^i(\mathcal{F}\langle x_0\underline{l}\rangle) \ge i$ for all $i > 0$. It is said to be *a M-regular sheaf* if $\mathrm{codim}_{\widehat{A}} V^i(\mathcal{F}\langle x_0\underline{l}\rangle) > i$ for all $i > 0$. It is said to *satisfy the index theorem with index 0, IT(0)* for short, if $V^i(\mathcal{F}\langle x_0\underline{l}\rangle)$ is empty for all $i \neq 0$.

If $\mathcal{F}\langle x_0\rangle$ is $\mathbb{Q}$-twisted coherent sheaf or, more generally, a $\mathbb{Q}$-twisted object of $\mathrm{D}^b(A)$ such that $V^i(\mathcal{F}\langle x_0\rangle)$ is empty for $i < 0$, such conditions can be equivalently stated described as follows:

THEOREM 5.1. – (a) *$\mathcal{F}\langle x_0\underline{l}\rangle$ is GV if and only if, for one (hence for all) representation $x_0 = \frac{a}{b}$*

$$\Phi_{\mathcal{P}^\vee}(\mu_b^*\mathcal{F}^\vee \otimes L^{-ab}) = R^g\Phi_{\mathcal{P}^\vee}(\mu_b^*\mathcal{F}^\vee \otimes L^{-ab})[-g].^{(8)}$$

If this is the case, we have

$$R^i\Phi_{\mathcal{P}}(\mu_b^*(\mathcal{F})\otimes L^{\mathrm{ab}}) = \mathcal{E}xt^i_{\mathcal{O}_{\widehat{A}}}(R^g\Phi_{\mathcal{P}^\vee}(\mu_b^*\mathcal{F}^\vee \otimes L^{-ab}), \mathcal{O}_{\widehat{A}}).$$

(b) *Assume that $\mathcal{F}\langle x_0\underline{l}\rangle$ is GV. Then it is M-regular if and only if the sheaf $R^g\Phi_{\mathcal{P}^\vee}(\mu_b^*\mathcal{F}^\vee \otimes L^{-ab})$ is torsion-free.*

(c) *Assume that $\mathcal{F}\langle x_0\underline{l}\rangle$ is GV, Then it is IT(0) if the sheaf $R^g\Phi_{\mathcal{P}^\vee}(\mu_b^*\mathcal{F}^\vee \otimes L^{-ab})$ is locally free. Equivalently*

$$\Phi_{\mathcal{P}}(\mu_b^*(\mathcal{F})\otimes L^{\mathrm{ab}}) = R^0\Phi_{\mathcal{P}}(\mu_b^*(\mathcal{F})\otimes L^{\mathrm{ab}}). \tag{5.1}$$

These results follow immediately from the same statements for coherent sheaves or objects in $\mathrm{D}^b(A)$, see e.g., the survey [21, §1], or [26, §3], where the subject is treated in much greater generality.

In this language the well known criteria of Hacon ([9]) can be stated as follows:

THEOREM 5.2. – (a) *$\mathcal{F}\langle x_0\underline{l}\rangle$ is GV if and only if $\mathcal{F}\langle (x_0+x)\underline{l}\rangle$ is IT(0) for sufficiently small $x\in\mathbb{Q}^+$. Equivalently $\mathcal{F}\langle x_0\underline{l}\rangle$ is GV if and only if $\mathcal{F}\langle (x_0+x)\underline{l}\rangle$ is IT(0) for all $x\in\mathbb{Q}^+$.*

(b) *If $\mathcal{F}\langle x_0\underline{l}\rangle$ is GV but not IT(0) then $\mathcal{F}\langle (x_0-x)\underline{l}\rangle$ is not GV for all $x\in\mathbb{Q}^+$.*

(c) *$\mathcal{F}\langle x_0\underline{l}\rangle$ is IT(0) if and only if $\mathcal{F}\langle (x_0-x)\underline{l}\rangle$ is IT(0) for sufficiently small $x\in\mathbb{Q}^+$.*

Proof. – (a) Let $x_0 = \frac{a}{b}$. We have that $\mathcal{F}\langle x_0\underline{l}\rangle$ is GV if and only if $\mu_b^*(\mathcal{F})\otimes L^{\mathrm{ab}}$ is GV. Hacon's criterion (see [26, Thm A]) states that this is the case if and only if

$$H^i(\mu_b^*(\mathcal{F})\otimes L^{\mathrm{ab}}\otimes \Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^{-k})[g]) = 0 \tag{5.2}$$

for all $i\neq 0$ and for all sufficiently big $k\in\mathbb{Z}$, where N is an ample line bundle on $\widehat{A}$. Equivalently (up to taking a higher lower bound for k),

$$\mu_b^*(\mathcal{F})\otimes L^{\mathrm{ab}}\otimes \Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^{-k})[g] \quad \text{is} \quad IT(0) \tag{5.3}$$

for sufficiently big k. We take as $N = L_\delta$ a line bundle representing the polarization $\underline{l}_\delta$ dual to $\underline{l}$ ([4] §14.4). By Prop 14.4.1 of loc. cit. we have that

$$\varphi_{\underline{l}}^*\underline{l}_\delta = d_1 d_g \underline{l} \tag{5.4}$$

and

$$\varphi_{\underline{l}_\delta}\circ\varphi_{\underline{l}} = \mu_{d_1 d_g}, \tag{5.5}$$

where $(d_1,\dots,d_g)$ is the type of $\underline{l}$. Combining with (1.5) we get

$$\mu_{d_1d_gk}^*\Phi_{\mathcal{P}}^{\widehat{A}\to A}(L_\delta^k) = \varphi_{\underline{l}}^*\varphi_{\underline{l}_\delta}^*\mu_k^*\Phi_{\mathcal{P}}^{\widehat{A}\to A}(L_\delta^k) = (\varphi_{\underline{l}}^*(L_\delta^{-k}))^{\oplus k^g\chi(\underline{l}_\delta)} = (L^{-d_1d_gk})^{\oplus k^g\chi(\underline{l}_\delta)}.$$

Loosely speaking, we can think of the vector bundle $\Phi_{\mathcal{P}}^{\widehat{A}\to A}(L_\delta^k)$ as representative of $(-\frac{1}{d_1d_gk}\underline{l})^{\oplus k^g\chi(\underline{l}_\delta)}$. It follows, after a little calculation, that (5.3), hence the fact that $\mathcal{F}\langle x_0\underline{l}\rangle$ is GV, is equivalent to the fact that $\mathcal{F}\langle (x_0+\frac{1}{d_1d_gk})\underline{l}\rangle$ is IT(0) for sufficiently big k. This is in turn equivalent to the fact that $\mathcal{F}\langle (x_0+x)\underline{l}\rangle$ is IT(0) for sufficiently small $x\in\mathbb{Q}^+$ because

(8) This condition is usually expressed by saying that $\mu_b^*\mathcal{F}^\vee\otimes L^{-ab}$ satisfies the Weak Index Theorem with index g.

the tensor product of an IT(0) (or, more generally, GV) sheaf and a locally free IT(0) sheaf is IT(0) ([27, Prop. 3.1]). This proves the first statement of (a). The second statement follows again from loc.cit.

(b) follows from (a).

(c) is proved as (a) using a similar Hacon's criterion telling that (5.1) is equivalent to the fact that $\mu_b^*(\mathcal{F}) \otimes L^{\mathrm{ab}} \otimes \Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^k)$ is IT(0) for sufficiently big k. □

Using the cohomological rank functions on the left neighborhood of a rational point, we have the following characterization of GV-sheaves and M-regular sheaves.

Proposition 5.3. – (a) *$\mathcal{F}\langle x_0\underline{l}\rangle$ is GV, if and only if $h^i_{\mathcal{F}}((x_0 - x)\underline{l}) = O(x^i)$ for sufficiently small $x \in \mathbb{Q}^+$, for all $i \geq 1$.*

(b) *$\mathcal{F}\langle x_0\underline{l}\rangle$ is M-regular, if and only if $h^i_{\mathcal{F}}((x_0 - x)\underline{l}) = O(x^{i+1})$ for sufficiently small $x \in \mathbb{Q}^+$, for all $i \geq 1$.*

Proof. – We may suppose that $x_0 = 0$. Then $\mathcal{F}$ is GV (resp. M-regular) is equivalent to say that $\operatorname{codim} R^i\Phi_{\mathcal{P}}(\mathcal{F}) \geq i$ (resp. $\operatorname{codim} R^i\Phi_{\mathcal{P}}(\mathcal{F}) > i$) for all $i \geq 1$ (see [26] Lemma 3.6). Then we conclude by Corollary 2.4. □

It turns out that, more generally, the notion of *gv*-index ([25] Def. 3.1) can be extended to the $\mathbb{Q}$-twisted setting and described via cohomological rank functions as in Proposition 5.3. We leave this to the reader.

It is likely that a sort converse of Proposition 4.4 holds, namely the rational critical points arise only in presence of non-empty jump loci, although not necessarily for the same cohomological index. A partial result in this direction is the following

Proposition 5.4. – *Let $x_0 \in \mathbb{Q}$. If the $\mathbb{Q}$-twisted sheaf $\mathcal{F}\langle x_0\underline{l}\rangle$ is GV but not IT(0) then $x_0 \in S_{\mathcal{F},\underline{l}}$. In fact it is the maximal element of $S_{\mathcal{F},\underline{l}}$.*

However notice that, given a coherent sheaf $\mathcal{F}$, in general there is no reason to expect that there is an $x_0 \in \mathbb{Q}$ such that the hypothesis of the proposition holds. In other words, the maximal critical point might be irrational.

Proof. – As before, we may assume that $x_0 = 0$ and we need to compare the coefficients of the two polynomials $P_1(x) = \chi_{\varphi_{\underline{l}}^* R^0\Phi_{\mathcal{P}}(\mathcal{F})}(x)$ and $P_2(x) = \chi_{\varphi_{\underline{l}}^* R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)}(x)$. By assumption, $\mathcal{F}$ is a GV sheaf, hence by Theorem 5.1(a) $R\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) = R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)[-g]$ and $R^i\Phi_{\mathcal{P}}(\mathcal{F}) = \mathcal{E}xt^i(R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee), \mathcal{O}_{\widehat{A}})$. Moreover the condition that $\mathcal{F}$ is GV but not IT(0) implies that $R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)$ is not locally free. Hence for some $i > 0$, $R^i\Phi_{\mathcal{P}}(\mathcal{F})$ is nonzero. Thus, $h^i_{\mathcal{F}}(x\underline{l})$ is nonzero for x in a left neighborhood of 0 and obviously $h^i_{\mathcal{F}}(x\underline{l}) = 0$ for x positive. Hence $x_0 \in S_{\mathcal{F},\underline{l}}$. □

Remark 5.5. – Under the above assumption, it is in general not true that x_0 is a critical point of $h^0_{\mathcal{F}}(x\underline{l})$ as shown by the following example. Let $(A, \underline{\theta})$ be a principally polarized abelian variety and let Θ be a theta-divisor. Let $\mathcal{F} = \mathcal{O}_A \oplus \mathcal{O}_\Theta(\Theta)$. Then $\mathcal{F}$ is GV and not IT(0). It is easy to see that

$$h^0_{\mathcal{F}}(x\underline{l}) = \begin{cases} (1+x)^g & \text{for } x \geq -1, \\ 0 & \text{for } x < -1. \end{cases}$$

However notice that $h^{g-1}_{\mathcal{F}}(x\underline{l}) = h^g_{\mathcal{F}}(x\underline{l}) = (-x)^g$ for $-1 \leq x \leq 0$.

6. Some integral properties of the coefficients

In this section we point out some interesting integrality properties of the polynomials involved in cohomological rank functions. We will use the results of §2 throughout.

LEMMA 6.1. – *Let $\mathcal{F} \in \mathrm{D}^b(A)$. Assume that $h^i_{\mathcal{F}}(x\underline{l}) = P(x)$ is a polynomial function for x in an interval $U_1 \subseteq \mathbb{R}$. Then all coefficients of $P(x)$ belong to $\frac{1}{g!}\mathbb{Z}$.*

Proof. – We already know that $P(x) \in \mathbb{Q}[x]$ is a polynomial of degree at most g. We may choose p sufficiently large such that there exists $q \in \mathbb{Z}$ such that the numbers $\frac{q}{p}, \dots, \frac{q+g}{p}$ and $\frac{q}{p+1}, \dots, \frac{q+g}{p+1}$ belong to U_1.

By definition, $a_i := P(\frac{q+i}{p}) = h^i_{\mathcal{F}}(\frac{q+i}{p}) = \frac{1}{p^{2g}} h^i_{\mathrm{gen}}(\mathcal{F} \otimes L^{p(q+i)}) \in \frac{1}{p^{2g}}\mathbb{Z}$. Then we know that

$$P(x) = \sum_{i=0}^{g} \frac{a_i}{\prod_{j\neq i}(\frac{i-j}{p})} \prod_{j\neq i}(x - \frac{q+j}{p}) = \sum_{i=0}^{g} \frac{a_i}{\prod_{j\neq i}(i-j)} \prod_{j\neq i}(px - q - j).$$

Hence all coefficients of $P(x)$ belong to $\frac{1}{g!}\frac{1}{p^{2g}}\mathbb{Z}$. Applying the same argument to $P(\frac{q}{p+1}), \dots, P(\frac{q+g}{p+1})$, we see that all coefficients of $P(x)$ belong to $\frac{1}{g!}\frac{1}{(p+1)^{2g}}\mathbb{Z}$. Hence they belong to $\frac{1}{g!}\mathbb{Z}$. □

REMARK 6.2. – By a slightly different argument, we can say something more. Let $\frac{a}{b} \in U_1$. Then by Corollary 2.4, we know that, for $x > 0$ small enough,

$$(6.1)\quad Q(x) := P(\frac{a}{b} - x) = x^g \frac{1}{\chi(\underline{l})} \chi_{\varphi^*_{\underline{l}} R^i \Phi_{\mathcal{P}}(\mu^*_b \mathcal{F} \otimes L^{\mathrm{ab}})}(\frac{1}{b^2} x\underline{l})$$
$$= \frac{1}{\chi(\underline{l})} \sum_{k=0}^{g} \big(ch_k(\varphi^*_{\underline{l}} R^i \Phi_{\mathcal{P}}(\mu^*_b \mathcal{F} \otimes L^{\mathrm{ab}})) \cdot \underline{l}^{g-k}\big)_A \frac{1}{b^{2g-2k}} x^k.$$

Hence the coefficient b_k of x^k of $Q(x)$ is $\frac{1}{\chi(\underline{l})}\big(ch_k(\varphi^*_{\underline{l}} R^i \Phi_{\mathcal{P}}(\mu^*_b \mathcal{F} \otimes L^{\mathrm{ab}})) \cdot \underline{l}^{g-k}\big)_A \frac{1}{b^{2g-2k}}$. Note that

$$\big(ch_k(\varphi^*_{\underline{l}} R^i \Phi_{\mathcal{P}}(\mu^*_b \mathcal{F} \otimes L^{\mathrm{ab}})) \cdot \underline{l}^{g-k}\big)_A = \big(\varphi^*_{\underline{l}} ch_k(R^i \Phi_{\mathcal{P}}(\mu^*_b \mathcal{F} \otimes L^{\mathrm{ab}})) \cdot \underline{l}^{g-k}\big)_A$$
$$= \frac{\deg \varphi_{\underline{l}}}{(d_1 d_g)^{g-k}} \big(ch_k(R^i \Phi_{\mathcal{P}}(\mu^*_b \mathcal{F} \otimes L^{\mathrm{ab}})) \cdot \underline{l}_\delta^{g-k}\big)_{\widehat{A}},$$

where $\underline{l}_\delta$ is the dual polarization ([4] §14.4) and the last equality holds because of (5.4). Moreover, since

$$(6.2)\qquad \chi(\underline{l}_\delta) = \frac{(d_1 d_g)^g}{\chi(\underline{l})}$$

we note that the class $[\underline{l}_\delta]^{g-k}$ belongs to $(g-k)!\frac{(d_1 d_g)^{g-k}}{d_g \cdots d_{k+1}} H^{2g-2k}(\widehat{A}, \mathbb{Z})$. On the other hand, it is clear that $[ch_k(R^i \Phi_{\mathcal{P}}(\mu^*_b \mathcal{F} \otimes L^{\mathrm{ab}}))] \in \frac{1}{k!} H^{2k}(\widehat{A}, \mathbb{Z})$. Thus $b_k \in (d_1 \cdots d_k)\frac{(g-k)!}{k!}\frac{1}{b^{2g-2k}}\mathbb{Z}$. From this computation, we see easily that the coefficient a_k of x^k in $P(x)$ belongs to $(d_1 \cdots d_k)\frac{(g-k)!}{k!}\mathbb{Z}$.

We have the following strange corollary.

COROLLARY 6.3. – *Let $\mathscr{F} \in \mathrm{D}^b(A)$. Then $b^g \mid h^i_{\mathrm{gen}}(\mu_b^* \mathscr{F} \otimes L^a)$ for all b such that $(b, g!) = 1$ and $a \in \mathbb{Z}$.*

Proof. – Since $h^i_{\mathscr{F}}(x\underline{l})$ is a polynomial of degree at most g whose coefficients belong to $\frac{1}{g!}\mathbb{Z}$, we have that $g! b^g h^i_{\mathscr{F}}(\frac{a}{b}\underline{l}) \in \mathbb{Z}$. As $(b, g!) = 1$, we conclude that $b^g \mid h^i_{\mathrm{gen}}(\mu_b^* \mathscr{F} \otimes L^a)$. □

7. GV-subschemes of principally polarized abelian varieties

Let $(A, \underline{\theta})$ be a g-dimensional principally polarized abelian variety. A subscheme X of A is called a *GV-subscheme* if its twisted ideal sheaf $\mathscr{I}_X(\Theta)$ is GV. This technical definition is motivated by the fact that the subvarieties $\pm W_d$ of Jacobians and $\pm F$, the Fano surface of lines of intermediate jacobians of cubic threefolds, are the only known examples of (non-degenerate) GV-subschemes. We summarize some basic results on the subject in use in the sequel. One considers the "theta-dual" of X, namely the cohomological support locus

$$V(X) := V^0(\mathscr{I}_X(\Theta)) = \{\alpha \in \widehat{A} | h^0(\mathscr{I}_X(\Theta) \otimes P_\alpha) > 0\},$$

equipped with its natural scheme structure ([24] §4). Let X be a geometrically non-degenerate GV-subscheme of pure dimension d. Then

(a) [28, Theorem 2(1)] *X and $V(X)$ are reduced and irreducible.*

(b) ([24]) *$V(X)$ is a geometrically non-degenerate GV-scheme of pure dimension* $g-d-1$ (the maximal dimension). *Moreover $V(V(X)) = X$ and both X and $V(X)$ are Cohen-Macaulay.*

(c) (loc. cit.) $\Phi_{\mathscr{P}}(\mathscr{O}_X(\Theta)) = \big(\mathscr{I}_{V(X)}(\Theta)\big)^\vee$. *Equivalently, by Grothendieck duality* (see (1.4)), $\Phi_{\mathscr{P}^\vee}(\omega_X(-\Theta)) = \mathscr{I}_{V(X)}(\Theta)[-d]$.

(d) (loc. cit.) *X has minimal class* $[X] = \frac{\underline{\theta}^{g-d}}{(g-d)!}$.

In loc. cit. it is conjectured that the converse of (d) holds. According to the conjecture of Debarre, this would imply that the only geometrically non-degenerate GV-subschemes are the subvarieties $\pm W_i$ and $\pm F$ as above. We refer to the Introduction for what is known in this direction.

Generalities on GV-subschemes. – We start with some general results on GV-subschemes, possibly of independent interest. The first proposition does not follow from Green-Lazarsfeld's Generic Vanishing Theorem because a GV-subscheme can be singular. It does follow, via Lemma 7.8 below, from the Generic Vanishing Theorem of [22] which works for *normal* Cohen-Macaulay subschemes of abelian varieties. However the following ad-hoc proof is much simpler.

PROPOSITION 7.1. – *Let X be a non-degenerate reduced GV-subscheme. Then its dualizing sheaf ω_X is a GV-sheaf.*

Proof. – By Hacon's criterion (5.2), it is enough to show that

$$H^i(\omega_X \otimes \Phi_{\mathscr{P}}(\mathcal{O}_A(-k\Theta))[g]) = 0 \quad \text{for } k \text{ sufficiently big.}$$

We have that $\Phi_{\mathscr{P}}(\mathcal{O}_A(-k\Theta))[g]$ is a vector bundle (in degree 0) which will be denoted, as usual, $\widehat{\mathcal{O}_A(-k\Theta)}$. We can write

$$H^i(\omega_X \otimes \widehat{\mathcal{O}_A(-k\Theta)}) = H^i(\omega_X(-\Theta) \otimes \widehat{\mathcal{O}_A(-k\Theta)}(\Theta)).$$

Applying the inverse of $\Phi_{\mathscr{P}}$, namely $\Phi_{\mathscr{P}^\vee[g]}$ (see (1.1)) to the last formula of (c) of this section we get that $\omega_X(-\Theta) = \Phi_{\mathscr{P}}(\mathscr{I}_{V(X)}(\Theta))[g-d]$. Therefore, by (1.7)

$$H^i((\omega_X(-\Theta)) \otimes \widehat{\mathcal{O}_A(-k\Theta)}(\Theta))) = H^{i+g-d}\Big(\mathscr{I}_{V(X)}(\Theta) \otimes \Phi_{\mathscr{P}}\big(\widehat{\mathcal{O}_A(-k\Theta)}(\Theta)\big)\Big).$$

Clearly $\widehat{\mathcal{O}_A(-k\Theta)}$ is an IT(0) sheaf for $k \geq 1$ ((1.5)). Therefore $\Phi_{\mathscr{P}}\big(\widehat{\mathcal{O}_A(-k\Theta)}(\Theta)\big)$ is a sheaf (locally free) in cohomological degree 0. Hence the above cohomology groups vanish for $i > 0$ because $\dim V(X) = g - d - 1$. □

Proposition 7.2. – *Let X be a non-degenerate reduced GV-subscheme. Then*

$$\Phi_{\mathscr{P}^\vee}(\omega_X) = \big(\Phi_{\mathscr{P}}(\mathscr{I}_{V(X)})\big)(-\Theta)[g-d].$$

Proof. – We have

$$\begin{aligned}\Phi_{\mathscr{P}}(\omega_X) &= \Phi_{\mathscr{P}}\big(\omega_X(-\Theta) \otimes \mathcal{O}_A(\Theta)\big)\\ &\overset{(1.3)}{=} \Phi_{\mathscr{P}}(\omega_X(-\Theta)) * \Phi_{\mathscr{P}}(\mathcal{O}_A(\Theta)[g]\\ &\overset{(c),(1.5)}{=} (-1)^* \mathscr{I}_{V(X)}(\Theta)[-d] * \mathcal{O}_A(-\Theta)[g]\\ &\overset{(1.6)}{=} (\Phi_{\mathscr{P}}(\mathscr{I}_{-V(X)}))(-\Theta)[g-d].\end{aligned}$$

□

Corollary 7.3. – *Let X be a non-degenerate reduced GV-subscheme. Then*

(7.1) $$R^i\Phi_{\mathscr{P}^\vee}(\omega_X) = 0 \quad \text{for } i \neq 0, d \quad \text{and} \quad R^d\Phi_{\mathscr{P}^\vee}(\omega_X) = k(\hat{e}).\ ^{(9)}$$

Proof. – This follows from the fact that also $V(X)$ is reduced Cohen-Macaulay ((b) of this section). Therefore, by Proposition 7.1, combined by the duality characterization of Theorem 5.2(a)

$$\Phi_{\mathscr{P}}(\mathcal{O}_{V(X)}) = R^{d-g-1}\Phi_{\mathscr{P}}(\mathcal{O}_{V(X)})[-(g-d-1)].$$

Hence it follows from the standard exact sequence $0 \to \mathscr{I}_{V(X)} \to \mathcal{O}_A \to \mathcal{O}_{V(X)} \to 0$ that $R^i\Phi_{\mathscr{P}}(\mathscr{I}_{V(X)}) = 0$ for $i \neq g-d, g$ and $R^g\Phi_{\mathscr{P}}(\mathscr{I}_{V(X)}) = k(\hat{e})[-g]$. Therefore the assertion follows from the previous proposition. □

Corollary 7.3 is quite strong. Its implications hold in a quite general context but, for sake of brevity, here we will stick to an ad-hoc treatment of the case of GV-subschemes.

Lemma 7.4. – *Let X be a subscheme of an abelian variety A. If X satisfies* (7.1) *then the natural maps $\Lambda^i H^1(\mathcal{O}_A) \to H^i(\mathcal{O}_X)$ are isomorphisms for all $i < d$ and injective for $i = d$.*

(9) The last assertion was proved under very general assumptions in [2, Prop. 6.1].

Proof. – In the first place we note that (as it is well known) for all $\mathcal{F} \in \mathrm{D}^b(A)$ we have a natural isomorphism

$$H^i(A, \mathcal{F}) \cong \mathrm{Ext}^{g+i}(k(\hat{e}), \Phi_{\mathcal{P}}(\mathcal{F})), \tag{7.2}$$

where $\hat{e}$ denotes the origin of $\widehat{A}$. Indeed, by the Fourier-Mukai equivalence,

$$\begin{aligned} H^i(A, \mathcal{F}) = \mathrm{Hom}_{D(A)}(\mathcal{O}_A, \mathcal{F}[i]) &\cong \mathrm{Hom}_{D(\widehat{A})}(k(\hat{e})[-g], \Phi_{\mathcal{P}}(\mathcal{F})) \\ &\cong \mathrm{Ext}^{g+i}(k(\hat{e}), \Phi_{\mathcal{P}}(\mathcal{F})). \end{aligned}$$

Applying (7.2) to $\mathcal{F} = \omega_X$ we are reduced to compute the hypercohomology spectral sequence

$$\mathrm{Ext}^{g+i-k}(k(\hat{e}), R^k\Phi_{\mathcal{P}}(\omega_X)) \Rightarrow \mathrm{Ext}^{g+i}(k(\hat{e}), \Phi_{\mathcal{P}}(\omega_X)) \cong H^i(\omega_X).$$

We recall that $\mathrm{Ext}^j_{\widehat{A}}(k(\hat{e}), k(\hat{e})) = \Lambda^j H^1(\mathcal{O}_A)$. Condition (7.1) makes the above spectral sequence very easy. In fact we get the maps

$$\begin{aligned} H^i(\omega_X) \cong \mathrm{Ext}^{g+i}(k(\hat{e}), \Phi_{\mathcal{P}}(\omega_X)) &\to \mathrm{Ext}^{g+i-d}(k(\hat{e}), R^d\Phi_{\mathcal{P}}(\omega_X)[d]) \\ &= \mathrm{Ext}^{g+i-d}(k(\hat{e}), k(\hat{e})) \\ &= \Lambda^{g+i-d} H^1(\mathcal{O}_A) \\ &= \Lambda^{d-i} H^1(\mathcal{O}_A)^\vee, \end{aligned}$$

which are isomorphisms for $i > 0$ and surjective for $i = 0$. The lemma follows by duality. $\square$

The Poincaré polynomial of a GV-subscheme. – As an application of cohomological rank functions we prove that the Hilbert polynomial of geometrically non-degenerate GV-subschemes is the conjectured one.

THEOREM 7.5. – *Let X be a geometrically non-degenerate GV-subscheme of dimension d of a principally polarized abelian variety $(A, \underline{\theta})$. Then*

$$\chi_{\mathcal{O}_X}(x\underline{\theta}) = \sum_{i=0}^{d} \binom{g}{i} (x-1)^i.$$

Proof. – We compute the functions $h^i_{\mathcal{O}_X}(x\underline{\theta})$ in a neighborhood of $x_0 = 1$. First we compute it in an interval $(1-\epsilon^-, 1]$ as in Corollary 2.4. By (b) and (c) of this section we have that $R^0\Phi_{\mathcal{P}}(\mathcal{O}_X(\Theta)) = \mathcal{O}_A(-\Theta)$, $R^d\Phi_{\mathcal{P}}(\mathcal{O}_X(\Theta)) = \omega_{V(X)}(-\Theta)$ and $R^i\Phi_{\mathcal{P}}(\mathcal{O}_X(\Theta)) = 0$ for $i \neq 0, d$ (see e.g., [24, Prop. 5.1(b)]). Therefore, in a small interval $[-\epsilon^-, 0]$

$$h^i_{\mathcal{O}_X(\Theta)}(y\underline{\theta}) = \begin{cases} (-y)^g \chi_{\mathcal{O}_A(-\Theta)}(-\frac{1}{y}) = y^g(1+\frac{1}{y})^g = (1+y)^g & \text{for } i = 0, \\ (-y)^g Q(-\frac{1}{y}) & \text{for } i = d, \\ 0 & \text{for } i \neq 0, d, \end{cases}$$

where Q is the Hilbert polynomial of the sheaf $\omega_{V(X)}(-\Theta)$, hence a polynomial of degree $g-d-1$. It follows that $(-y)^g Q(-\frac{1}{y}) = y^{d+1}T(y)$, where T is a polynomial of degree $g-d-1$ such that $T(0) \neq 0$. Setting $x = 1+y$ we get that in the interval $(1-\epsilon^-, 1]$ [10]

$$h^i_{\mathcal{O}_X}(x\underline{\theta}) = \begin{cases} x^g & \text{for } i=0 \\ (x-1)^{d+1}T(x-1) & \text{for } i=d, \text{where } \deg T = g-d-1 \text{and } T(1)\neq 0 \\ 0 & \text{for } i \neq 0, d. \end{cases}$$

Writing x^g as its Taylor expansion centered at $x_0 = 1$ this yields the equality of polynomials

$$\chi_{\mathcal{O}_X}(x\underline{\theta}) = \sum_{i=0}^{g} \binom{g}{i}(x-1)^i + (-1)^d(x-1)^{d+1}T(x-1).$$

It follows that $\chi_{\mathcal{O}_X}(x\underline{\theta})$, which is a polynomial of degree d, is the Taylor expansion of x^g at order d. □

In the following proposition we compute the cohomological rank functions (with respect to the polarization $\underline{\theta}$) of the structure sheaf of a GV-scheme. In particular, this answers to Question 8.10 of [3], asking, in the present terminology, for the cohomological rank functions of the structure sheaf of a curve in its Jacobian.

Proposition 7.6. – *In the same hypotheses of the previous theorem*

$$h^0_{\mathcal{O}_X}(x\underline{\theta}) = \begin{cases} 0 & \text{for } x \leq 0, \\ x^g & \text{for } x \in [0,1], \\ \chi_{\mathcal{O}_X}(x\underline{\theta}) = \sum_{i=0}^{d}\binom{g}{i}(x-1)^i & \text{for } x \geq 1. \end{cases}$$

Proof. – The assertion for $x \leq 0$ is obvious. The assertion for $x \geq 1$ follows from the fact that $\mathcal{O}_X(\Theta)$ is a GV-sheaf (in fact M-regular), this last assertion being well known, as it follows at once from the definition of GV-subscheme and the exact sequence

$$0 \to \mathcal{I}_X(\Theta) \to \mathcal{O}_A(\Theta) \to \mathcal{O}_X(\Theta) \to 0.$$

Therefore, by Theorem 5.2(a), $\mathcal{O}_X\langle(1+x)\underline{\theta}\rangle$ is IT(0) for $x > 0$. In the proof of the previous theorem, we computed the function $h^0_{\mathcal{O}_X}(x\underline{\theta}) = x^g$ for x in an interval $(1-\epsilon^-, 1]$. Therefore, to conclude the proof, we need to show that we can take $\epsilon^- = 1$. By Proposition 7.1, the dualizing sheaf $\omega_{V(X)}$ is a GV-sheaf. Hence, again by Theorem 5.2(a), $\omega_{V(X)}\langle x\underline{\theta}\rangle$ is IT(0) for $x > 0$. Therefore both $R^0\Phi_{\mathcal{P}}(\mathcal{O}_X(\Theta))\langle n\underline{\theta}\rangle = \mathcal{O}_A(-\Theta)\langle n\underline{\theta}\rangle$ and $R^d\Phi_{\mathcal{P}}(\mathcal{O}_X(\Theta))\langle n\underline{\theta}\rangle = \omega_{V(X)}(-\Theta)\langle n\underline{\theta}\rangle$ are IT(0) for $n > 1$. Therefore one can take $\epsilon^- = 1$ (see Remark 2.5). □

As a consequence of Corollary 7.3 and Theorem 7.5 we get

Theorem 7.7. – *Let X be a geometrically non-degenerate d-dimensional GV-subscheme of a g-dimensional p.p.a.v. A. Then*

$$h^i(\mathcal{O}_X) = \binom{g}{i} \quad \textit{for all } i \leq d.$$

[10] Actually in Proposition 7.6 below it will be shown that $\epsilon^- = 1$, but this is not necessary for the present Theorem.

Proof. – Lemma 7.4 implies that the natural map $\Lambda^i H^1(\mathcal{O}_A) \to H^i(\mathcal{O}_X)$ is an isomorphism for $i < d$ and injective for $i = d$. By Theorem 7.5 the coefficient of degree 0 of the Poincaré polynomial, namely $\chi(\mathcal{O}_X)$, is equal to $\sum_{i=0}^{d}(-1)^i\binom{g}{i}$. The result follows. □

If one believes that the conjectures mentioned at the beginning of this section are true, then geometrically non-degenerate GV-subschemes should be normal with rational singularities. On a somewhat different note, we take the opportunity to prove some partial results in this direction, using the results of [28] and [5].

LEMMA 7.8. – *Let $(A, \underline{\theta})$ be an indecomposable PPAV. Assume that X is a geometrically non-degenerate GV-subscheme. Then X is normal.*

Proof. – We know that X is reduced and irreducible by (a) of this section. Since an irreducible theta divisor is smooth in codimension 1, by the argument in [5, Theorem 4.1], GV-subschemes of A are smooth in codimension 1. Since X is Cohen-Macaulay, we conclude that X is normal by Serre's criterion. □

COROLLARY 7.9. – *Let $(A, \underline{\theta})$ be an indecomposable PPAV. Assume that X is a geometrically non-degenerate GV-subscheme of dimension* 2. *Then X has rational singularities.*

Proof. – Fix $\mu : X' \to X$ a resolution of singularities. By Lemma 7.8, we only need to prove that $\mu_*\omega_{X'} = \omega_X$ to conclude that X has rational singularities.

We first claim that $h^1(\mathcal{O}_{X'}) = h^0(\Omega^1_{X'}) = g$. Assume the contrary. Then the Albanese variety $A_{X'}$ has dimension $h^1(\mathcal{O}_{X'}) > g$. We consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "a_{X'}"] \arrow[d, "\mu"'] \arrow[dr, "\tau"] & A_{X'} \arrow[d] \\ X \arrow[r, hook] & A. \end{tikzcd}$$

Since X is normal, $\mu_*\mathcal{O}_{X'} = \mathcal{O}_X$. Hence $h^1(\mathcal{O}_{X'}) = h^1(\mathcal{O}_X) + h^0(X, R^1\mu_*\mathcal{O}_{X'}) > g$. Thus $h^0(X, R^1\mu_*\mathcal{O}_{X'}) > 0$ and hence there exists an irreducible curve C on X' which is contracted by μ and $a_{X'}|_C$ is generically finite onto its image. In particular, there exists a holomorphic 1-form $\omega_0 \in H^0(X', \Omega^1_{X'})$ such that $(\omega_0)_{|C}$ is non-zero. Pick $p \in C$ a general point and consider the following local calculation around p. Let x, y be local analytic coordinates of X' around p and assume that C is defined by $y = 0$. We may assume that $\tau(p)$ is the origin of A and $\tau(x, y) = (f_1(x, y), \dots, f_g(x, y))$ in an analytic neighborhood of p. Let m be the multiplicity of C in the fiber $\mu^*(0)$. Then the holomorphic functions f_i can be written as $y^m g_i(x, y)$ with g_i holomorphic around p. For each holomorphic 1-form $\omega \in H^0(A, \Omega^1_A)$, we write $\tau^*\omega = h_1dx + h_2dy$ in a neighborhood of p. Then $y^m \mid h_1$ and $y^{m-1} \mid h_2$. Thus for any $s \in \tau^*H^0(A, \Omega^2_A) \subset H^0(X', K_{X'})$, the corresponding divisor $D(s)$ has multiplicity of C $\geq 2m - 1$. On the other hand, since $\omega_0|_C \neq 0$, writing locally $\omega_0 = g_1dx + g_2dy$ around p, we have that $(g_1)|_C$ is non-zero. Hence there exists $t \in \omega_0 \wedge \tau^*H^0(A, \Omega^1_A) \subset H^0(X', K_{X'})$ such as the corresponding divisor $D(t)$ whose multiplicity of C is $m - 1$. But then we have a contradiction, since, by a result of Schreieder mentioned in the Introduction, the natural map $H^0(X', K_{X'}) \simeq \tau^*H^0(A, \Omega^2_A)$ is an isomorphism ([28, Theorem 2]). This proves what we claimed. It follows that the natural map $H^1(\mathcal{O}_A) \to H^1(\mathcal{O}_{X'})$ is an isomorphism.

Consider the short exact sequence

$$0 \to \mu_* \omega_{X'} \to \omega_X \to \tau \to 0.$$

By Theorem 7.7 and the result of Schreieder it follows that $H^0(\mu_* \omega_{X'}) \to H^0(\omega_X)$ is an isomorphism. By Theorem 7.7 and the above claim it follows that the map $H^1(\mu_* \omega_{X'}) \to H^1(\omega_X)$ is an isomorphism. Thus $h^0(\tau) = 0$ hence the cohomological support locus $V^0(\tau)$ is strictly contained in $\widehat{A}$. On the other hand, we know that ω_X is M-regular by Corollary 7.3. This, together with the fact that $\mu_* \omega_{X'}$ is GV ([9]) yields that τ is M-regular. Since $V^0(\tau)$ is strictly contained in $\widehat{A}$, this implies (as it is well known, see e.g., [21, Lemma 1.12(b)]) that $\tau = 0$. □

Corollary 7.10. – *In the hypothesis of the previous corollary, let X' be any desingularization of X. Then the induced morphism $\tau : X' \to A$ is the Albanese morphism of X'.*

Proof. – In view of the fact that $h^1(\mathcal{O}_{X'}) = g = \dim A$, it is enough to prove that τ does not factor through any non-trivial isogeny. This means that if $\alpha \in \widehat{A}$ is such that $\tau^* P_\alpha$ is trivial then $\alpha = \hat{e}$. But this follows from the last part of Lemma 7.3, which implies by base change that the cohomological support locus $V^d(\omega_X) = \{\hat{e}\}$ and Lemma 7.9. Alternatively, one can use the results of [16]. □

8. Cohomological rank functions of the ideal of one point, multiplication maps of global sections, and normal generation of abelian varieties

We refer to the Introduction for a general presentation of the contents of this section. Let A be an abelian variety and $\underline{l}$ and $\underline{n}$ be polarizations on A (in our applications $\underline{n}$ will be a multiple of $\underline{l}$). Assume moreover that $\underline{n}$ is basepoint free. Let N be an ample and basepoint free line bundle representing $\underline{n}$. We consider the evaluation bundle of N, defined by the exact sequence

$$0 \to M_N \to H^0(N) \otimes \mathcal{O}_A \to N \to 0. \tag{8.1}$$

Finally, let $p \in A$. We consider the cohomological rank functions $h^i_{\mathcal{I}_p}(x\underline{l})$ and $h^i_{M_N}(x\underline{n})$.[11] In both cases for $x \geq 0$ the functions are zero for $i \geq 2$ and for $x \leq 0$ they are zero for $i \neq 1, g$. We consider their maximal critical points, namely

$$\beta(\underline{l}) = \inf\{x \in \mathbb{Q} | h^1_{\mathcal{I}_p}(x\underline{l}) = 0\},$$
$$s(\underline{n}) = \inf\{x \in \mathbb{Q} | h^1_{M_N}(x\underline{n}) = 0\}.$$

As it is easy to see, the problem illustrated by Remark 5.5 about Proposition 5.4 does not occur for these two sheaves. Hence if $x_0 \in \mathbb{Q}$ is such that $\mathcal{I}_p\langle x_0\underline{l}\rangle$ (resp. $M_N\langle x_0\underline{n}\rangle$) is GV but not IT(0) then $\beta(\underline{l})$ (resp. $s(\underline{n})$) is a critical point of $h^i_{\mathcal{I}_p}(x\underline{l})$ (resp. $h^i_{M_N}(x\underline{n})$) for $i = 0, 1$, in fact the maximal one. In any case, for $x \in \mathbb{Q}$, the fact that $x > \beta(\underline{l})$ (resp. $y > s(\underline{n})$) is equivalent to the fact that $\mathcal{I}_p\langle x\underline{l}\rangle$ (resp. $M_N\langle y\underline{n}\rangle$) is IT(0).

---

(11) Note that they don't depend respectively on p and on N.

Let us spell what the IT(0) (resp. GV) condition means for the above $\mathbb{Q}$-twisted sheaves. For $x = \frac{a}{b} \in \mathbb{Q}^{>0}$, the fact that $\mathcal{I}_p\langle x\underline{l}\rangle$ is IT(0) means that

$$h^1(\mu_b^*(\mathcal{I}_p) \otimes L_\alpha^{\mathrm{ab}}) = 0$$

for all line $\alpha \in \widehat{A}$, where as usual L_α^{ab} denotes $L^{\mathrm{ab}} \otimes P_\alpha$. This means that the finite scheme $p + \mu_b^{-1}(0)$ imposes independent conditions to the line bundles L_α^{ab} for all $\alpha \in \widehat{A}$. Since the L_α^{ab}'s are all translates of the same line bundle this means that *for all* $p \in A$ the finite scheme $p + \mu_b^{-1}(0)$ imposes independent conditions to the global sections of the line bundle L^{ab} (hence the same happens for *all* line bundles L_α^{ab}). This condition can be interpreted as basepoint-freeness for the fractional polarization $x\underline{l}$. Note that if $x \in \mathbb{Z}$, writing $x = \frac{xb}{b}$ one finds back the usual basepoint-freeness. In turn the fact that $\mathcal{I}_p\langle x\underline{l}\rangle$ is GV but not IT(0) means that for all α in a proper closed subset of $\widehat{A}$ the finite scheme $p + \mu_b^{-1}(0)$ does not impose independent conditions to the global sections of the line bundle L_α^{ab}. As above this means that for p in a proper subset of A the finite scheme $p + \mu_b^{-1}(0)$ does not impose independent conditions to the global sections of L^{ab} (hence the same property holds for all line bundles L_α^{ab}). Again, for $x \in \mathbb{Z}$ one finds back the usual notion of base points and base locus. It follows that $\mathcal{I}_p(L)$ is any case GV and it is IT(0) if and only if $\underline{l}$ is basepoint free. In other words: $\beta(\underline{l}) \leq 1$ and equality holds if and only if $\underline{l}$ has base points.

Similarly, for $y = \frac{a}{b}$, the fact that $M_N\langle y\underline{n}\rangle$ is IT(0) (resp. GV) means that

$$h^1(\mu_b^*(M_N) \otimes N_\alpha^{\mathrm{ab}}) = 0$$

for all (resp. for general) $\alpha \in \widehat{A}$. Pulling back the exact sequence (8.1) via μ_b and tensoring with L_α^{ab} this has the meaning mentioned in the introduction, namely that the multiplication maps obtained by composing with the natural inclusion $H^0(N) \hookrightarrow H^0(\mu_b^* N)$

$$H^0(N) \otimes H^0(N_\alpha^{\mathrm{ab}}) \to H^0(\mu_b^*(N) \otimes N_\alpha^{\mathrm{ab}}) \tag{8.2}$$

are surjective for all (resp. for general) $\alpha \in \widehat{A}$. The above maps (8.2) can be thought as the multiplication maps of global sections of N and of a representative of the rational power $N^{\frac{a}{b}}$ (twisted by P_α)

Proposition 8.1. – *Let $(A, \underline{n})$ be a polarized abelian variety and assume that $\underline{n}$ is basepoint free. Let $p \in A$. For $i = 0, 1$ and $y < 1$*

$$h^i_{\mathcal{I}_p}(y\underline{n}) = \frac{(1-y)^g}{\chi(\underline{n})} h^i_{M_N}((-1 + \frac{1}{1-y})\underline{n}).$$

Consequently

$$s(\underline{n}) = -1 + \frac{1}{1-\beta(\underline{n})} = \frac{\beta(\underline{n})}{1-\beta(\underline{n})}. \tag{8.3}$$

Proof. – We can assume that $p = e$ (the origin of A). The essential point of the proof is that

$$\varphi_{\underline{n}}^*(R^0\Phi_{\mathcal{P}}(\mathcal{I}_e(N))) = M_N \otimes N^{-1}. \tag{8.4}$$

Indeed, by the exact sequence $0 \to \mathcal{I}_e(N) \to N \to N \otimes k(e) \to 0$ it follows that $R^0\Phi_{\mathcal{P}}(\mathcal{I}_e(N))$ is the kernel of the map

$$R^0\Phi_{\mathcal{P}}(N) = \widehat{N} \xrightarrow{f} R^0\Phi_{\mathcal{P}}(N \otimes k(e)) = \mathcal{O}_{\widehat{A}}.$$

By (1.5) the map $\varphi_{\underline{n}}^*(f)$ is identified to a map $H^0(N)\otimes N^{-1}\to \mathcal{O}_A$ which is easily seen to be the evaluation map tensored with N^{-1}.

Next, we notice that, since the polarization $\underline{n}$ is assumed to be basepoint free, we have that

$$\Phi_{\mathcal{P}}(\mathcal{I}_e(N)) = R^0\Phi_{\mathcal{P}}(\mathcal{I}_e(N)). \tag{8.5}$$

To prove this, we first notice that $H^i(\mathcal{I}_e\otimes N_\alpha)=0$ for all $\alpha\in\widehat{A}$ and $i>1$. By base change this implies that the support $R^i\Phi_{\mathcal{P}}(\mathcal{I}_e(N))$ is equal to

$$V^1(\mathcal{I}_e(L)) = \{\alpha\in\widehat{A} | h^1(\mathcal{I}_e\otimes N\otimes P_\alpha)>0\},$$

which is non-empty if and only if $\underline{n}$ has base points. This proves (8.5).

Therefore, by Proposition 2.3 and degeneration of the spectral sequence computing the hypercohomology, we have that for $i=0,1$ and $t<0$

$$h^i_{\mathcal{I}_e(N)}(t\underline{n}) = \frac{(-t)^g}{\chi(\underline{n})} h^i_{\varphi_{\underline{n}}^* R^0\Phi_{\mathcal{P}}(\mathcal{I}_0(N))}(-\frac{1}{t}\underline{n}) \overset{(8.4)}{=} \frac{(-t)^g}{\chi(\underline{n})} h^i_{M_N}((-1-\frac{1}{t})\underline{n}).$$

The first statement of the proposition follows setting $y=1+t$. The second statement follows from the first one. □

Applying the previous proposition to divisible polarizations $\underline{n}=h\underline{l}$ we get Theorem D of the Introduction, namely

Corollary 8.2. – *Let $(A,\underline{l})$ be a polarized abelian variety and let h be an integer such that $h\underline{l}$ is basepoint free* (hence $h\geq 2$, and $h\geq 1$ if $\underline{l}$ is basepoint-free). *Then*

$$s(h\underline{l}) = \frac{\beta(\underline{l})}{h-\beta(\underline{l})}. \tag{8.6}$$

Consequently:

(a)

$$s(h\underline{l}) \leq \frac{1}{h-1}$$

and equality holds if and only if $\underline{l}$ has base points.

(b) *Assume that $\underline{l}$ is base point free. Then $s(\underline{l})<1$ if and only if $\beta(\underline{l})<\frac{1}{2}$. In particular, if $\beta(\underline{l})<\frac{1}{2}$ then $\underline{l}$ is normally generated.*

Proof. – (a) By definition, $h^i_{\mathcal{I}_e}(x(h\underline{l})) = h^i_{\mathcal{I}_e}((xh)\underline{l})$. Hence $\beta(h\underline{l}) = \frac{1}{h}\beta(\underline{l})$. By the previous proposition,

$$s(h\underline{l}) = \frac{\beta(h\underline{l})}{1-\beta(h\underline{l})} = \frac{\beta(\underline{l})}{h-\beta(\underline{l})}.$$

The last statement follows from the fact that $\beta(\underline{l})\leq 1$ and equality holds if and only if $\underline{l}$ has base points.

(b) The first assertion follows immediately from (8.6). Concerning the last assertion, we have that $s(\underline{l})<1$ if and only if the multiplication maps

$$H^0(L)\otimes H^0(L_\alpha)\to H^0(L_\alpha^2)$$

are surjective for all $\alpha\in\widehat{A}$. A well known argument (e.g., [13], proof of Thm 6.8(c) and Cor. 6.9, or [4], proof of Theorem 7.3.1) proves that this implies that L_α is normally generated for all $\alpha\in\widehat{A}$. □

Item (b) as well as the case $h = 2$ of item (a) of the above corollary have been already commented in the Introduction. Here we note that the proposition implies that, for an integer $h \geq 2$ and $\frac{a}{b} \geq \frac{1}{h-1}$ the "fractional" multiplication maps of global sections

$$H^0(L^h) \otimes H^0(L_\alpha^{hab}) \to H^0(\mu_b^*(L^h) \otimes L_\alpha^{hab})$$

are surjective for general $\alpha \in \widehat{A}$ and in fact for all $\alpha \in \widehat{A}$ as soon as $\frac{a}{b} > \frac{1}{h-1}$ or $\underline{l}$ is basepoint free. This is much stronger than the known results on the subject. For example, for $h = 3$ Koizumi's theorem on projective normality, in a slightly stronger version ([13] Cor.6.9, [4] Th. 7.3.1) tells that the above maps are surjective for all $\alpha \in \widehat{A}$ for $b = 1$ and $a = \frac{2}{3}$ while the corollary asserts that the same happens for $\frac{a}{b} > \frac{1}{2}$. Moreover, for the critical value $\frac{a}{b} = \frac{1}{2}$, the corollary tells that the maps

$$H^0(L^3) \otimes H^0(L_\alpha^6) \to H^0(\mu_2^*(L^3) \otimes L_\alpha^6)$$

are surjective for general $\alpha \in \widehat{A}$ and in fact for all $\alpha \in \widehat{A}$ as soon as $\underline{l}$ is basepoint free. For arbitrary h the same happens for the "critical" maps

$$H^0(L^h) \otimes H^0(L_\alpha^{h(h-1)}) \to H^0(\mu_{h-1}^*(L^h) \otimes L_\alpha^{h(h-1)}).$$

Note that, when $\underline{l}$ is a principal polarization, the dimension of the source of the above maps is equal to the dimension of the target, namely $(h^2(h-1))^g$.

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GLOBAL REGULARITY FOR THE 3D FINITE DEPTH CAPILLARY WATER WAVES

BY XUECHENG WANG

ABSTRACT. – In this paper, we prove global regularity, scattering, and the non-existence of small traveling waves for the $3D$ capillary waves system in the flat bottom setting for smooth localized small initial data.

To construct global solutions, we highly exploit the symmetric structures inside the capillary waves system and control both a low order weighted norm and a high order weighted norm of the profile of a good substitution variable over time to show that, although the nonlinear solution itself doesn't decay sharply at rate $1/(1+t)$ over time, the "$1+\alpha$" derivatives of the nonlinear solution indeed decay sharply, where α is some fixed positive number.

RÉSUMÉ. – Dans cet article, on démontre la régularité globale, la dispersion des solutions et la non-existence des petites ondes progressives pour un système d'equations des ondes capillaires en dimension 3 avec des petites données initiales régulières et localisées, dans le cas des fonds plats.

Pour construire des solutions globales, on exploite les structures symétriques du système d'ondes capillaires et contrôle à la fois les évolutions des deux normes avec poids du profil d'une bonne variable substitutive, l'une d'ordre petit et l'autre d'ordre grand. En conséquence, on montre que les dérivées d'ordre $1+\alpha$ de la solution non-linéaire décroissent rapidement au taux de $1/(1+t)$, bien que la solution elle-même ne décroisse pas aussi rapidement, où α est un nombre positif fixé.

1. Introduction

1.1. The set-up of problem and previous results

We study the evolution of a constant density irrotational inviscid fluid, e.g., water, inside a time dependent domain $\Omega(t) \subset \mathbb{R}^3$, which has a fixed flat bottom Σ and a moving interface $\Gamma(t)$. Above the water region $\Omega(t)$ is vacuum. We neglect the gravity effect and only consider the surface tension effect in this paper. The problem under consideration is also known as the capillary waves system.

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After normalizing the depth of $\Omega(t)$ to be "1," we can represent $\Omega(t)$, $\Gamma(t)$, and Σ in the Eulerian coordinates as follows,

$$\Omega(t) := \{(x,y) : x \in \mathbb{R}^2, -1 \le y \le h(t,x)\},$$
$$\Gamma(t) := \{(x, h(t,x) : x \in \mathbb{R}^2\}, \quad \Sigma := \{(x,-1) : x \in \mathbb{R}^2\},$$

where $h(t,x)$ represents the height of interface, which will be a small perturbation of zero.

Let "u" and "p" denote the velocity and the pressure of the fluid respectively. Then the evolution of fluid can be described by the free boundary Euler equation as follows,

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \nabla \cdot u = 0, \quad \nabla \times u = 0, \quad \text{in } \Omega(t). \tag{1.1}$$

The free surface $\Gamma(t)$ moves with the normal component of the velocity according to the kinematic boundary condition as follows,

$$\partial_t + u \cdot \nabla \text{ is tangent to } \cup_t \Gamma(t).$$

The pressure p satisfies the Young-Laplace equation as follows,

$$p = \sigma H(h), \quad \text{on } \Gamma(t),$$

where "σ" denotes the surface tension coefficient, which will be normalized to be one, and $H(h)$ represents the mean curvature of the interface, which is given as follows,

$$H(h) = \nabla \cdot \Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \Big).$$

Lastly, the following Neumann type boundary condition holds on the bottom Σ,

$$u \cdot \vec{\mathbf{n}} = 0, \quad \text{on } \Sigma.$$

Because the bottom is assumed to be fixed, the fluid cannot go through the bottom. This explains why the above boundary condition holds.

Since the velocity field is irrotational, we can represent it in terms of a velocity potential ϕ. Let ψ be the restriction of the velocity potential on the boundary $\Gamma(t)$, more precisely, $\psi(t,x) := \phi(t,x,\ h(t,x))$. From the divergence free condition and the boundary conditions, we can derive the Laplace equation with two boundary conditions as follows,

$$(\partial_y^2 + \Delta_x)\phi = 0, \quad \frac{\partial \phi}{\partial \vec{\mathbf{n}}}\big|_{\Sigma} = 0, \quad \phi\big|_{\Gamma(t)} = \psi. \tag{1.2}$$

Hence, we can reduce the study of the motion of fluid in $\Omega(t)$ to the study of the evolution of the height function "$h(t,x)$" and the restricted velocity potential "$\psi(t,x)$" as follows,

$$\begin{cases} \partial_t h = G(h)\psi, \\ \partial_t \psi = H(h) - \frac{1}{2}|\nabla \psi|^2 + \dfrac{(G(h)\psi + \nabla h \cdot \nabla \psi)^2}{2(1+|\nabla h|^2)}, \end{cases} \tag{1.3}$$

where $G(h)\psi = \sqrt{1+|\nabla h|^2}\,\mathcal{N}(h)\psi$ and $\mathcal{N}(h)\psi$ is the Dirichlet-Neumann operator at the interface $\Gamma(t)$. See e.g., [42] for the derivation of the system (1.3).

The capillary waves system (1.3) has the conserved energy and the conserved momentum as follows, see e.g., [7],

$$\mathcal{H}(h(t),\psi(t)) := \Big[\int_{\mathbb{R}^2} \frac{1}{2}\psi(t) G(h(t))\psi(t) + \frac{|\nabla h(t)|^2}{1+\sqrt{1+|\nabla h(t)|^2}} dx \Big] = \mathcal{H}(h(0),\psi(0)), \tag{1.4}$$

$$\int_{\mathbb{R}^2} h(t,x)dx = \int_{\mathbb{R}^2} h(0,x)dx. \tag{1.5}$$

From [34, Lemma 3.4], we know that

(1.6) (Flat bottom setting) :

$$\Lambda_{\leq 2}[G(h)\psi] = |\nabla|\tanh|\nabla|\psi - \nabla\cdot(h\nabla\psi) - |\nabla|\tanh|\nabla|(h|\nabla|\tanh|\nabla|\psi),$$

$$\text{(Flat bottom setting)}: \quad \Lambda_{\leq 2}[\partial_t\psi] = \Delta h - \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}(|\nabla|\tanh|\nabla|\psi)^2, \tag{1.7}$$

where $\Lambda_{\leq 2}[\mathcal{N}]$ denotes the linear terms and the quadratic terms of the nonlinearity $\mathcal{N}$.

From the above Taylor expansions, in the small solution regime, the conserved Hamiltonian in (1.4) tells us that the L^2-norm of $(\nabla_x h, |\nabla|\sqrt{\tanh|\nabla|}\psi)$ doesn't change much over time. More precisely, the following approximation holds,

$$\frac{1}{4}\big(\|\nabla h(t)\|_{L^2}^2 + \||\nabla|P_{\leq 1}[\psi(t)]\|_{L^2}^2 + \||\nabla|^{1/2}P_{\geq 1}[\psi(t)]\|_{L^2}^2\big) \leq \mathcal{H}(h(t),\psi(t))$$

$$= \mathcal{H}(h(0),\psi(0)) \leq 4\big(\|\nabla h(t)\|_{L^2}^2 + \||\nabla|P_{\leq 1}[\psi(t)]\|_{L^2}^2 + \||\nabla|^{1/2}P_{\geq 1}[\psi(t)]\|_{L^2}^2\big). \tag{1.8}$$

There is an extensive literature on the study of the water waves system. Without being exhaustive, we only discuss some previous works here and refer readers to the references therein.

Previous results on the local existence of the water waves system. – Due to the quasilinear nature of the water waves systems, to obtain the local existence, it is very important to get around the losing derivatives issue. Early works of Nalimov [32] and Yosihara [41] considered the local well-posedness of the small perturbation of a flat interface such that the Rayleigh-Taylor sign condition holds. It was first discovered by Wu [37, 38] that the Rayleigh-Taylor sign condition holds without the smallness assumptions in the infinite depth setting. She showed the local existence for arbitrary size of initial data in Sobolev spaces. After the breakthrough of Wu's work, there are many important works devoted to improve the understanding of local well-posedness of the full water waves system and the free boundary Euler equations. Christodoulou-Lindblad [10] and Lindblad [31] considered the gravity waves with vorticity. Beyer-Gunter [8] considered the effect of surface tension. Lannes [30] considered the finite depth setting. See also Shatah-Zeng [33], and Coutand-Shkoller [11]. It turns out that local well-posedness also holds even if the curvature of the interface is unbounded and the bottom is very rough even without regularity assumption, only a finite separation condition is required, see the works of Alazard-Burq-Zuily [1, 2] for more detailed and precise description of this result.

Previous results on the long time behavior of the water waves system. – The long time behavior of the water waves system is more difficult and challenging. To study the long time behavior, the low frequency part of the solution plays an essential role. It is very interesting to see that the water waves systems in different settings have very different behavior at the low frequency part. Even for a small perturbation of static solution and flat interface, we only have few results so far. Note that it is possible to develop the so-called "splash-singularity" for a large perturbation, see [9] and references therein for more details.

We first discuss previous results in the infinite depth setting. The first long-time result for the water waves system is due to the work of Wu [39], where she proved the almost global existence of the $2D$gravity waves for small initial data. Subsequently, Germain-Masmoudi-Shatah [17] and Wu [40] proved the global existence for the $3D$ gravity waves system, which is the first global regularity result for the water waves system. Global existence of the $3D$ capillary waves was also obtained, see Germain-Masmoudi-Shatah [18]. For the $2D$ gravity waves system, it is highly nontrivial to bypass the almost global existence. As first pointed out by Ionescu-Pusateri [27] and independently by Alazard-Delort [3, 4], we have to modify the profile appropriately first to prove the global existence. The nonlinear solution possesses the modified scattering property instead of the usual scattering. Later, a different interesting proof of the almost global existence was obtained in the holomorphic coordinates by Hunter-Ifrim-Tataru [22], then Ifrim-Tataru [23] improved this result and gave another interesting proof of the global existence. The author [35] considered the infinite energy solution of the gravity waves in $2D$, which removed the momentum assumption assumed in previous results. Global existence of the capillary waves system in $2D$ was also obtained. See Ionescu-Pusateri [28, 29] and Ifrim-Tataru [23]. For the $3D$ gravity-capillary waves with any possible positive gravity effect constant and positive surface tension coefficient, Deng-Ionescu-Pausader-Pusateri [14] proved global existence for small localized initial data in the infinite depth setting.

Now, we move on to the finite depth setting. The behavior of the water waves system in the finite depth setting is more delicate than the infinite depth setting due to three factors listed as follows, the presence of small traveling waves, the more complicated structure at low frequencies, and less favorable quadratic terms.

Roughly speaking, the existence of small (in L^2 sense) traveling waves for the water waves system in different settings can be summarized as follows. From previous results [40, 17, 14] on the $3D$ water waves system in the infinite depth setting, we know that there is no small traveling waves regardless the size of σ/g. However, we do know the existence of small traveling waves for the $3D$ gravity-capillary waves system in the flat bottom setting as long as $\sigma/g > 1/3$, see [12]. From the recent work of the author [36], we know that there is no small traveling wave for the 3D gravity waves system in the flat bottom setting, i.e., $\sigma/g = 0$. So far, it is still not clear whether there exist small traveling waves for the $3D$ gravity-capillary waves system in the flat bottom setting if $0 < \sigma/g \leq 1/3$.

On the long time behavior side. Only results on the gravity waves system were obtained. The large time existence was obtained by Alvarez-Samaniego-Lannes [6] for the $3D$ finite depth gravity waves system. Recently, the author [34, 36] showed that the $3D$ gravity waves system admits global solutions for small smooth localized initial data in the flat bottom setting. For the $2D$ gravity waves system in the flat bottom setting, Harrop-Griffiths-Ifrim-Tataru [21] showed that the lifespan of the solution is at least of size $1/\epsilon^2$ if the small initial data is of size ϵ.

1.2. Main difficulties for the capillary waves system in the flat bottom settings

Note that the linear operator of the Dirichlet-Neumann operator changes with respect to the depth of water region. To help readers understand the main difficulties of the capillary waves in the finite depth setting, we compare the capillary waves system in the infinite depth setting and the flat bottom setting with the depth of water region normalized to be one. Intuitively speaking, we have the following two types of dispersive equations,

$$\text{(Infinite depth setting)} \qquad (\partial_t + i|\nabla|^{3/2})u = \mathcal{N}_1(u), \tag{1.9}$$

$$\text{(Flat bottom setting)} \qquad (\partial_t + i|\nabla|^{3/2}\sqrt{\tanh|\nabla|})u = \mathcal{N}_2(u). \tag{1.10}$$

The main new difficulties of the $3D$ capillary waves in the flat bottom setting, which are caused by the difference of linear operators in two settings at low frequencies, can be summarized by the following two facts.

(i) The nonlinearity of (1.10) doesn't have null structure at low frequencies, which does appear in the infinite depth setting. Intuitively speaking, the presence of null structure stabilizes the nonlinear effect. Hence, we expect a stronger nonlinear effect at low frequencies, which makes the global regularity problem more delicate in the flat bottom setting.

(ii) A new type of time resonance set appears for the capillary waves system in the flat bottom setting. The long time accumulated effect caused by the new time resonance set has not been carefully studied before. Given the fact that there exists a finite time blow up solution for a similar equation but with a different nonlinearity, we expect that the nonlinear effect caused by the new type of time resonance set is very delicate.

For the sake of readers, we provide more detailed discussion about the existence of null structure at low frequencies in two different settings here. Note that

(1.11)

$$\text{(Infinite depth setting)}: \quad \Lambda_{\leq 2}[\partial_t h] = \Lambda_{\leq 2}[G(h)\psi] = |\nabla|\psi - \nabla\cdot(h\nabla\psi) - |\nabla|(h|\nabla|\psi),$$

(1.12)

$$\text{(Infinite depth setting)}: \quad \Lambda_{\leq 2}[\partial_t\psi] = \Delta h - \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}(|\nabla|\psi)^2.$$

From (1.11) and (1.12), it is easy to check that the symbols of quadratic terms vanish if the output frequency of quadratic terms is zero. Moreover, if the frequency of the height function "$h(t)$" is zero, then the symbol of quadratic terms in "$\partial_t h(t)$" also vanishes. Unfortunately, we lose all these favorable cancelations for the capillary waves system (1.3) in the flat bottom setting. From (1.6) and (1.7), it is easy to check that the symbols of quadratic terms in the corresponding scenarios don't vanish in the flat bottom setting.

Due to the lack of null structures at low frequencies in the flat bottom setting, we expect much stronger nonlinear effect for the finite depth capillary waves. One way to capture the nonlinear effect is to study the growth of the profile of the solution, which is the pull back of the nonlinear solution along the linear flow, with respect to time.

For simplicity and also for intuitive purpose, we study a relevant toy model of the capillary waves system (1.3). More precisely, we consider the long time behavior of the following toy model,

$$\text{(Toy model)}: \quad (\partial_t - i\Delta)v = Q_1(v,\bar{v}) + Q_2(v,v) + Q_3(\bar{v},\bar{v}), \quad v:\mathbb{R}_t\times\mathbb{R}^2_x \longrightarrow \mathbb{C}, \tag{1.13}$$

where the symbols $q_i(\xi-\eta,\eta)$ of the quadratic terms $Q_i(.,.)$, $i \in \{1,2,3\}$, satisfy the following estimate,

(1.14)
$$\|\mathscr{F}^{-1}[q_i(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)]\|_{L^1} \leq C\min\{2^{2\max\{k_1,k_2\}},1\}, \quad i\in\{1,2,3\},$$

where C is some absolute constant.

The toy model (1.13) is derived by only keeping the quadratic terms of (1.3), which are expected to be the leading terms in the small data regime, and replacing the linear operator $|\nabla|^{3/2}\sqrt{\tanh|\nabla|}$ by the leading operator $|\nabla|^2$ at low frequencies. The estimate of symbol in (1.14) captures the facts that there are at least two derivatives inside (1.3) and the size of symbol is "1" in both 1×1 (sizes of two input frequencies) $\to 0$ (size of the output frequency) type interaction and the $1\times 0\to 1$ type interaction.

It turns out that the toy model (1.13), which is a $2D$ quadratic Schrödinger equation, is already a very delicate problem due to the presence of $v\bar{v}$ type nonlinearity. Even the quadratic Schrödinger equation in $3D$ is not completely solved.

If without the $v\bar{v}$ type quadratic term, then the $1\times 1\to 0$ type interaction is actually not bad. Note that the phases are all of size 1 in the $1\times 1\to 0$ type interaction if there is no $v\bar{v}$ type quadratic term. The high oscillation of phase in time will also stabilize the growth of the profile in a neighborhood of zero frequency even without the smallness arose from the symbol. We refer readers to the works of Germain-Masmoudi-Shatah [15, 16] for more detailed discussion.

To capture the nonlinear effect of $v\bar{v}$ type quadratic term in the toy model (1.13), we study the growth of profile $g(t) := e^{-it\Delta}v(t)$ over time, which gives us a sense of what the dispersion of the nonlinear solution "$v(t)$" will be. From the Duhamel's formula, we have

(1.15)
$$\begin{aligned}\widehat{g}(t,\xi) = \widehat{g}(0,\xi) + \int_0^t \int_{\mathbb{R}^2} \big(& e^{i2s\xi\cdot\eta} q_1(\xi-\eta,\eta)\widehat{g}(s,\xi-\eta)\widehat{\bar{g}}(s,\eta)\\ &+ e^{i2s\eta\cdot(\xi-\eta)} q_2(\xi-\eta,\eta)\widehat{g}(s,\xi-\eta)\widehat{g}(s,\eta)\\ &+ e^{is(|\xi|^2+|\xi-\eta|^2+|\eta|^2)} q_3(\xi-\eta,\eta)\widehat{\bar{g}}(s,\xi-\eta)\widehat{\bar{g}}(s,\eta)\big) d\eta ds.\end{aligned}$$

We start from the first iteration by replacing "$g(s)$" on the right side of (1.15) with the initial data $g(0)$, whose frequency is localized around "1". As a result, intuitively speaking, the following rough estimate holds in a small neighborhood of zero,

$$ct \leq |\widehat{g}(t,\xi)| \leq Ct, \qquad \text{when } |\xi| \leq c/t, \tag{1.16}$$

where c and C are some absolute constants and the time "t" is very large.

Due to the nonlinear nature of the problem, the growth of profile at low frequencies will trigger the growth of profile at other modes of frequencies. Therefore, it is reasonable to expect that a certain instability could possibly happen. Recently, Ikeda and Inui [24] showed that there exists a class of small L^2 initial data such that the solution of the quadratic Schrödinger equation with $v\bar{v}$ type nonlinearity blows up within a polynomial time in both $2D$ and $3D$.

This intuition, which comes from the first Picard iteration in (1.16), says that the nonlinear solution behaves differently from a linear flow. It says few precise information about the nonlinear solution itself. In this paper, our goal is not trying to classify all possible outcomes for different types of nonlinearities inside the toy model (1.13). Because it is a very delicate problem, we should not expect a universal answer. Instead, our goal is to exploit some hidden structures inside the capillary water waves (1.3) and show that the $3D$ capillary waves system (1.3) admits global solution for small localized initial data.

1.3. Main result

In this paper, we show that the solution of the capillary waves system (1.3) globally exists and scatters to a linear solution in a weak normed space for small initial data. More precisely, our main theorem is stated as follows,

THEOREM 1.1. – *Let $N_0 = 2000$, $\delta \in (0, 10^{-9}]$, and $\alpha = 1/10$. Assume that the initial data $(h_0, \psi_0) \in H^{N_0+1/2}(\mathbb{R}^2) \times H^{N_0+1/2}(\mathbb{R}^2)$ satisfies the following smallness condition,*

$$\|(h_0, \psi_0)\|_{H^{N_0+1/2}} + \sum_{\Gamma \in \{L, \Omega\}} \|(\Gamma h_0, \Gamma \psi_0)\|_{H^{10+1/2}} + \sum_{\Gamma^1, \Gamma^2 \in \{L, \Omega\}} \|(\Gamma^1 \Gamma^2 h_0, \Gamma^1 \Gamma^2 \psi_0)\|_{H^{1/2}} \leq \epsilon_0,$$

where ϵ_0 is a sufficiently small constant, $\Omega := x^{\perp} \cdot \nabla_x$ and $L := x \cdot \nabla_x + 2$. Then there exists a unique global solution for the capillary water waves system (1.3) *with initial data (h_0, ψ_0). Moreover, the solution scatters to a corresponding linear solution in a homogeneous Sobolev space $\dot{H}^{\alpha+\delta}$ and the following estimate holds,*

(1.17)
$$\sup_{t \in [0,T]} (1+t)^{-\delta} \|(\tilde{\Lambda} h, \psi)(t)\|_{H^{N_0}} + (1+t)\Big[\sum_{k \in \mathbb{Z}} 2^{(1+\alpha)k + 6k_+} \|P_k[(h, \psi)(t)]\|_{L^\infty}\Big] \leq C\epsilon_0,$$

where C is some absolute constant and $\tilde{\Lambda} := |\nabla|^{1/2}(\tanh|\nabla|)^{-1/2}$.

REMARK 1.1. – From (1.17), we know that the solution decays over time. This fact implies that there is no small traveling waves for the $3D$ capillary waves system (1.3) in the flat bottom setting, i.e., $\sigma/g = \infty$.

1.4. Main ideas of proof

The idea of proving global existence for the $3D$ finite depth capillary waves system (1.3) is classic, which is iterating the local existence result by controlling both the energy and the dispersion of the nonlinear solution over time.

The whole argument depends on the dispersion estimate of the nonlinear solution, which is very delicate. The main difficulty and the delicacy come from the complicated and large time resonance set associated with the quadratic terms. More precisely, the time resonance sets of the quadratic terms are defined as follows,

$$\mathscr{T}_{\mu,\nu} := \{(\xi, \eta) : \quad \Lambda(|\xi|) - \mu\Lambda(|\xi - \eta|) - \nu\Lambda(|\eta|) = 0\}, \quad \mu, \nu \in \{+, -\}.$$

As a typical example, the following approximation holds at low frequencies for the case when $\mu = +$ and $\nu = -$,

(1.18)
$$\mathscr{T}_{+,-} \cap \{(\xi,\eta) : |\xi|, |\eta| \leq 2^{-10}\} \approx \{(\xi,\eta) : |\xi|, |\eta| \leq 2^{-10}, \Lambda(|\xi|) - \Lambda(|\xi-\eta|) + \Lambda(|\eta|) \approx 2\xi\cdot\eta \approx 0\}.$$

Note that the time resonance set is almost everywhere since it is possible that "$\xi \cdot \eta = 0$" no matter what the sizes of $|\xi|$ and $|\eta|$ are.

Recall (1.16). Since the growth mode happens at a small neighborhood of zero, it is reasonable to expect that the spatial derivatives, which provide smallness at low frequencies, compensate the L^∞_x decay rate of the nonlinear solution. To capture this expectation, we aim to prove the sharp decay rate for certain derivatives of the nonlinear solution instead of the nonlinear solution itself.

Now, the first question is how many derivatives we associate with the nonlinear solution to obtain the sharp decay rate. To answer this question, we need to keep a basic principle in mind. Generally speaking, the more derivatives we associate with the solution the less information we can tell about the solution itself. Recall (1.16). Intuitively speaking, because of the accumulated effect of the $1 \times 1 \to 0$ type interaction, it is unlikely that the "1−" derivatives of the profile of the nonlinear solution do not grow over time. Therefore, in practice, we expect that $1+\alpha$ derivatives of solution decay sharply, where α is a small positive number.

Now, the real question is whether we can close the argument and show that our expectation indeed holds globally in time. The argument that we will present is very complicated and technical. There are two main ingredients that are very essential to the validity to the argument: (i) there are requisite symmetric structures inside the finite depth capillary wave system (1.3); (ii) the bulk scenario, which is nontrivial to justify and will be clear later, is the accumulated effect of the $t^{-1/2} \times t^{-1/2} \to t^{-1/2}$ type interaction. Recall (1.14), there are two derivatives in total at low frequencies. Hence, the accumulated effect of the $t^{-1/2} \times t^{-1/2} \to t^{-1/2}$ type interaction is compensated by the symbol of quadratic terms. As a result, the bulk scenario is not an issue.

We discuss some main ideas and strategies used in the bootstrap argument with more details as follows.

1.4.1. *Energy estimate: controlling the high frequency part of solution.* – We first point out that the difference of the high frequency part between the infinite depth setting and the flat bottom setting is very little. Thanks to the works of Alazard-Métivier [5] and Alazard-Burq-Zuilly [1, 2], by using the method of paralinearization and symmetrization, we can find a pair of good unknown variables, such that the equations satisfied by the good unknown variables have symmetries inside, which help us to avoid losing derivatives in the energy estimate.

Recall that we expect that the decay rate of $1+\alpha$ derivatives of solution is sharp. However, within our expectation, the L^∞_x-norm of the nonlinear solution itself in the worst scenario is only $(1+t)^{-1/2+\delta}$. As a result, a rough $L^2 - L^\infty$ type energy estimate is not sufficient to close the energy estimate. Hence, we need to pay special attention to the low frequency part of the input putted in L^∞-type space. To this end, an important step is to understand the structure of the low frequency part of the Dirichlet-Neumann operator, which has been studied in details in [34].

We first state our desired energy estimate and then explain the main intuitions behind. We expect that the following new type of energy estimate holds,

$$\Big|\frac{d}{dt}E(t)\Big| \le CE(t)\big(\|(h(t),\psi(t))\|_{W^{6,1+\alpha}} + \|(h(t),\psi(t))\|_{W^{6,1}}\|(h(t),\psi(t))\|_{W^{6,0}}\big), \tag{1.19}$$

where C is some absolute constant and the $W^{\gamma,b}$ type function space is defined as follows,

$$\|f\|_{W^{\gamma,b}} := \sum_{k\in\mathbb{Z}}(2^{\gamma k} + 2^{bk})\|P_k f\|_{L^\infty}, \quad b < \gamma. \tag{1.20}$$

Note that the desired new type of energy estimate (1.19) is sufficient to show that the energy only grows sub-polynomially as long as the nonlinear solution decays sharply in $W^{6,1+\alpha}$. To derive the new type energy estimate (1.19), besides the quadratic terms, we also need to pay special attention to the low frequency part of the cubic terms.

Now, we provide an intuitive explanation about why the desired estimate (1.19) holds. Note that the following three facts hold: (i) there are at least two derivatives in total inside the quadratic terms; (ii) we don't lose derivatives after utilizing symmetries during the energy estimates; (iii) the total number of derivatives doesn't decrease in this process. As a result, intuitively speaking, there are only two possible scenarios, which are listed as follows: (i) Including the High $\times$ High type interaction, there are at least two derivatives associated with the input with relatively smaller frequency; (ii) Smooth error terms. In other words, the high order Sobolev-norm of those terms can be controlled by their L^2-norms. Therefore, we can put the input with larger frequency in L^∞ and put the other input in L^2. In whichever scenario, the input putted in L^∞ type space always associates with two spatial derivatives, which explains the first estimate in (1.19). A very similar intuition also holds for cubic and higher order terms, which leads to the second part of (1.19).

1.4.2. *The dispersion estimate: sharp decay rate of the* $1+\alpha$ *derivatives of solution.* – To carry out the analysis of decay estimate, we first identify a good substitution variable, which has the same decay rate as the original solution. Instead of proving the dispersion estimate for the original variable, our goal is reduced to prove the sharp decay estimate for the good substitution variable over time.

We divide the rest of this subsubsection into three parts. (i) In the first part, we explain how to find such a good substitution variable. (ii) In the second part, we explain some main ideas in the estimate of the lower order weighted norm. Our goal is to prove that, under the assumption that the high order weighted norm only grows sub-polynomially over time, the low order weighted norm of the profile doesn't grow over time, which implies that the decay rate of $1+\alpha$ derivatives of the nonlinear solution is sharp. (iii) In the third part, we explain main ideas behind the estimate of high order weighted norm and show that it indeed grows only sub-polynomially over time.

A good substitution variable. – The good substitution variable is obtained by using the normal form transformation that removes some nonlinearities, which associate with phases that are highly oscillating in time. As a result, the equation satisfied by the good substitution variable has less terms, which simplify the whole argument.

For simplicity, we consider the toy model (1.13) to illustrate the main idea behind. Define the profile of $u(t)$ as $f(t) := e^{it\Lambda}u(t)$, as a result of direct computation, we have

$$\widehat{f}(t,\xi) = \widehat{f}(0,\xi) + \sum_{\mu,\nu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} e^{is\Phi^{\mu,\nu}(\xi,\eta)} q_{\mu,\nu}(\xi-\eta,\eta)\widehat{f^{\mu}}(s,\xi-\eta)\widehat{f^{\nu}}(s,\eta)d\eta ds,$$

where $f^+ := f =: P_+[f]$, $f^- := \bar{f} =: P_-[f]$, $q_{\mu,\nu}(\xi-\eta,\eta)$ is the symbol of $u^{\mu}u^{\nu}$ type quadratic term, and the phases $\Phi^{\mu,\nu}(\xi,\eta)$, $\mu,\nu\in\{+,-\}$, are defined as follows,

$$\Phi^{\mu,\nu}(\xi,\eta) = |\xi|^2 - \mu|\xi-\eta|^2 - \nu|\eta|^2, \quad \mu,\nu\in\{+,-\}.$$

Note that

$$\nabla_\eta\Phi^{+,+}(\xi,\eta) = -(\eta-\xi)-\eta \Longrightarrow \nabla_\eta\Phi^{+,+}(\xi,\xi/2) = 0.$$

Therefore, we can't do integration by parts in "η" around a small neighborhood of $(\xi,\xi/2)$ (space resonance set) . Fortunately, $(\xi,\xi/2)$ doesn't belong to the time resonance set. From the explicit formula, it is easy to check the validity of the following estimate,

$$\Phi^{+,+}(\xi,\xi/2) = |\xi|^2 - 2(|\xi|/2)^2 = |\xi|^2/2.$$

Very similarly, we can verify that the following estimate holds when $|\eta| \leq 2^{-10}|\xi|$ and $\mu = -$ or $|\xi| \leq 2^{-10}|\eta|$, $\mu\nu = +$,

$$2^{-2}\max\{|\xi|^2,|\eta|^2\} \leq |\Phi^{\mu,\nu}(\xi,\eta)| \leq 2^2\max\{|\xi|^2,|\eta|^2\}.$$

Since the associated phases are relatively large, we refer those cases as the high-oscillation-in-time cases.

To take the advantage of the high oscillation in time for these scenarios, we can use a normal form transformation to remove the high oscillation in time cases as follows,

$$v := u + \sum_{\mu,\nu\in\{+,-\}} A_{\mu,\nu}(u^{\mu},u^{\nu}), \quad a_{\mu,\nu}(\xi-\eta,\eta) = \sum_{k\in\mathbb{Z}} \frac{iq_{\mu,\nu}(\xi-\eta,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)} \tag{1.21}$$

$$\times\big(\psi_{\leq k-10}(\eta-\xi/2)\psi_k(\xi) + \frac{1-\mu}{2}\psi_k(\xi)\psi_{\leq k-10}(\eta) + \frac{1+\mu\nu}{2}\psi_k(\xi)\psi_{\geq k+10}(\eta)\big),$$

where $a_{\mu,\nu}(\xi-\eta,\eta)$, $\mu,\nu\in\{+,-\}$, are the symbol of quadratic terms $A_{\mu,\nu}(.,.)$. Note that there are at least two derivatives inside the symbol, which cover the loss of dividing the phase. As a result, the normal form transformation is not singular.

The discussion so far is restricted to the toy model (1.13). For the capillary waves system (1.3), we use similar ideas not only for quadratic terms, but also for cubic terms and quartic terms, see (4.20). Please refer to Subsection 4.1 for more details.

The low order weighted norm. – We first define the low order weighted norm Z_1-norm and the high order weighted norm Z_2-norm as follows,

$$\|g\|_{Z_1} := \sum_{k\in\mathbb{Z}}\sum_{j\geq -k_-} \|g\|_{B_{k,j}}, \quad \|g\|_{B_{k,j}} := (2^{(1+\alpha)k} + 2^{10k_+})2^j\|\varphi_j^k(x)P_k g(x)\|_{L^2}, \tag{1.22}$$

$$\|g\|_{Z_2} := \sum_{\Gamma^1,\Gamma^2\in\{L,\Omega\}} \|\Gamma^1\Gamma^2 g\|_{L^2} + \|\Gamma^1 g\|_{L^2}, \tag{1.23}$$

where $\varphi_j^k(x)$ is defined in (2.1), which is first introduced in the work of Ionescu-Pausader [25]. An advantage of using this type of space is that it not only localizes the frequency but also localizes the spatial concentration. The atomic space of this type has been successfully used in many dispersive PDEs, see [14, 13, 19, 25, 26, 36].

Define the profile of the good substitution variable $v(t)$ as $g(t) := e^{it\Lambda}v(t)$. From the linear dispersion estimates (2.10) and (2.11) in Lemma 2.7, to prove the sharp decay rate, it would be sufficient to prove that the Z_1-norm of the profile $g(t)$ doesn't grow over time. Now, under the assumption that the Z_2-norm only grows polynomially, we explain some main ideas of how to prove that Z_1-norm doesn't grow over time.

Note that the High $\times$ High type interaction is not an issue because we put a very high order weighted (i.e., $2^{(1+\alpha)k}$) in the definition of Z_1-norm, see (1.22). It remains to consider the High $\times$ Low type interaction, e.g., $|\eta| \le 2^{-10}|\xi|$. As a typical example of the bulk threshold case in the High $\times$ Low type interaction, we consider the case when $|\eta| \in [2^{-10}/t, 2^{10}/t], |\xi| \in [2^{-10}, 2^{10}]$. To get around the difficulty caused by the lack of null structure and the growth of profile around the small neighborhood of zero frequency (see(1.16)), we analyze more carefully the source of the growth mode inside the nonlinearity of the capillary waves system (1.3).

Recall (1.5). We know that $\widehat{h}(t,0)$ is conserved over time. Moreover, a simple Fourier analysis shows that $|\widehat{\psi}(t,0)| \le 2^{10}|t|\epsilon_0$, where ϵ_0 is the size of initial data. These two facts motivate us to expect that the source of trouble should be the restricted velocity potential $\widehat{\psi}(t,\eta)$ instead of the height function $\widehat{h}(t,\eta)$, i.e., the size of $\widehat{h}(t,\eta)$ should be much smaller than $\widehat{g}(t,\eta)$ when $|\eta| \le 2^{10}/t$, where time "t" is very large. As a matter of fact, we do have a better estimate for $\widehat{h}(t,\eta)$, see (5.15) in Lemma 5.3, which says that $\widehat{h}(t,\eta)$ grows at most at rate "$t^{2\delta}$" with respect to time if $|\eta| \le 2^{10}/t$. Recall again (1.6) and (1.7), we know that there is at least one spatial derivative associated with the velocity potential "$\psi(t)$". Therefore, if the velocity potential "$\psi(t)$" has the small frequency "η," the associated spatial derivative contributes to the smallness of $|\eta|$. To sum up, either the symbol contributes to the smallness of "$|\eta|$" or the input with smaller frequency is the height function "$h(t)$". In whichever case, the bulk threshold case $|\eta| \in [2^{-10}/t, 2^{10}/t], |\xi| \in [2^{-10}, 2^{10}]$ is not an issue.

For the non-threshold case, we do integration by parts in "η" once to take the advantage of the gap between the threshold case and the non-threshold case. For the High $\times$ High type interaction, the gap is created by the extra "$2^{\alpha k}$" we put in the definition of Z_1-norm. For the High $\times$ Low type and Low $\times$ High type interactions, the gap is created by the observation on the source of growth mode that we made in the above discussion. The gain of decay rate from the gap between the threshold case and the non-threshold case is more than the loss from the growth rate of the high order weighted norm. This fact leads to the conclusion that the Z_1-norm of the profile doesn't grow in time.

The high order weighted norm. – Now, we explain some essential ideas that make it possible to conclude that the high order weighted norm (Z_2-norm) of the profile grows at most rate "t^{δ}" with respect to time.

The losing derivatives issue. – Note that the high frequency part of the nonlinear solution is controlled by the high order energy (H^{N_0}-norm) of the nonlinear solution, which only grows at most at rate t^{δ} with respect to time. As a result, we only have to consider the case when the sizes of all frequencies are less than $t^{5/1900}$, which is only a minor growth rate. Therefore, the losing derivatives issue can be reduced to the losing time decay rate issue. For the cubic and higher order terms, we use the extra decay rate to cover the loss of losing derivatives. Since the "$1/t$" decay rate for the quadratic terms is critical to close the argument, we can't afford the loss of time decay rate.

To get around this issue, we notice that it would be sufficient to avoid losing derivatives at the quadratic level if the losing derivatives issue is only relevant at the quadratic level. After carefully studying the explicit quadratic terms of the capillary waves system (1.3), we study the system of equations satisfied by $(h, \psi - T_{|\nabla|\tanh|\nabla|\psi} h)$, which is the truncation up to quadratic level of the good unknown variable found in the paralinearization and symmetrization process, instead of the system of equations satisfied by (h, ψ). As a result, after utilizing the symmetric structure for the good substitution variable, the quadratic term doesn't lose derivatives.

The insufficient decay rates issue. – Now, we explain the main ideas used in two typical scenarios in which it is not obvious to obtain the critical "$1/t$" decay rate over time.

Recall that we applied the vector field "$L := x \cdot \nabla + 2$" (equivalently, "$-\xi \cdot \nabla_\xi$" on the Fourier side) on the profile of the nonlinear solution in the definition of Z_2-norm. A drawback of using the "L" vector field is that we face a loss of "t" when "L" hits the phases of nonlinearities. To be more precise and as a typical example, we consider the High $\times$ Low type interaction of the quadratic term. Note that (see (6.17) and (6.19) for more details), the following decomposition holds if the vector field "L" hits the phase.

$$\xi \cdot \nabla_\xi \Phi^{+,\nu}(\xi, \eta) = O(1)\Phi^{+,\nu}(\xi, \eta) + O(|\eta|^2), \quad \text{when } |\eta| \leq 2^{-10}|\xi|. \tag{1.24}$$

For the first term on the right hand side of (1.24), which is comparable with the phase function $\Phi^{+,\nu}(\xi, \eta)$, we can take the advantage of oscillation in time by doing integration by parts in time once first and then take the advantage of the space oscillation in "η". For the second term of (1.24), the smallness of "$|\eta|^2$" acts like null structures, which allow us to obtain sharp L^∞_x decay rate even after taking the advantage of the space oscillation in "η".

Another typical scenario with insufficient time decay rate is the case when all the vector fields hit the input with the largest frequency. Since the L^∞_x-norm of the nonlinear solution itself only decays at rate $(1 + t)^{-1/2+\delta}$, a rough $L^2 - L^\infty$ type estimate is not sufficient to close the argument. To get around this issue, we use the hidden symmetric structure inside the capillary waves system. As a result, a similar decomposition as in (1.24) holds for the symbol of quadratic terms after utilizing the symmetric structure inside the capillary waves system. Therefore, the aforementioned strategy is also applicable for the case we are considering. For the cubic terms, similar to the desired energy estimate (1.19), the symmetric structure inside the capillary waves system (1.3) also plays an essential role.

1.5. The outline of this paper

In Section 2, we introduce notations and some basic lemmas that will be used constantly. In Section 3, we prove a new type of energy estimate by using the method of paralinearization and symmetrization and paying special attention to the low frequency part. In Section 4, we identify a good substitution variable to carry out the estimate of weighted norms. In Section 5, we prove that the low order weighted norm doesn't grow over time under the assumptions that the high order weight norm only grows appropriately and a good control of the quintic and higher order remainder term is available. In Section 6, we prove that the high order weighted norm only grows appropriately under the assumption that we have a good control on the quintic and higher order remainder term. In Section 7, we first prove some fixed time weighted norm estimates, which were taken for granted in Section 5 and 6, and then estimate the quintic and higher order reminder terms by using a fixed point type argument.

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2. Preliminaries

For $p \in \mathbb{N}_+$, then we use $\Lambda_p(\mathcal{N})$ to denote the p-th order terms of the nonlinearity $\mathcal{N}$. Also, we use notation $\Lambda_{\geq p}[\mathcal{N}]$ to denote the p-th and higher orders terms, i.e., $\Lambda_{\geq p}[\mathcal{N}] := \sum_{q\geq p} \Lambda_q[\mathcal{N}]$. For example, $\Lambda_2[\mathcal{N}]$ denotes the quadratic term of $\mathcal{N}$ and $\Lambda_{\geq 2}[\mathcal{N}]$ denotes the quadratic and higher order terms of $\mathcal{N}$. If there is no special annotation, Taylor expansions are in terms of the height function "$h(t)$" and the restricted velocity potential "$\psi(t)$".

We fix an even smooth function $\tilde{\psi} : \mathbb{R} \to [0,1]$ supported in $[-3/2, 3/2]$ and equals to 1 in $[-5/4, 5/4]$. For any $k \in \mathbb{Z}$, we define

$$\psi_k(x) := \tilde{\psi}(x/2^k) - \tilde{\psi}(x/2^{k-1}), \qquad \psi_{\leq k}(x) := \tilde{\psi}(x/2^k) = \sum_{l\leq k} \psi_l(x),$$
$$\psi_{\geq k}(x) := 1 - \psi_{\leq k-1}(x),$$

and use P_k, $P_{\leq k}$ and $P_{\geq k}$ to denote the projection operators by the Fourier multipliers ψ_k, $\psi_{\leq k}$ and $\psi_{\geq k}$ respectively. We use $f_k(x)$ to abbreviate $P_k f(x)$. We use both $\widehat{f}(\xi)$ and $\mathcal{F}(f)(\xi)$ to denote the Fourier transform of f, which is defined as follows,

$$\mathcal{F}(f)(\xi) = \int e^{-ix\cdot\xi} f(x)dx.$$

We use $\mathcal{F}^{-1}(g)$ to denote the inverse Fourier transform of $g(\xi)$. For an integer $k \in \mathbb{Z}$, we use k_+ to denote $\max\{k, 0\}$ and use k_- to denote $\min\{k, 0\}$. We use f^+ and $P_+(f)$ to denote f itself and use f^- and $P_-(f)$ to denote the complex conjugate $\bar{f}$ of f.

Recall the Z_1-normed space and the Z_2-normed space we defined in (1.22) and (1.23). The spatial localization function $\varphi_j^k(x)$ used there is defined as follows,

$$\varphi_j^k(x) := \begin{cases} \psi_{(-\infty,-k]}(x) \text{ if } k+j = 0 \text{ and } k \leq 0, \\ \psi_{(-\infty,0]}(x) \quad \text{ if } j = 0 \text{ and } k \geq 0, \\ \psi_j(x) \quad \text{ if } k+j \geq 1 \text{ and } j \geq 1. \end{cases} \tag{2.1}$$

For any $k \in \mathbb{Z}$, $j \geq -k_-$, we define

$$f_{k,j} := P_{[k-2,k+2]}[\varphi_j^k(x) P_k f].$$

For two localized functions $f(x), g(x) \in L^2$, we use the convention that the symbol $q(.,.)$ of a bilinear form $Q(.,.)$ is defined in the following sense throughout this paper,

$$\mathcal{F}[Q(f,g)](\xi) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \widehat{f}(\xi-\eta)\widehat{g}(\eta)q(\xi-\eta,\eta)d\eta. \tag{2.2}$$

Very similarly, the symbol $c(.,.,.)$ of a trilinear form $C(f,g,h)$ is defined in the following sense:

$$\mathcal{F}[C(f,g,h)](\xi) = \frac{1}{16\pi^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \widehat{f}(\xi-\eta)\widehat{g}(\eta-\sigma)\widehat{h}(\sigma)c(\xi-\eta,\eta-\sigma,\sigma)d\eta d\sigma.$$

Define a class of symbol and its associated norms as follows,

$$\begin{aligned}
\mathcal{S}^\infty &:= \{m : \mathbb{R}^4 \text{ or } \mathbb{R}^6 \to \mathbb{C}, m \text{ is continuous and } \quad \|\mathcal{F}^{-1}(m)\|_{L^1} < \infty\},\\
\|m\|_{\mathcal{S}^\infty} &:= \|\mathcal{F}^{-1}(m)\|_{L^1},\\
\|m(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} &:= \|m(\xi,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty},\\
\|m(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} &:= \|m(\xi,\eta,\sigma)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}.
\end{aligned}$$

Lemma 2.1. – *For $i \in \{1,2,3\}$, $f \in W^{i+1,\infty}(\mathbb{R}^{2i})$, there exists an absolute constant $C \in \mathbb{R}_+$ such that the following estimate holds,*

$$\Big\|\int_{\mathbb{R}^{2i}} f(\xi_1,\ldots,\xi_i)\prod_{j=1}^{i} e^{ix_j\cdot\xi_j}\psi_{k_j}(\xi_j)d\xi_1\cdots d\xi_i\Big\|_{L^1_{x_1,\ldots,x_i}} \leq \sum_{m=0}^{i+1}\sum_{j=1}^{i} C2^{mk_j}\|\partial_{\xi_j}^m f\|_{L^\infty}. \tag{2.3}$$

Lemma 2.2. – *Assume that $m, m' \in S^\infty$, $f,g,h \in L^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$, $p,q,r,s \in [1,\infty]$, then there exists an absolute constant $C \in \mathbb{R}_+$, such that the following estimates hold,*

$$\|m\cdot m'\|_{S^\infty} \leq C\|m\|_{S^\infty}\|m'\|_{S^\infty}, \tag{2.4}$$

$$\Big\|\mathcal{F}^{-1}\big[\int_{\mathbb{R}^2} m(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\big]\Big\|_{L^p} \leq C\|m\|_{\mathcal{S}^\infty}\|f\|_{L^q}\|g\|_{L^r}, \tag{2.5}$$

if $1/p = 1/q + 1/r$,

$$\Big\|\mathcal{F}^{-1}\big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} m'(\xi,\eta,\sigma)\widehat{f}(\xi-\eta)\widehat{h}(\sigma)\widehat{g}(\eta-\sigma)d\eta d\sigma\big]\Big\|_{L^p} \leq C\|m'\|_{\mathcal{S}^\infty}\|f\|_{L^q}\|g\|_{L^r}\|h\|_{L^s}, \tag{2.6}$$

where $1/p = 1/q + 1/r + 1/s$.

Definition 2.3. – Given $\rho \in \mathbb{N}_+, \rho \geq 0$ and $m \in \mathbb{R}$, we use $\Gamma^m_\rho(\mathbb{R}^2)$ to denote the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^2 \times (\mathbb{R}^2/\{0\})$, which are C^∞ with respect to ξ for $\xi \neq 0$. Moreover, for any $\alpha \in \mathbb{N}^2_+$, there exists a constant $C_\alpha(a)$, which only depends on "α" and the symbol $a(x,\xi)$ itself, such that the following estimate holds for the symbol $a(x,\xi)$,

$$\forall|\xi| \geq 1/2, \|\partial_\xi^\alpha a(.,\xi)\|_{W^{\rho,\infty}} \leq C_\alpha(a)(1+|\xi|)^{m-|\alpha|},$$

where $W^{\rho,\infty}$ is the usual Sobolev space. For a symbol $a \in \Gamma^m_\rho$, we define its norm as follows,

$$M^m_\rho(a) := \sup_{|\alpha|\leq 2+\rho}\sup_{|\xi|\geq 1/2}\|(1+|\xi|)^{|\alpha|-m}\partial_\xi^\alpha a(.,\xi)\|_{W^{\rho,\infty}}.$$

DEFINITION 2.4. – (i) We use $\dot{\Gamma}^m_\rho(\mathbb{R}^2)$ to denote the subspace of $\Gamma^m_\rho(\mathbb{R}^2)$, which consists of symbols that are homogeneous of degree m in ξ.

(ii) If $a = \sum_{0\leq j<\rho} a^{(m-j)}$, where $a^{(m-j)} \in \dot{\Gamma}^{m-j}_{\rho-j}(\mathbb{R}^2)$, then we call $a^{(m)}$ and $a^{(m-1)}$ as the principal symbol and the subprincipal symbol of a respectively.

(iii) An operator T is said to be of order m, $m \in \mathbb{R}$, if for all $\mu \in \mathbb{R}$, it's bounded from $H^\mu(\mathbb{R}^2)$ to $H^{\mu-m}(\mathbb{R}^2)$. We use S^m to denote the set of all operators of order m.

For $a, f \in L^2$ and a pseudo differential operator $\tilde{a}(x,\xi)$, we define the operator $T_a f$ and $T_{\tilde{a}} f$ as follows,

$$T_a f = \mathscr{F}^{-1}[\int_{\mathbb{R}} \widehat{a}(\xi-\eta)\theta(\xi-\eta,\eta)\widehat{f}(\eta)d\eta], \; T_{\tilde{a}} f = \mathscr{F}^{-1}[\int_{\mathbb{R}} \mathscr{F}_x(\tilde{a})(\xi-\eta,\eta)\theta(\xi-\eta,\eta)\widehat{f}(\eta)d\eta], \tag{2.7}$$

where the cut-off function is defined as follows,

$$\theta(\xi-\eta,\eta) = \begin{cases} 1 \text{ when } |\xi-\eta| \leq 2^{-10}|\eta|, \\ 0 \text{ when } |\xi-\eta| \geq 2^{10}|\eta|. \end{cases} \tag{2.8}$$

LEMMA 2.5. – *Let $m \in \mathbb{R}$ and $\rho > 0$ and let $a \in \Gamma^m_\rho(\mathbb{R}^d)$, if we denote $(T_a)^*$ as the adjoint operator of T_a and denote $\bar{a}$ as the complex conjugate of a, then we know that, $(T_a)^* - T_{a^*}$ is of order $m-\rho$, where*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha \partial_x^\alpha \bar{a}.$$

Moreover, the operator norm of $(T_a)^* - T_{a^*}$ is bounded by $M^m_\rho(a)$.

Proof. – See [1, Theorem 3.10]. □

LEMMA 2.6. – *Let $m \in \mathbb{R}$ and $\rho > 0$, if given symbols $a \in \Gamma^m_\rho(\mathbb{R}^d)$ and $b \in \Gamma^{m'}_\rho(\mathbb{R}^d)$, we define*

$$a\sharp b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a \partial_x^\alpha b,$$

then for all $\mu \in \mathbb{R}$, there exists a constant K such that

$$\|T_a T_b - T_{a\sharp b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K M^m_\rho(a) M^{m'}_\rho(b). \tag{2.9}$$

We have the following lemma on the L^∞_x decay estimate of the linear solution associated with the capillary wave system (1.3).

LEMMA 2.7. – *For $f \in L^1(\mathbb{R}^2)$ and any $\theta \in [0,1]$, there exists an absolute constant C and a constant C_θ which only depends on θ such that the following L^∞_x-type estimates hold,*

$$\|e^{it\Lambda}P_k f\|_{L^\infty_x} \leq C(1+|t|)^{-1}2^{k/2}\|f\|_{L^1}, \quad \text{if } k \geq 0. \tag{2.10}$$

$$\|e^{it\Lambda}P_k f\|_{L^\infty_x} \leq C_\theta(1+|t|)^{-\frac{1+\theta}{2}}2^{\frac{(1-\theta)k}{2}}\|f\|_{L^1}, \quad \text{if } k \leq 0. \tag{2.11}$$

Proof. – After checking the expansion of the phase, see (6.13), we can apply the main result in [20, Theorem 1:(a)&(b)] directly to derive above results. □

3. The energy estimate

The goal of this section is to prove that the energy of solution grows at most at rate $(1+t)^{\delta}$ over time. We first state our bootstrap assumption as follows,

$$\sup_{t\in[0,T]} (1+t)^{-\delta}\|(\tilde{\Lambda} h(t),\psi(t)\|_{H^{N_0}} + (1+t)\|(h(t),\psi)(t)\|_{W^{6,1+\alpha}} \leq \epsilon_1 := \epsilon_0^{5/6}, \tag{3.1}$$

where $\tilde{\Lambda} := |\nabla|^{1/2}(\tanh|\nabla|)^{-1/2}$ and the function space $W^{6,1+\alpha}$ was defined in (1.20).

The main result of this section is summarized as the following proposition.

PROPOSITION 3.1. – *Under the bootstrap assumption* (3.1)*, there exists an absolute constant C such that the following energy estimate holds for any $t \in [0,T]$,*

$$\|(\tilde{\Lambda} h(t),\psi(t))\|_{H^{N_0}}^2$$

$$\leq C\big[\epsilon_0^2 + \int_0^t \|(\tilde{\Lambda} h(s),\psi(s))\|_{H^{N_0}}^2\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,0}}\|(h,\psi)\|_{W^{6,1}}\big)ds\big]. \tag{3.2}$$

We separate this section into three parts: (i) Firstly, we introduce main results and briefly explain main ideas of the paralinearization process for the capillary waves system (1.3). (ii) Secondly, with the highlighted structures of losing derivative inside the system (1.3), we symmetrize the system (1.3) such that it doesn't lose derivatives during the energy estimate. (iii) Lastly, we use the symmetrized system to prove the desired new energy estimate (3.2).

3.1. Paralinearization of the full system

Most of this section has been studied in details in [34]. Here we only briefly introduce related main results and main ideas behind those results. Please refer to [34] for more detailed discussions.

To perform the paralinearization process, we need some basic estimates of the Dirichlet-Neumann operator, which are obtained from analyzing the velocity potential inside the water region "$\Omega(t)$".

Recall that the velocity potential $\phi(t,x)$ satisfies the following Laplace equation with two boundary conditions as follows,

$$\Delta\phi = 0, \quad \phi|_{\Gamma(t)} = \psi(t), \quad \partial_{\vec{n}}\phi|_{\Sigma} = 0. \tag{3.3}$$

To simplify analysis, we map the water region "$\Omega(t)$" into the strap $\mathcal{S} := \mathbb{R}\times[-1,0]$ by doing change of coordinates as follows,

$$(x,y) \longrightarrow (x,z), \quad z := \frac{y-h(t,x)}{1+h(t,x)}.$$

We define $\varphi(t,z) := \phi(t,z+h(t,x))$. From (3.3), we have

$$P\varphi := [\Delta_x + \tilde{a}\partial_z^2 + \tilde{b}\cdot\nabla\partial_z + \tilde{c}\partial_z]\varphi = 0, \quad \varphi|_{z=0} = \psi, \quad \partial_z\varphi|_{z=-1} = 0, \tag{3.4}$$

where

$$\tilde{a} = \frac{(y+1)^2|\nabla h|^2}{(1+h)^4} + \frac{1}{(1+h)^2} = \frac{1+(z+1)^2|\nabla h|^2}{(1+h)^2}, \tag{3.5}$$

$$\tilde{b} = -2\frac{(y+1)\nabla h}{(1+h)^2} = \frac{-2(z+1)\nabla h}{1+h}, \quad \tilde{c} = \frac{-(z+1)\Delta_x h}{(1+h)} + 2\frac{(z+1)|\nabla h|^2}{(1+h)^2}, \tag{3.6}$$

$$G(h)\psi = [-\nabla h \cdot \nabla \phi + \partial_y \phi]\big|_{y=h} = \frac{1+|\nabla h|^2}{1+h}\partial_z \varphi\big|_{z=0} - \nabla\psi \cdot \nabla h. \tag{3.7}$$

Hence, to study the Dirichlet-Neumann operator, it is sufficient to study the only nontrivial part of $G(h)\psi$, which is $\partial_z \varphi\big|_{z=0}$.

From (3.4), we can derive the following fixed point type formulation for $\nabla_{x,z}\varphi$, which provides a good way to analyze and estimate the Dirichlet-Neumann operator in the small data regime. More precisely, we have

$$\begin{aligned}\nabla_{x,z}\varphi = &\Big[\Big[\frac{e^{-(z+1)|\nabla|} + e^{(z+1)|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}}\Big]\nabla\psi, \frac{e^{(z+1)|\nabla|} - e^{-(z+1)|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}}|\nabla|\psi\Big] \\ &+ [\mathbf{0}, g_1(z)] + \int_{-1}^{0} [K_1(z,s) - K_2(z,s) - K_3(z,s)](g_2(s) + \nabla \cdot g_3(s))ds \\ &+ \int_{-1}^{0} K_3(z,s)|\nabla|\mathrm{sign}(z-s)g_1(s) - |\nabla|[K_1(z,s) + K_2(z,s)]g_1(s)\, ds,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}K_1(z,s) := \Big[&\frac{\nabla}{2|\nabla|}\frac{e^{-z|\nabla|} - e^{z|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}} e^{(s-1)|\nabla|} + \frac{\nabla}{2|\nabla|}e^{(z+s)|\nabla|}, \\ &-\frac{1}{2}\frac{e^{z|\nabla|} + e^{-z|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}} e^{(s-1)|\nabla|} + \frac{1}{2}e^{(z+s)|\nabla|}\Big],\end{aligned} \tag{3.9}$$

$$K_2(z,s) := \Big[\frac{\nabla}{2|\nabla|}\frac{e^{-z|\nabla|} - e^{z|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}} e^{-(s+1)|\nabla|}\ , \ -\frac{1}{2}\frac{e^{z|\nabla|} + e^{-z|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}} e^{-(s+1)|\nabla|}\Big], \tag{3.10}$$

$$K_3(z,s) = \Big[\frac{\nabla}{2|\nabla|}e^{-|z-s||\nabla|}\ , \ \frac{1}{2}e^{-|z-s||\nabla|}\mathrm{sign}(s-z)\Big], \tag{3.11}$$

$$g_1(z) = \frac{2h + h^2 - (z+1)^2|\nabla h|^2}{(1+h)^2}\partial_z \varphi + \frac{(z+1)\nabla h \cdot \nabla\varphi}{1+h}, \quad g_1(-1) = 0, \tag{3.12}$$

$$g_2(z) = \frac{(z+1)|\nabla h|^2 \partial_z \varphi}{(1+h)^2} - \frac{\nabla h \cdot \nabla \varphi}{1+h}, \quad g_3(z) = \frac{(z+1)\nabla h \partial_z \varphi}{1+h}. \tag{3.13}$$

In the small data regime, the fixed point type formulation (3.8) is sufficient to derive the L^2-type and L^∞-type estimates for $\nabla_{x,z}\varphi$ as summarized in the following lemma.

LEMMA 3.2. – *Assume that* $(h,\psi) \in H^{N_0+1/2}(\mathbb{R}^2) \times H^{N_0+1/2}(\mathbb{R}^2)$ *and given any* $k, \gamma \in \mathbb{R}$ *s.t.,* $k \le N_0 - 1$ *and* $\gamma \le N_0 - 3$. *Under the bootstrap assumption* (3.1)*, there exists an absolute constant* C *such that the following estimates hold for the derivative of velocity potential "*$\nabla_{x,z}\varphi$*"*

inside the mapped water region,

(3.14)
$$\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k} \leq C\big[\|\nabla\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W^0}}\big],$$

(3.15)
$$\|\nabla_x\varphi\|_{L^\infty_z\widetilde{W^\gamma}} \leq C\|\nabla\psi\|_{\widetilde{W^\gamma}}, \quad \|\partial_z\varphi\|_{L^\infty_z\widetilde{W^\gamma}} \leq C\big[\|\psi\|_{W^{\gamma,1+\alpha}} + \|h\|_{\widetilde{W^{\gamma+1}}}\|\nabla\psi\|_{\widetilde{W^\gamma}}\big],$$

(3.16)
$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z\widetilde{W^\gamma}} \leq C\|\nabla\psi\|_{\widetilde{W^\gamma}}\|h\|_{\widetilde{W^{\gamma+1}}},$$

(3.17)
$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \leq C\big[\|h\|_{\widetilde{W^1}}\||\nabla|\psi\|_{H^k} + \|\nabla\psi\|_{\widetilde{W^0}}\|h\|_{H^{k+1}}\big],$$
$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z L^2} \leq C\big[\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|(h,\psi)\|_{H^2}\big],$$

where the L^∞_x-type function space $\widetilde{W^\gamma}$ is defined as follows,

$$\widetilde{W^\gamma} := \{f : \|f\|_{\widetilde{W^\gamma}} := \|P_{\leq 0}[f](x)\|_{L^\infty_x} + \sum_{k\in\mathbb{Z},k\geq 1} 2^{\gamma k}\|P_k(x)\|_{L^\infty_x} < \infty\}.$$

Proof. – Thanks to the small data regime, above estimates can be obtained from the fixed point type formulation in (3.8) by using a fixed point type argument. With minor modifications, the proof of above estimates are almost same as the proof of Lemma 3.3 in [34]. □

During the paralinearization process, we usually omit good error terms, which do not lose derivatives. For simplicity, we define the equivalence relation "$\approx$" as follows,

$$A \approx B, \quad \text{if and only if } A - B \text{ is a good error term in the sense of (3.18)},$$

(3.18)
$$\|\text{good error term}\|_{H^k} \leq C\big[\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,0}}\|(h,\psi)\|_{W^{6,1}}\big]\big(\|h\|_{H^k} + \|\psi\|_{H^{(k-1/2)_+}}\big),$$

where C is an absolute constant and $0 \leq k \leq N_0$.

As a result of paralinearization in Alazard-Burq-Zuily [1], modulo the good error terms, we can identify the principal part of the Dirichlet-Neumann operator as in the following lemma.

LEMMA 3.3. – *Under the smallness condition* (4.49)*, the following equivalence relation in the sense of* (3.18) *holds,*

(3.19)
$$G(h)\psi \approx T_\lambda\omega - T_V\cdot\nabla h, \quad \omega := \psi - T_B h,$$
$$B \overset{\text{abbr}}{=} B(h)\psi = \frac{G(h)\psi + \nabla h\cdot\nabla\psi}{1+|\nabla h|^2}, \quad V \overset{\text{abbr}}{=} V(h)\psi = \nabla\psi - B\nabla h,$$
$$\lambda = \lambda^{(1)} + \lambda^{(0)}, \quad \lambda^{(1)} := \sqrt{(1+|\nabla h|^2)|\xi|^2 - (\nabla h\cdot\xi)^2},$$
$$\lambda^{(0)} = \frac{1+|\nabla h|^2}{2\lambda^{(1)}}\big(\nabla\cdot\big(\frac{\lambda^{(1)} + i\nabla h\cdot\xi}{1+|\nabla h|^2}\nabla h\big) + i\nabla_\xi\lambda^{(1)}\cdot\nabla\big(\frac{\lambda^{(1)} + i\nabla h\cdot\xi}{1+|\nabla h|^2}\big)\big),$$

where "ω" is the so-called good unknown variable and $\lambda^{(1)}$ and $\lambda^{(0)}$ are the principal symbol and sub-principal of the Dirichlet-Neumann operator respectively.

Proof. – The detailed proof of above lemma can be found in Alazard-Burq-Zuily [1, Proposition 3.14]. Only minor modifications are required. □

As a result of paralinearization in [1], modulo the good error terms, we can identify the bulk terms of the nonlinearity of "$\partial_t \psi$" in the capillary wave system (1.3) as in the following lemma.

LEMMA 3.4. – *Under the bootstrap assumption* (3.1)*, the following equivalence relation in the sense of* (3.18) *holds,*

$$H(h) \approx -T_l h, \qquad l = l^{(2)} + l^{(1)}, \qquad l^{(2)} = (1+|\nabla h|^2)^{-1/2}\Big(|\xi|^2 - \frac{(\nabla h\cdot\xi)^2}{1+|\nabla h|^2}\Big),$$
$$l^{(1)} = \frac{-i}{2}(\nabla_x\cdot\nabla_\xi)l^{(2)},$$

$$\frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla h\cdot\nabla\psi + G(h)\psi)^2}{1+|\nabla h|^2} \approx T_V\cdot\nabla\omega - T_B G(h)\psi. \tag{3.20}$$

Proof. – See [1, Lemma 3.25 & Lemma 3.26]. □

3.2. Symmetrization of the full system

In this subsection, we use the results obtained in the paralinearization process to find out the good substitution variables such that the system of equations satisfied by the good substitution variables has requisite symmetries inside.

Recall (1.3) and results in Lemma 3.3 and Lemma 3.4, we have

$$\begin{cases} \partial_t h \approx T_\lambda\omega - T_V\cdot\nabla h \\ \partial_t\psi \approx -T_l h + T_B G(h)\psi - T_V\cdot\nabla\omega. \end{cases} \tag{3.21}$$

The symmetrization process, which is only relevant at the high frequency part, is same as what Alazard-Burq-Zuily did in [1]. We first state the main results and then briefly explain main ideas behind.

Intuitively speaking, the symmetrization process can be summarized as seeking two good substitution variables $(U_1, U_2) = (T_p h, T_q w)$ with good unknown symbols $p(x,\xi)$ and $q(x,\xi)$ to be determined such that the system of equations satisfied by (U_1, U_2) has requisite symmetries such that it doesn't lose derivatives during energy estimate. As a result of the symmetrization process in Alazard-Burq-Zuily [1], the good substitution variables and their associated symbols are given as follows,

$$\begin{aligned} U_1 &= \tilde\Lambda(h + T_{p|\xi|^{-1/2}-1}h), & U_2 &= \omega + T_{q-1}\omega, \\ \omega &= \psi - T_B h, & \tilde\Lambda &= |\nabla|^{1/2}(\tanh|\nabla|)^{-1/2}, \end{aligned} \tag{3.22}$$

where

$$p = p^{(1/2)} + p^{(-1/2)}, \quad q = (1+|\nabla h|^2)^{-1/2}, \tag{3.23}$$

(3.24)
$$p^{(1/2)} = (1+|\nabla h|^2)^{-5/4}\sqrt{\lambda^{(1)}}, \quad p^{(-1/2)} = \frac{1}{\gamma^{(3/2)}}\big[ql^{(1)} - \gamma^{(1/2)}p^{(1/2)} + i\nabla_\xi\gamma^{(3/2)}\cdot\nabla_x p^{(1/2)}\big],$$

$$\gamma = \sqrt{l^{(2)}\lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}}\frac{\mathrm{Re}\lambda^{(0)}}{2} - \frac{i}{2}\big(\nabla_\xi \cdot \nabla_x\big)\sqrt{l^{(2)}\lambda^{(1)}} - |\xi|^{3/2}. \tag{3.25}$$

Note that, in the sense of losing derivatives, U_1 and U_2 are equivalent to $T_p h$ and $T_q \omega$. Here, we pulled out and emphasized the leading linear terms.

From (3.21) and (3.22), we can derive the system of equations satisfied by U_1 and U_2 as follows,

$$\begin{cases} \partial_t U_1 = \Lambda U_2 + T_\gamma U_2 - T_V \cdot \nabla U_1 + \mathfrak{R}_1, \\ \partial_t U_2 = -\Lambda U_1 - T_\gamma U_1 - T_V \cdot \nabla U_2 + \mathfrak{R}_2, \end{cases} \tag{3.26}$$

where $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are good error terms in the sense of (3.18), i.e., the following estimate holds for the error terms for some absolute constant C,

$$\|\mathfrak{R}_1\|_{H^{N_0}} + \|\mathfrak{R}_2\|_{H^{N_0}} \leq C\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|(\tilde{\Lambda} h,\psi)\|_{H^{N_0}}. \tag{3.27}$$

Very importantly, the symbol "$\gamma(x,\xi)$" satisfies the following equivalence relation,

$$T_\gamma \sim (T_\gamma)^*, \tag{3.28}$$

where the equivalence relation "$\sim$" is defined in the following sense,

$$T_{a_1} \sim T_{a_2}, \text{ iff } \|T_{a_1} f - T_{a_2} f\|_{H^k} \leq C_k\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|f\|_{H^k},$$

where $k \in \mathbb{R}_+$ and C_k is some constant that only depends on "k". From the above equivalence relation, we can verify that the system (3.26) indeed has requisite symmetries for avoiding losing derivatives.

Now, we explain why the good unknown symbols $p(x,\xi)$ and $q(x,\xi)$ are given as (3.23) and (3.24). Recall that $\lambda \in \Gamma_5^1$ and $l \in \Gamma_5^2$. To obtain the system (3.26) from the system (3.21), naturally, we are seeking $p \in \Gamma_5^{1/2}$, $q \in \Gamma_5^0$ and $\lambda \in \Gamma_5^{3/2}$ such that the equivalence relation (3.28) and the following two equivalence relations hold at the same time,

$$T_p T_\lambda \sim T_{\gamma+|\xi|^{3/2}} T_q, \quad T_q T_l \sim T_{\gamma+|\xi|^{3/2}} T_p. \tag{3.29}$$

From Lemma 2.5, we have

$$\big(T_\gamma\big)^* \sim T_{\lambda^*}, \quad \lambda^* = \gamma^{(3/2)} + \overline{\gamma^{(1/2)}} + \frac{1}{i}\nabla_\xi \cdot \nabla_x \gamma^{(3/2)}. \tag{3.30}$$

Hence, (3.28) can be reformulated as follows,

$$T_\gamma \sim T_{\gamma^{(3/2)} + \overline{\gamma^{(1/2)}} + \frac{1}{i}\nabla_\xi \cdot \nabla_x \gamma^{(3/2)}}. \tag{3.31}$$

By using Lemma 2.6, we can derive six equations about the principal symbols and sub-principal symbols of $p(x,\xi)$, $q(x,\xi)$, and $\gamma(x,\xi)$ from the three equivalence relations in (3.29) and (3.31). After solving those equations, one can see that the principal symbols and sub-principal symbols of $p(x,\xi)$, $q(x,\xi)$, and $\gamma(x,\xi)$ are given as in (3.23), (3.24), and (3.25). For more detailed computations, please refer to [1, Subsection 4.2].

From the bootstrap assumption (3.1) and estimates in Lemma 3.2, the following estimate holds,

$$\|U_1 - \tilde{\Lambda} h\|_{H^{N_0}} + \|U_2 - \psi\|_{H^{N_0}} \leq C(\|h\|_{W^{6,1}} + \|\psi\|_{W^{6,1}})\|(\tilde{\Lambda} h,\psi)\|_{H^{N_0}} \leq C\epsilon_1^2, \tag{3.32}$$

where C is an absolute constant. From the above estimate (3.32), we know that the difference of energy between (U_1, U_2) and $(\tilde{\Lambda} h, \psi)$ is a higher order smallness. Therefore, to control the energy of $(\tilde{\Lambda} h, \psi)$ over time, it would be sufficient to control the energy of (U_1, U_2) over time.

3.3. Energy estimate

We define the energy as follows,

$$E_{N_0}(t) := \|U_1(t)\|_{L^2}^2 + \|U_2(t)\|_{L^2}^2 + \|U_1^{N_0}(t)\|_{L^2}^2 + \|U_2^{N_0}(t)\|_{L^2}^2, \tag{3.33}$$

where

$$U_1^{N_0}(t) = T_\beta U_1(t), \quad U_2^{N_0}(t) = T_\beta U_2(t), \quad \beta := \big(\gamma^{(3/2)} + |\xi|^{3/2}\big)^{2N_0/3}, \tag{3.34}$$

where $\gamma^{(3/2)}(x,\xi)$ is the principal symbol of $\gamma(x,\xi)$, which is defined in (3.25). Note that, from the above definition, the following equality holds,

$$\partial_\xi \beta \partial_x \big(\gamma^{(3/2)} + |\xi|^{3/2}\big) = \partial_\xi \big(\gamma^{(3/2)} + |\xi|^{3/2}\big) \partial_x \beta.$$

Hence, very importantly, the operator as follows is an operator of order zero,

$$T_\beta T_{\gamma + |\xi|^{3/2}} - T_{\gamma+|\xi|^{3/2}} T_\beta.$$

REMARK 3.1. – To estimate the high order Sobolev norm, we use the variable $T_\beta U_i$ instead of using $|\nabla|^{N_0} U_i$ because the commutator $[T_{|\xi|^{N_0}}, T_\gamma]$ is of order $1/2$, which causes the loss of derivatives. The idea of using the good variables $T_\beta U_1$ and $T_\beta U_2$ comes from the work of Alazard-Burq-Zuily [1].

Recall the definition (3.34) and the system (3.26). As a result of direct computation, we can derive the system of equations satisfied by $U_1^{N_0}$ and $U_2^{N_0}$ as follows,

$$\begin{cases} \partial_t U_1^{N_0} = \Lambda U_1^{N_0} + T_\gamma U_2^{N_0} - T_V \cdot \nabla U_1^{N_0} + \mathfrak{R}_1^{N_0}, \\ \partial_t U_2^{N_0} = -\Lambda U_1^{N_0} - T_\gamma U_2^{N_0} - T_V \cdot \nabla U_2^{N_0} + \mathfrak{R}_2^{N_0}, \end{cases} \tag{3.35}$$

where the good remainder terms $\mathfrak{R}_1^{N_0}$ and $\mathfrak{R}_1^{N_0}$ satisfy the following estimate,

$$\|\mathfrak{R}_1^{N_0}\|_{L^2} + \|\mathfrak{R}_2^{N_0}\|_{L^2} \le C\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}} \|(h,\psi)\|_{W^{6,0}}\big) \|(\tilde{\Lambda} h, \psi)\|_{H^{N_0}}, \tag{3.36}$$

where C is some absolute constant. Recall (3.33). From the bootstrap assumption (3.1), it is easy to see that the following estimate holds,

$$\begin{aligned} c_2\big(\|\tilde{\Lambda} h(t)\|_{H^{N_0}}^2 + \|\psi(t)\|_{H^{N_0}}^2\big) &\le c_1\big(\|U_1(t)\|_{H^{N_0}}^2 + \|U_2(t)\|_{H^{N_0}}^2\big) \le E_{N_0}(t) \\ &\le C_1\big(\|U_1(t)\|_{H^{N_0}}^2 + \|U_2(t)\|_{H^{N_0}}^2\big) \\ &\le C_2\big(\|\tilde{\Lambda} h(t)\|_{H^{N_0}}^2 + \|\psi(t)\|_{H^{N_0}}^2\big), \end{aligned} \tag{3.37}$$

where c_i and C_i, $i \in \{1,2\}$, are some absolute constants.

Recall the systems of equations in (3.26) and (3.27). From the estimates (3.35) and (3.36) and the $L^2 - L^\infty$ type bilinear estimate, we have

$$
\begin{aligned}
\Big|\frac{d}{dt}E_{N_0}(t)\Big| &\le C_1\Big[\|(U_1(t),U_2(t))\|_{H^{N_0}}\|(\mathfrak{R}_1(t),\mathfrak{R}_2(t),\mathfrak{R}_1^{N_0}(t),\mathfrak{R}_2^{N_0}(t))\|_{L^2}\\
&\quad+\Big|\int_{\mathbb{R}^2}U_1\big(-T_V\cdot\nabla U_1\big)+U_2\big(-T_V\cdot\nabla U_2\big)\\
&\qquad+U_1^{N_0}\big(-T_V\cdot\nabla U_1^{N_0}\big)+U_2^{N_0}\big(-T_V\cdot\nabla U_2^{N_0}\big)dx\Big|\\
&\quad+\Big|\int_{\mathbb{R}^2}U_1(T_\lambda U_2)-U_2(T_\lambda U_1)+U_1^{N_0}(T_\lambda U_2^{N_0})-U_2^{N_0}(T_\lambda U_1^{N_0})\Big|\Big]\\
&\le C_2\Big[\big(\|(h,\psi)\|_{W^{6,1+\alpha}}+\|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|(U_1,U_2)\|_{H^{N_0}}^2\\
&\quad+\Big|\int_{\mathbb{R}^2}U_1(T_\lambda-(T_\lambda)^*)U_2+U_1^{N_0}(T_\lambda-(T_\lambda)^*)U_2^{N_0}dx\Big|\Big]\\
&\le C_3\big(\|(h,\psi)\|_{W^{6,1+\alpha}}+\|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|(U_1,U_2)\|_{H^{N_0}}^2,
\end{aligned}
\tag{3.38}
$$

where $C_i, i\in\{1,2,3\}$, are some absolute constants. Note that, in the above estimate, we used the following facts, which are direct results from (3.30),

$$
\Lambda_1[\gamma]=|\xi|^{1/2}\big(\frac{1}{2}\Delta h-\frac{\xi}{|\xi|}\cdot\nabla_x(\nabla h\cdot\frac{\xi}{|\xi|})\big),\quad M_5^0(\Lambda_{\ge 2}[\gamma]-\Lambda_{\ge 2}[\gamma^*])\le C\|h\|_{W^{6,1}}^2,
\tag{3.39}
$$

where C is some absolute constant. The first equality in the above equality (3.39) is derived from the explicit formula of γ in (3.25). Note that $\Lambda_1[\gamma]$ only depends on the second derivative of h, which explains why we can gain $(1+\alpha)$ derivatives at the low frequency part for the input putted in L^∞-type space.

Combining the estimates (3.38) and (3.37), it is easy to see that the desired estimate (3.2) in Proposition 3.1 holds. Hence finishing the proof of Proposition 3.1.

4. The set-up of the weighted norm estimates

By using the linear dispersion estimates in Lemma 2.7, we reduce the study of the dispersion estimate of the nonlinear solution to the study of the weighted norms of the profile of the nonlinear solution.

In this section, we mainly introduce the set-up of the weighted norms (the Z_1-norm and the Z_2-norm) estimates, which includes two main steps as follows: (i) We identify a good substitution variable, which allows us to study and control properly the evolution of the weighted norms of the good substitution variable over time. (ii) We reduce our goal of proving the sharp dispersion estimate to two desired estimates inside a fixed dyadic time interval.

4.1. A good substitution variable

To avoid losing derivatives at the quadratic level, we use the following variable instead of the velocity potential "ψ" itself,

$$\tilde{\psi} := \psi - T_{|\nabla|\tanh|\nabla|\psi}h,$$

which is the linear and quadratic terms of the good unknown variable "ω" defined in (3.22). Hence, instead of working on the system of equations satisfied by (h,ψ), we work on the system of equations satisfied by $(h,\tilde{\psi})$.

From (1.6) and (1.7), as a result of direct computations, we obtain the following equalities,

$$\begin{aligned}\Lambda_{\leq 2}[\partial_t h] &= |\nabla|\tanh|\nabla|\tilde{\psi} + |\nabla|\tanh|\nabla|\big(T_{|\nabla|\tanh|\nabla|\tilde{\psi}}h\big)\\ &\quad - \nabla\cdot(h\nabla\tilde{\psi}) - |\nabla|\tanh|\nabla|(h|\nabla|\tanh|\nabla|\tilde{\psi})\end{aligned}\tag{4.1}$$

$$\begin{aligned}\Lambda_{\leq 2}[\partial_t\tilde{\psi}] &= \Delta h - \frac{1}{2}|\nabla\tilde{\psi}|^2 + \frac{1}{2}||\nabla|\tanh|\nabla|\tilde{\psi}|^2\\ &\quad - T_{|\nabla|\tanh|\nabla|\tilde{\psi}}|\nabla|\tanh|\nabla|\tilde{\psi} - T_{|\nabla|\tanh|\nabla|\Delta h}h.\end{aligned}\tag{4.2}$$

We remark that the Taylor expansions in (4.1), (4.2) and also in the rest of paper are all in terms of $(h,\tilde{\psi})$.

Next, we reduce the system of equations satisfied by h and $\tilde{\psi}$ into a quasilinear equation satisfied by $u=\tilde{\Lambda}h+i\tilde{\psi}$, where $\tilde{\Lambda}=|\nabla|^{1/2}\big(\tanh|\nabla|\big)^{-1/2}$. Very naturally, we have

$$h=\tilde{\Lambda}^{-1}\Big(\frac{u+\bar{u}}{2}\Big),\quad \tilde{\psi}=c_+u+c_-\bar{u},\quad c_\mu:=-\mu i/2.\tag{4.3}$$

There, from (1.3), (4.1), and (4.2), we can derive the equation satisfied by u as follows,

$$\begin{aligned}(\partial_t+i\Lambda)u &= \sum_{\mu,\nu\in\{+,-\}} Q_{\mu,\nu}(u^\mu,u^\nu) + \sum_{\tau,\kappa,\iota\in\{+,-\}} C_{\tau,\kappa,\iota}(u^\tau,u^\kappa,u^\iota)\\ &\quad + \sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}} D_{\mu_1,\mu_2,\nu_1,\nu_2}(u^{\mu_1},u^{\mu_2},u^{\nu_1},u^{\nu_2}) + \mathscr{R},\end{aligned}\tag{4.4}$$

where $\mathscr{R}$ denotes the quintic and higher order terms. From (4.1), (4.2), and (4.3), we can obtain the detailed formulas of quadratic terms as follows,

$$\begin{aligned}Q_{\mu,\nu}(u^\mu,u^\nu) &= -\frac{c_\nu}{2}\tilde{\Lambda}\partial_x(\tilde{\Lambda}^{-1}u^\mu\partial_x u^\nu)\\ &\quad - \frac{c_\nu}{2}\tilde{\Lambda}|\nabla|\tanh|\nabla|(\tilde{\Lambda}^{-1}u^\mu|\nabla|\tanh|\nabla|u^\nu - T_{|\nabla|\tanh|\nabla|u^\nu}\tilde{\Lambda}^{-1}u^\mu)\\ &\quad + \frac{ic_\mu c_\nu}{2}\big[-\nabla u^\mu\cdot\nabla u^\nu + |\nabla|\tanh|\nabla|u^\mu|\nabla|\tanh|\nabla|u^\nu\\ &\quad - T_{|\nabla|\tanh|\nabla|u^\mu}|\nabla|\tanh|\nabla|u^\nu - T_{|\nabla|\tanh|\nabla|u^\nu}|\nabla|\tanh|\nabla|u^\mu\big]\\ &\quad - \frac{i}{4}T_{|\nabla|\tanh|\nabla|\Delta u^\nu}u^\mu,\quad \mu,\nu\in\{+,-\}.\end{aligned}\tag{4.5}$$

We gave the detailed formulas of quadratic terms $Q_{\mu,\nu}(.,.)$, $\mu,\nu\in\{+,-\}$, because the precise detailed formulas help us to verify a symmetric structure that we will reveal later.

Define the profile of the solution $u(t)$ as $f(t):=e^{it\Lambda}u(t)$. From (4.4), we have

$$\partial_t\widehat{f}(t,\xi) = \sum_{(\mu,\nu)\in\{+,-\}}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}q_{\mu,\nu}(\xi-\eta,\eta)\widehat{f^\mu}(t,\xi-\eta)\widehat{f^\nu}(\eta)d\eta$$

$$+\sum_{\tau,\kappa,\iota\in\{+,-\}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}c_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{f^\tau}(t,\xi-\eta)\widehat{f^\kappa}(t,\eta-\sigma)\widehat{f^\iota}(t,\sigma)d\eta d\sigma$$

$$+\sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}d_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\widehat{f^{\mu_1}}(t,\xi-\eta)$$

$$\times \widehat{f^{\mu_2}}(t,\eta-\sigma)\widehat{f^{\nu_1}}(t,\sigma-\kappa)\widehat{f^{\nu_2}}(t,\kappa)d\eta d\sigma d\kappa + e^{it\Lambda(\xi)}\widehat{\mathscr{R}}(t,\xi), \tag{4.6}$$

where the phases $\Phi^{\mu,\nu}(\xi,\eta)$, $\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$, and $\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$ are defined as follows,

$$\Phi^{\mu,\nu}(\xi,\eta) = \Lambda(|\xi|) - \mu\Lambda(|\xi-\eta|) - \nu\Lambda(|\eta|), \quad \Lambda(|\xi|) := |\xi|^{3/2}\sqrt{\tanh|\xi|}, \tag{4.7}$$

$$\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma) = \Lambda(|\xi|) - \tau\Lambda(|\xi-\eta|) - \kappa\Lambda(|\eta-\sigma|) - \iota\Lambda(|\sigma|), \tag{4.8}$$

$$\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa) = \Lambda(|\xi|) - \mu_1\Lambda(|\xi-\eta|) - \mu_2\Lambda(|\eta-\sigma|) - \nu_1\Lambda(|\sigma-\kappa|) - \nu_2\Lambda(|\kappa|). \tag{4.9}$$

From (4.5), we write explicitly the symbol $q_{\mu,\nu}(\xi-\eta,\eta)$ of $Q_{\mu,\nu}(u^\mu,u^\nu)$ in the sense of (2.2) as follows,

$$\begin{aligned} q_{\mu,\nu}(\xi-\eta,\eta) = \Big(& \frac{c_\nu\tilde{\lambda}(|\xi|^2)}{2\tilde{\lambda}(|\xi-\eta|^2)}\big(\xi\cdot\eta - |\xi||\eta|\tanh(|\xi|)\tanh(|\eta|)\big) \\ &+ \frac{ic_\mu c_\nu}{2}\big((\xi-\eta)\cdot\eta + |\xi-\eta||\eta| \times \tanh(|\xi-\eta|)\tanh(|\eta|)\big)\Big)\tilde{\theta}(\eta,\xi-\eta) \\ + \Big(&\frac{c_\mu\tilde{\lambda}(|\xi|^2)}{2\tilde{\lambda}(|\eta|^2)}\big((\xi-\eta)\cdot\xi - |\xi-\eta||\xi|\tanh(|\xi|)\tanh(|\xi-\eta|)\big) \\ &+ \frac{c_\nu\tilde{\lambda}(|\xi|^2)}{2\tilde{\lambda}(|\xi-\eta|^2)}\xi\cdot\eta + ic_\mu c_\nu(\xi-\eta)\cdot\eta + \frac{i}{4}|\eta|^2(\tanh|\eta|)^2\Big)\theta(\eta,\xi-\eta), \end{aligned} \tag{4.10}$$

where

$$\tilde{\lambda}(\xi) := |\xi|^{1/4}(\tanh(\sqrt{|\xi|}))^{-1/2}, \quad \tilde{\lambda}(\xi) = 1 + \frac{|\xi|}{6} + O(|\xi|^2), \quad \text{if } |\xi| \le 2^{-10},$$

$$\tilde{\theta}(\eta,\xi-\eta) := 1 - \theta(\eta,\xi-\eta) - \theta(\xi-\eta,\eta). \tag{4.11}$$

Note that, in (4.10), we switched the roles of $\xi-\eta$ and η when $|\xi-\eta| \le 2^{-10}|\eta|$. As a result, the following estimate holds inside the support of the symbol $q_{\mu,\nu}(\xi-\eta,\eta)$, $\mu,\nu\in\{+,-\}$,

$$k_2 \le k_1 + 10, \quad \text{where } \eta \in \mathrm{supp}(\psi_{k_2}(x)), \xi-\eta \in \mathrm{supp}(\psi_{k_1}(\xi-\eta)). \tag{4.12}$$

From the estimate (2.3) in Lemma 2.1 and the detailed formula of the symbol $q_{\mu,\nu}(\xi-\eta,\eta)$ in (4.10), the following rough estimate holds for some absolute constant C,

$$\|q_{\mu,\nu}(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathscr{S}^\infty} \le C2^{2k_1}, \quad \mu,\nu\in\{+,-\}. \tag{4.13}$$

Moreover, from the explicit formula in (4.10), we can identify the leading part "$c(\xi)$" of $q_{+,\nu}(\xi-\eta,\eta)$ for the case when $|\eta| \le 2^{-10}|\xi|$ as follows,

$$c(\xi) := \frac{c_+}{2}\tilde{\lambda}(|\xi|^2)|\xi|^2\big(1-\tanh(|\xi|)^2\big). \tag{4.14}$$

After subtracting $c(\xi)$ from the symbol $q_{+,\nu}(.,.)$, from the estimate (2.3) in Lemma 2.1, the following improved estimate holds for some absolute constant C,

$$\|\big(q_{+,\nu}(\xi-\eta,\eta) - c(\xi)\big)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathscr{S}^\infty} \le C2^{k_2+k_1}, \quad \text{if } k_2 \le k_1 - 10. \tag{4.15}$$

Since it is also very essential to identify the symmetric structure inside the cubic terms, which will play an important role in the Z_2-norm estimate of the cubic terms, we summarize some properties of the symbols of cubic terms of the Dirichlet-Neumann operator as in the following lemma.

LEMMA 4.1. – *After writing the cubic term $\Lambda_3[B(h)\psi]$ in terms of u and $\bar{u}$ via the equality* (4.3)*, we do dyadic decompositions for all inputs and rearrange inputs such that the following unique decomposition holds*

$$\Lambda_3[B(h)\psi] = \sum_{\mu,\nu,\tau\in\{+,-\}} C'_{\mu,\nu,\tau}(u^{\mu}, u^{\nu}, u^{\tau}),$$

where the first input u^{μ} of cubic term $C'_{\mu,\nu,\tau}(u^{\mu}, u^{\nu}, u^{\tau})$ has the largest scale of dyadic localization among three inputs. Then there exists an absolute constant C such that the following estimates hold for the symbol $c'_{\mu,\nu,\tau}(\xi,\eta,\sigma)$ of the cubic term $C'_{\mu,\nu,\tau}(u^{\mu}, u^{\nu}, u^{\tau})$,

$$\|c'_{\mu,\nu,\tau}(\xi,\eta,\sigma)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty} \leq C 2^{2k_1+2k_{1,+}}. \tag{4.16}$$

$$\Big\|\Big(c'_{\mu,\nu,\tau}(\xi,\eta,\sigma) - \frac{c_\mu}{4} d(\xi)\Big)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\Big\|_{\mathcal{S}^\infty} \leq C 2^{\max\{k_2,k_3\}+3k_{1,+}}, \tag{4.17}$$

if $k_2, k_3 \leq k_1 - 10$, where the detailed formula of $d(\xi)$ is given in (4.19)*. Moreover, there exists an absolute constant C such that the following rough estimate holds for the symbol of quartic terms $\Lambda_4[B(h)\psi]$,*

$$\begin{aligned} \|d_{\mu_1,\nu_1,\mu_2,\nu_2}(\xi,\eta,\sigma,\kappa)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma-\kappa)\psi_{k_4}(\kappa)\|_{\mathcal{S}^\infty} \\ \leq C 2^{2\max\{k_1,\dots,k_4\}+3\max\{k_1,\dots,k_4\}_+}. \end{aligned} \tag{4.18}$$

Proof. – Note that the detailed formulas of symbols of cubic terms and quartic terms can be derived from iterating the fixed point type formulation of $\nabla_{x,z}\varphi$ in (3.8). To prove (4.16) and (4.18), it is sufficient to prove that the corresponding estimates hold for $\Lambda_3[g_i(z)]$ and $\Lambda_4[g_i(z)]$, $i \in \{1,2,3\}$. From (3.12) and (3.13), we have

$$\Lambda_2[g_1(z)] = 2h\Lambda_1[\partial_z\varphi] + (z+1)\nabla h \cdot \Lambda_1[\nabla\varphi],$$
$$\Lambda_2[g_2(z)] = -\nabla h \cdot \Lambda_1[\nabla\varphi], \quad \Lambda_2[g_3(z)] = (z+1)\nabla h \Lambda_1[\partial_z\varphi].$$

Recall that

$$\Lambda_1[\nabla_{x,z}\varphi] = \bigg[\Big[\frac{e^{-(z+1)|\nabla|} + e^{(z+1)|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}}\Big]\nabla\psi, \frac{e^{(z+1)|\nabla|} - e^{-(z+1)|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}}|\nabla|\psi\bigg].$$

From the above formula and the formulation (3.8), we know that there are two derivatives inside $\Lambda_2[\nabla_{x,z}\varphi(z)]$ at low frequencies. We can keep doing the iteration process to check the minimal number and the maximal number of derivatives inside $\Lambda_3[\nabla_{x,z}\varphi]$. For example, from (3.12) and (3.13), we obtain the cubic terms of $g_i(z)$, $i \in \{1,2,3\}$, as follows,

$$\Lambda_3[g_1(z)] = 2h\Lambda_2[\partial_z\varphi] + \big(-3h^2 - (z+1)^2|\nabla h|^2\big)\Lambda_1[\partial_z\varphi] - (z+1)h\nabla h \cdot \nabla\varphi,$$
$$\Lambda_3[g_2(z)] = (z+1)|\nabla h|^2\Lambda_1[\partial_z\varphi] + h\nabla h \cdot \Lambda_1[\nabla\varphi] - \nabla h \cdot \Lambda_2[\nabla\varphi],$$
$$\Lambda_3[g_3(z)] = (z+1)\nabla h\Lambda_2[\partial_z\varphi] - (z+1)h\nabla h\Lambda_1[\partial_z\varphi].$$

Recall that there are at least two derivatives inside $\Lambda_2[\nabla_{x,z}\varphi]$. Hence, we know that there are at least two derivatives and at most four derivatives in total inside $\Lambda_3[\nabla_{x,z}\varphi]$. Following the same strategy, we know that there are at least two derivatives and at most five derivatives inside $\Lambda_4[\nabla_{x,z}\varphi]$. There two facts imply that our desired estimates (4.16) and (4.18) hold.

Next, we prove our desired estimate (4.17). We first identify the bulk term, in which all derivatives act on the input that has the largest scale of dyadic localization of frequencies.

With this principle in mind, recall (3.12) and (3.13), we know that the bulk term only appears in $g_1(z)$, which is $T_{(2h+h^2)/(1+h)^2}\partial_z\varphi$. Recall the fixed point formulation (3.8). We know that the bulk term of $\Lambda_2[\partial_z\varphi(z)]$, in which all derivatives act on the input with the largest scale of dyadic localization of frequencies, is given as follows,

$$\int_{-1}^{0}\big(-e^{-|z-s||\nabla|}-e^{(z+s)|\nabla|}+\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}(e^{(s-1)|\nabla|}+e^{-(s+1)|\nabla|})\big) \\ \times\frac{e^{s+1}|\nabla|-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|^2\big(T_h\psi\big)ds+2\frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|(T_h\psi).$$

In the same spirit, we can derive the bulk term of $\Lambda_3[\partial_z\varphi(z)|_{z=0}]$, in which all derivatives act on the input with the largest scale of dyadic localization of frequencies, is given as follows,

$$C(h,h,\psi)=\mathscr{F}^{-1}\big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widehat{\psi}(\xi-\eta)\widehat{h}(\eta-\sigma)\widehat{h}(\sigma)d(\xi)\theta(\sigma,\xi)\theta(\eta-\sigma,\xi)d\eta d\sigma\big],$$

where

$$\begin{aligned} d(\xi):=2\int_{-1}^{0}\int_{-1}^{0}&\big(\frac{(e^{-(z+1)|\xi|}-e^{(z+1)|\xi|})}{e^{-|\xi|}+e^{|\xi|}}\big)\frac{e^{(s+1)|\xi|}-e^{-(s+1)|\xi|}}{e^{|\xi|}+e^{-|\xi|}}\\ &\times\big(\frac{(e^{z|\xi|}+e^{-z|\xi|})(e^{(s-1)|\xi|}+e^{-(s+1)|\xi|})}{e^{-|\xi|}+e^{|\xi|}}-e^{-|z-s||\xi|}-e^{(z+s)|\xi|}\big)|\xi|^3dsdz\\ &-\int_{-1}^{0}\big(\frac{(e^{-(s+1)|\xi|}-e^{(s+1)|\xi|})}{e^{-|\xi|}+e^{|\xi|}}\big)^2|\xi|^2ds+\tanh(|\xi|)|\xi|. \end{aligned} \tag{4.19}$$

After removing the bulk term, by definition, there is at least one derivative acts on the input, which doesn't have the largest scale of dyadic localization of frequencies, for the rest of terms. This fact implies that our desired estimate (4.17) holds. □

With the previous preparation, which improves our understanding of the equation satisfied by u in (4.4), we are now ready to find a good substitution variable. We seek a good substitution variable as follows,

$$\begin{aligned} v(t)=u(t)&+\sum_{\mu,\nu\in\{+,-\}}A_{\mu,\nu}(u^{\mu}(t),u^{\nu}(t))+\sum_{\tau,\kappa,\iota\in\{+,-\}}B_{\tau,\kappa,\iota}(u^{\tau}(t),u^{\kappa}(t),u^{\iota}(t))\\ &+\sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}E_{\mu_1,\mu_2,\nu_1,\nu_2}(u^{\mu_1}(t),u^{\mu_2}(t),u^{\nu_1}(t),u^{\nu_2}(t)), \end{aligned} \tag{4.20}$$

where quadratic terms $A_{\mu,\nu}(.,.)$, cubic terms $B_{\tau,\kappa,\iota}(.,.,.)$, and quartic terms $E_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.,.)$ are to be determined. From the equation satisfied by $u(t)$ in (4.4) and definition of $v(t)$ in (4.20), as a result of direct computation, we have

(4.21)

$$\begin{aligned} (\partial_t+i\Lambda)v=&\sum_{\mu,\nu\in\{+,-\}}\tilde{Q}_{\mu,\nu}(v^{\mu}(t),v^{\nu}(t))+\sum_{\tau,\kappa,\iota\in\{+,-\}}\tilde{C}_{\tau,\kappa,\iota}(v^{\tau}(t),v^{\kappa}(t),v^{\iota}(t))\\ &+\sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}\tilde{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),v^{\mu_2}(t),v^{\nu_1}(t),v^{\nu_2}(t))+\mathscr{R}_1(t), \end{aligned}$$

where $\mathscr{R}_1(t)$ is the quintic and higher order terms. The quadratic terms and cubic terms are given as follows,

$$\tilde{Q}_{\mu,\nu}(v^{\mu},v^{\nu}) = Q_{\mu,\nu}(v^{\mu},v^{\nu}) + i\Lambda(A_{\mu,\nu}(v^{\mu},v^{\nu})) - i\mu A_{\mu,\nu}(\Lambda v^{\mu},v^{\nu}) - i\nu A_{\mu,\nu}(v^{\mu},\Lambda v^{\nu}), \tag{4.22}$$

$$\begin{aligned}\tilde{C}_{\tau,\kappa,\iota}(v^{\tau},v^{\kappa},v^{\iota}) := \widehat{C}_{\tau,\kappa,\iota}(v^{\tau},v^{\kappa},v^{\iota}) &+ i\Lambda(B_{\tau,\kappa,\iota}(v^{\tau},v^{\kappa},v^{\iota})) - i\tau B_{\tau,\kappa,\iota}(\Lambda v^{\tau},v^{\kappa},v^{\iota})\\ &- i\kappa B_{\tau,\kappa,\iota}(v^{\tau},\Lambda v^{\kappa},v^{\iota}) - i\iota B_{\tau,\kappa,\iota}(v^{\tau},v^{\kappa},\Lambda v^{\iota}),\end{aligned} \tag{4.23}$$

where the cubic term $\widehat{C}_{\tau,\kappa,\iota}(v^{\tau},v^{\kappa},v^{\iota})$ is the unique cubic term associated with the following equality, such that the scales of dyadic localized frequencies of inputs v^{τ}, v^{κ}, and v^{ι} are ordered in a descending manner after we rearrange the inputs,

$$\begin{aligned}\sum_{\tau,\kappa,\iota\in\{+,-\}} \widehat{C}_{\tau,\kappa,\iota}(v^{\tau},v^{\kappa},v^{\iota}) = &\sum_{\tau,\kappa,\iota\in\{+,-\}} C_{\tau,\kappa,\iota}(v^{\tau},v^{\kappa},v^{\iota})\\ &+ \sum_{\mu,\nu,\mu_1,\nu_1\in\{+,-\}} A_{\mu,\nu}(P_{\mu}[Q_{\mu_1,\nu_1}(v^{\mu_1},v^{\nu_1})],v^{\nu})\\ &+ A_{\mu,\nu}(v^{\nu},P_{\nu}[Q_{\mu_1,\nu_1}(v^{\mu_1},v^{\nu_1})])\\ &- \tilde{Q}_{\mu,\nu}\big(P_{\mu}\big(A_{\mu_1,\nu_1}(v^{\mu_1},u^{\nu_1})\big),v^{\nu}\big)\\ &- \tilde{Q}_{\mu,\nu}\big(v^{\mu},P_{\nu}\big(A_{\mu_1,\nu_1}(v^{\mu_1},v^{\nu_1})\big)\big).\end{aligned} \tag{4.24}$$

More precisely, the following estimate holds inside the support of symbol $\hat{c}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$ of the trilinear operator $\widehat{C}_{\tau,\kappa,\iota}(.,.,.)$,

(4.25)
$$k_3 \le k_2 \le k_1, \qquad \text{where } \sigma\in \mathrm{supp}(\psi_{k_3}(x)), \eta-\sigma\in\mathrm{supp}(\psi_{k_2}(x)), \xi-\eta\in\mathrm{supp}(\psi_{k_1}(x)).$$

Similarly, for any $\mu_1,\mu_2,\nu_1,\nu_2 \in \{+,-\}$, the quartic term $\tilde{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),v^{\mu_2}(t), v^{\nu_1}(t),v^{\nu_2}(t))$ in (4.21) is given as follows,

(4.26)
$$\begin{aligned}\tilde{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),v^{\mu_2}(t),v^{\nu_1}(t),v^{\nu_2}(t)) = &\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),v^{\mu_2}(t),v^{\nu_1}(t),v^{\nu_2}(t))\\ &+ i\Lambda(E_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),v^{\mu_2}(t),v^{\nu_1}(t),v^{\nu_2}(t)))\\ &- i\mu_1 E_{\mu_1,\mu_2,\nu_1,\nu_2}(\Lambda v^{\mu_1}(t),v^{\mu_2}(t),v^{\nu_1}(t),v^{\nu_2}(t))\\ &- i\mu_2 E_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),\Lambda v^{\mu_2}(t),v^{\nu_1}(t),v^{\nu_2}(t))\\ &- i\nu_1 E_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),v^{\mu_2}(t),\Lambda v^{\nu_1}(t),v^{\nu_2}(t))\\ &- i\nu_2 E_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),v^{\mu_2}(t),v^{\nu_1}(t),\Lambda v^{\nu_2}(t)),\end{aligned}$$

where $\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1},v^{\mu_2},v^{\nu_1},v^{\nu_2})$ is the unique decomposition associated with the quartic terms such that the scales of dyadic localization of frequencies of all inputs are ordered in a descending manner. More precisely, the following estimate holds inside the support of symbol $\hat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)$ of the multilinear operator $\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.,.)$,

$$\begin{aligned}k_4\le k_3\le k_2\le k_1, \qquad &\text{where } \kappa\in\mathrm{supp}(\psi_{k_4}(x)), \sigma-\kappa\in\mathrm{supp}(\psi_{k_3}(x)),\\ &\eta-\sigma\in\mathrm{supp}(\psi_{k_2}(x)), \qquad \xi-\eta\in\mathrm{supp}(\psi_{k_1}(x)),\end{aligned} \tag{4.27}$$

The detail formula of $\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.,.)$ can be obtained explicitly from $A_{\mu,\nu}(.,.)$, $B_{\tau,\kappa,\iota}(.,.,.)$, $Q_{\mu,\nu}(.,.,.)$, $C_{\tau,\kappa,\iota}(.,.,.)$, and the quartic terms $D_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.,.)$ in (4.4). Since its detailed formula is not necessary in later argument, for simplicity, we omit it here.

Now, we are ready to determine the normal formal transformation defined in (4.20).

Firstly, we consider the quadratic terms. Recall (4.7). Note that, if $|\eta| \leq 2^{-10}|\xi|$, $\mu = -$ or if $|\xi| \leq 2^{-10}|\eta|$, $\mu\nu = +$, the size of phase "$\Phi^{\mu,\nu}(\xi,\eta)$" is comparable to "$\max\{\Lambda(|\eta|), \Lambda(|\xi|)\}$," which is relatively big. Moreover, if (ξ,η) lies inside a small neighborhood of $(\xi, \xi/2)$(the space resonance set), the size of phase is also relatively big. More precisely, the following estimate holds,

$$c\Lambda(|\xi|) \leq |\Phi^{\mu,\nu}(\xi,\eta)| \leq C\Lambda(|\xi|), \quad \text{if} \quad |\eta - \xi/2| \leq 2^{-10}|\xi|,$$

where c and C are some absolute constants.

To take the advantage of the fact that the phase is highly oscillating with respect to time in the aforementioned scenarios, we use the normal form transformation $A_{\mu,\nu}(.,.)$ by choosing the symbol $a_{\mu,\nu}(.,.)$ defined as follows,

(4.28)
$$\begin{aligned} a_{\mu,\nu}(\xi-\eta,\eta) = \sum_{k_2\in\mathbb{Z}} \frac{iq_{\mu,\nu}(\xi-\eta,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)} \psi_{k_2}(\eta)\big(\psi_{\leq k_2-10}(\eta-\xi/2)\psi_{\leq k_2+9}(\xi-\eta)\psi_{\geq k_2-9}(\xi) \\ + \mathbf{1}_{\{-\}}(\mu)\psi_{\geq k_2+10}(\xi-\eta) + \mathbf{1}_{\{+\}}(\mu\nu)\psi_{\leq k_2-10}(\xi)\psi_{\leq k_2+9}(\xi-\eta)\big), \end{aligned}$$

where $\mathbf{1}_S(\cdot)$ denotes the characteristic function of set S and the phase $\Phi^{\mu,\nu}(\xi,\eta)$ satisfies the following estimate inside the support of $a_{\mu,\nu}(\xi-\eta,\eta)$,

$$\begin{aligned} (4.29) \quad c\max\{|\xi|,|\eta|\}^2(1+\max\{|\xi|,|\eta|\})^{-1/2} &\leq |\Phi^{\mu,\nu}(\xi,\eta)| \\ &\leq C\max\{|\xi|,|\eta|\}^2(1+\max\{|\xi|,|\eta|\})^{-1/2}, \end{aligned}$$

where c and C are some absolute constants.

Next, we proceed to consider the cubic terms. Recall (4.8). Note that, for $\tau,\kappa,\iota \in \{+,-\}$, the phase $\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is relatively big in the scenarios listed as follows,

- If $\tau = -$ and $|\eta|, |\sigma| \leq 2^{-10}|\xi|$.
- If $|\eta - \xi/2| \leq 2^{-10}|\xi|$ and $\sigma \leq 2^{-10}|\xi|$.
- If $|\eta - 2\xi/3| \leq 2^{-10}|\xi|$ and $|\sigma - \xi/3| \leq 2^{-10}|\xi|$, i.e., $(\xi-\eta, \eta-\sigma, \sigma)$ is close to $(\xi/3, \xi/3, \xi/3)$, which is the space resonance in η and σ set.
- If $|\xi - \eta + \tau\xi| \leq 2^{-10}|\xi|$, $|\eta - \sigma + \kappa\xi| \leq 2^{-10}|\xi|$, and $|\sigma + \iota\xi| \leq 2^{-10}|\xi|$, i.e., $(\xi-\eta, \eta-\sigma, \sigma)$ is very close to $(-\tau\xi, -\kappa\xi, -\iota\xi)$, which is the space resonance in η and σ set, where $(\tau,\kappa,\iota) \in \widetilde{S} := \{(+,-,-), (-,+,-), (-,-,+)\}$. See the proof of Lemma 5.7 for more details.

To take the advantage of the high oscillation of phase with respect to time in aforementioned scenarios, we use the normal form transformation $B_{\tau,\kappa,\iota}(.,.,.)$ by choosing the symbol

$b_{\tau,\kappa,\iota}(.,.,.)$ defined as follows,

(4.30)

$$\begin{aligned}b_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma) = \frac{i\widehat{c}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)}{\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)} &\times \sum_{k\in\mathbb{Z}} \psi_k(\xi)\big(\mathbf{1}_{\widetilde{S}}((\tau,\kappa,\iota))\psi_{\le k-10}((1+\tau)\xi-\eta)\psi_{\le k-10}(\sigma+\iota\xi)\\ &+\psi_{\le k-10}(\eta-2\xi/3)\psi_{\le k-10}(\sigma-\xi/3)\\ &+\psi_{\le k-10}(\eta-\xi/2)\psi_{\le k-10}(\sigma)\\ &+\mathbf{1}_{\{-\}}(\tau)\psi_{\le k-10}(\eta-\sigma)\psi_{\le k-10}(\sigma)\big),\end{aligned}$$

where $\widehat{c}_{\tau,\kappa,\iota}(.,.,.)$ is the associated symbol of cubic term $\widehat{C}_{\tau,\kappa,\iota}(.,.,.)$ which is defined in (4.24) and the phase "$\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$" satisfies the following estimate inside the support of the symbol $\widehat{c}_{\tau,\kappa,\iota}(.,.,.)$,

$$(4.31)\quad \begin{aligned}&|\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)|\\ &\qquad\in \max\{|\xi|,|\eta-\sigma|,|\sigma|\}^2(1+\max\{|\xi|,|\eta-\sigma|,|\sigma|\})^{-1/2}[c,C],\end{aligned}$$

where where c and C are some absolute constants.

Lastly, we consider the quartic terms. Note that the phase $\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$, which is defined in (4.9), is relatively big if $|\eta|,|\sigma|,|\kappa| \le 2^{-10}|\xi|$, $\mu_1 = -$ or if $|\eta-\xi/2|,|\sigma|,|\kappa|\le 2^{-10}|\xi|$. Hence, we use the normal form transformation $E_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.,.)$ by choosing its symbol $e_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.)$ defined as follows,

$$\begin{aligned}e_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa) = \frac{i\widehat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)}{\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\sum_{k\in\mathbb{Z}}\psi_k(\xi)\end{aligned}$$

(4.32)

$$\times\big(\psi_{\le k-10}(\eta-\xi/2)\psi_{\le k-10}(\sigma-\kappa)\psi_{\le k-10}(\kappa)+\mathbf{1}_{\{-\}}(\mu_1)\psi_{\le k-10}(\eta)\psi_{\le k-10}(\sigma-\kappa)\psi_{\le k-10}(\kappa)\big),$$

where $\widehat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.,.)$ is the associated symbol of quartic term $\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.,.)$ and the phase $\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$ satisfies the following estimate inside the support of the symbol $\widehat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(.,.,.,.)$,

$$(4.33)\quad \begin{aligned}&|\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)|\\ &\qquad\in \max\{|\xi|,|\eta-\sigma|,|\sigma-\kappa|,|\kappa|\}^2(1+\max\{|\xi|,|\eta-\sigma|,|\sigma-\kappa|,|\kappa|\})^{-1/2}[c,C],\end{aligned}$$

where c and C are some absolute constants.

From the estimates (4.29), (4.31), and (4.33, the estimate (2.3) in Lemma 2.1, and the estimate (4.13), the following estimate holds for some absolute constant C,

$$(4.34)\quad \begin{aligned}&\|a_{\mu,\nu}(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty}\\ &\quad+\|b_{\tau,\kappa,\iota}(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}\\ &\quad+\|e_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\\ &\qquad\times\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma-\kappa)\psi_{k_4}(\kappa)\|_{\mathcal{S}^\infty}\\ &\le C2^{k_{1,+}}.\end{aligned}$$

With the above determined normal form transformation, now we are ready to study the time evolution of the profile $g(t) := e^{it\Lambda}v(t)$ associated with $v(t)$. Recall (4.21). As a result of direct computations, we obtain the following equality,

$$\begin{aligned}\partial_t g(t,\xi)\psi_k(\xi) = &\sum_{\mu,\nu\in\{+,-\}}\sum_{k_1,k_2\in\mathbb{Z},k_2\leq k_1+10} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\\ &+\sum_{\tau,\kappa,\iota\in\{+,-\}}\sum_{k_3\leq k_2\leq k_1} T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)\\ &+\sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}\sum_{k_4\leq k_3\leq k_2\leq k_1} K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi) + e^{it\Lambda(\xi)}\widehat{\mathcal{R}_1}(t,\xi)\psi_k(\xi),\end{aligned}\tag{4.35}$$

where

$$B^{\mu,\nu}_{k,k_1,k_2}(t,\xi) := \int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(\eta)\psi_k(\xi)d\eta,\tag{4.36}$$

$$\begin{aligned}T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi) = &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^{\tau}_{k_1}}(t,\xi-\eta)\widehat{g^{\kappa}_{k_2}}(t,\eta-\sigma)\\ &\times\widehat{g^{\iota}_{k_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma,\end{aligned}\tag{4.37}$$

$$\begin{aligned}K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi) = &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\\ &\times\widehat{g^{\mu_1}_{k_1}}(t,\xi-\eta)\widehat{g^{\mu_2}_{k_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)\psi_k(\xi)d\eta d\sigma d\kappa,\end{aligned}\tag{4.38}$$

where

$$\begin{aligned}\tilde{q}_{\mu,\nu}(\xi-\eta,\eta) = &\sum_{k_2\in\mathbb{Z}} q_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_2}(\eta)\big(\psi_{\geq k_2-9}(\xi-2\eta)\psi_{\leq k_2+4}(\xi-\eta)\psi_{\geq k_2-5}(\xi)\\ &+\frac{1+\mu}{2}\psi_{\geq k_2+5}(\xi-\eta)+\frac{(1-\mu\nu)}{2}\psi_{\leq k_2-5}(\xi)\psi_{\leq k_2+4}(\xi-\eta)\big),\end{aligned}\tag{4.39}$$

$$\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma) = \hat{c}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma) + ib_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma),\tag{4.40}$$

$$\begin{aligned}&\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\\ &\quad= \hat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\\ &\qquad+ie_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa).\end{aligned}\tag{4.41}$$

Recall the fact that we rearranged inputs such that the scales of dyadic localization of inputs are ordered in a descending manner, see (4.12), (4.25) and (4.27). It explains why we have $k_2\leq k_1+10$, $k_3\leq k_2\leq k_1$ and $k_4\leq k_3\leq k_2\leq k_1$ in (4.35).

Note that, from (4.39), the following equalities hold if $|\eta|\leq 2^{-10}|\xi|$,

$$\tilde{q}_{-,\nu}(\xi-\eta,\eta)=0,\quad \tilde{q}_{+,\nu}(\xi-\eta,\eta)=q_{+,\nu}(\xi-\eta,\eta).\tag{4.42}$$

Moreover, if $|\xi|\leq 2^{-10}|\eta|$, we have

$$\tilde{q}_{\mu,\mu}(\xi-\eta,\eta)=0,\qquad \mu\in\{+,-\}.\tag{4.43}$$

Recall (4.40) and (4.41). From the rough estimates of the symbols of the cubic terms in (4.16) and the quartic terms in (4.18), the following rough estimates hold for some absolute constant C,

$$\|\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \le C2^{2k_1+3k_{1,+}}, \tag{4.44}$$

(4.45)
$$\|\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_1'}(\sigma-\kappa)\psi_{k_2'}(\kappa)\|_{\mathcal{S}^\infty} \le C2^{2k_1+4k_{1,+}}.$$

In later high order weighted norm estimate, we will also need to use the hidden symmetry inside the symbol $\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$ when $|\sigma|,|\eta|\le 2^{-10}|\xi|$. To this end, we identify the leading symbol inside $\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$ first. From (4.30) and (4.40), we know that we only have to consider the case when $\tau=+$ and the leading part of $\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$ is same as the leading part of $\tilde{c}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$. Recall (4.24) and (4.28).If $k_2,k_3\le k_1-10$, then the following estimate holds,
(4.46)
$$\|(\tilde{d}_{+,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)-e(\xi))\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty} \le C2^{\max\{k_2,k_3\}+k_1+4k_{1,+}}.$$

where C is some absolute constant and $e(\xi)$ is given as follows,

$$e(\xi) := \frac{c_+}{4}\tilde{\lambda}(|\xi|^2)d(\xi) - \frac{ic(\xi)^2}{\Lambda(|\xi|)}, \tag{4.47}$$

where "$d(\xi)$" is defined in (4.19). We remark that the first part of $e(\xi)$ comes from the cubic term $C_{\tau,\kappa,\iota}(u^\tau,u^\kappa,u^\iota)$ in (4.24), see (4.17) in Lemma 4.1 and the second part of $e(\xi)$ comes from the composition of quadratic terms and the normal form transformation in (4.24).

4.2. Further reduction of the dispersion estimate

In this subsection, we first show that the dispersion rate of the nonlinear solution $v(t)$ and $u(t)$ are comparable in $W^{6,1+\alpha}$ space and then reduce the control of the dispersion rate of $v(t)$ into the control of weighted norms for the profile $g(t)$ of $v(t)$ in a fixed dyadic time interval.

LEMMA 4.2. – *Under the bootstrap assumption* (3.1)*, the following estimate holds,*

$$\sup_{t\in[0,T]}(1+t)\|v(t)-u(t)\|_{W^{6,1+\alpha}} + \|v(t)-u(t)\|_{H^{N_0-10}} \le \epsilon_0. \tag{4.48}$$

Proof. – From the $L^\infty-L^\infty$ type bilinear estimate in (2.5), the estimate of symbols in (4.34) and the $L^\infty\to L^2$ type Sobolev embedding, the following estimate holds for some absolute constant C,

$$\|v(t)-u(t)\|_{W^{6,1+\alpha}} \le C\|u(t)\|^{4/3}_{W^{6,1+\alpha}}\|u(t)\|^{2/3}_{H^{N_0}} \le C(1+t)^{-6/5}\epsilon_1^2 \le (1+t)^{-6/5}\epsilon_0.$$

From the L^2-L^∞ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\|v(t)-u(t)\|_{H^{N_0-10}} \le C\|u(t)\|_{H^{N_0}}\|u(t)\|_{W^{4,0}} \le C\epsilon_1^2 \le \epsilon_0. \qquad \square$$

Therefore, to control the dispersion rate of the nonlinear solution $u(t)$, now it would be sufficient to control the weighted norms of the profile $g(t)$ of the nonlinear solution $v(t)$. Recall the definitions of Z_1-norm and Z_2-norm in (1.22) and (1.23), we expect that the Z_1-norm of the profile $g(t)$ doesn't grow and the Z_2-norm of the profile only grows appropriately, which leads us to the following bootstrap assumption for some $T' \in (0, T]$,

$$\sup_{t\in[0,T']} (1+t)\|e^{-it\Lambda} g(t)\|_{W^{6,1+\alpha}} + \|g(t)\|_{Z_1} + (1+t)^{-\tilde{\delta}}\|g(t)\|_{Z_2} \leq \epsilon_1 = \epsilon_0^{5/6}, \tag{4.49}$$

where $\tilde{\delta} := 400\delta$.

To close the bootstrap argument, it would be sufficient to prove that there exists some absolute constant "C" such that the following estimates hold for any $t_1, t_2 \in [2^{m-1}, 2^m] \subset [0, T']$, $m \in \mathbb{Z}_+$,

$$\|g(t_2) - g(t_1)\|_{Z_1} \leq C 2^{-\delta m}\epsilon_0, \tag{4.50}$$

$$\|g(t_2)\|_{Z_2}^2 - \|g(t_1)\|_{Z_2}^2 \leq C 2^{2\tilde{\delta} m}\epsilon_0. \tag{4.51}$$

The proof of the desired estimate (4.50) is postponed to the Section 5 and the proof of the desired estimate (4.51) is postponed to the Section 6.

5. The low order weighted norm estimate

In this section, we mainly prove (4.50) under the bootstrap assumption (4.49). Recall (4.35). Note that, from the estimate (7.13) in Lemma 7.4, the low order weighted norm of the quintic and higher order remainder term $\widehat{\mathcal{R}_1}(t, \xi)$ is controlled. In the first subsection, we estimate the low order weight norm (Z_1-norm) of the quadratic terms $B^{\mu,\nu}_{k,k_1,k_2}(\xi, \eta)$ in details. In the last subsection, we estimate the Z_1-norm of the cubic terms $T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t, \xi)$ and quartic terms $K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t, \xi)$ at the same time because the methods we will use for cubic terms and quartic terms are very similar.

5.1. The Z_1-norm estimate of quadratic terms

Recall (4.35). Based on the possible size of k_1 and k_2, we separate into two cases, which are the High-High type interaction and the High-Low type interaction.

The main result for the High-High type interaction is summarized in the following lemma.

LEMMA 5.1. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for any* $\mu, \nu \in \{+, -\}$*, and any* $t_1, t_2 \in [2^{m-1}, 2^m]$*,*

$$\sum_{k\in\mathbb{Z}} \sum_{j\geq -k_-} \sum_{k_1,k_2\in\mathbb{Z}, |k_1-k_2|\leq 10} \left\| \mathcal{F}^{-1}\left[\int_{t_1}^{t_2} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)dt\right]\right\|_{B_{k,j}} \leq C2^{-\delta m}\epsilon_0, \tag{5.1}$$

where C *is some absolute constant.*

Proof. – Recall (1.22) and (4.36). Note that, from the $L^2 \to L^1$ type Sobolev embedding and $L^2 - L^2$ type estimate, the following rough estimate holds for any $\mu, \nu \in \{+,-\}$,

$$\|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B_{k,k_1,k_2}^{\mu,\nu}(t,\xi)dt]\|_{B_{k,j}} \leq \sup_{t\in[2^{m-1},2^m]} C2^{(2+\alpha)k+m+j+2k_1+10k_{1,+}}\|g_{k_1}(t)\|_{L^2}\|g_{k_2}(t)\|_{L^2}$$
$$\leq C2^{(2+\alpha)k+m+j+(2-2\alpha)k_1-(N_0-12)k_{1,+}}\epsilon_0,$$

where C is some absolute constant. From the above estimate, we can first rule out the case when $k \leq -(1+\delta)(m+j)/(2+\alpha)$ or $k_1 \leq -(1+\delta)(m+j)/(4-\alpha)$ or $k_1 \geq (m+j)/(N_0-30)$. As a result, it is sufficient to consider the case when k and k_1 are restricted in the following range,

(5.2)
$$-(1+\delta)(m+j)/(2+\alpha) \leq k \leq k_1 \leq (m+j)/(N_0-30), \quad k_1 \geq -(1+\delta)(m+j)/(4-\alpha).$$

Recall again (4.36). After doing spatial localizations for two inputs, the following decomposition holds,

$$\mathcal{F}^{-1}\big[B_{k,k_1,k_2}^{\mu,\nu}(t,\xi)\big](x) = \sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}} \mathcal{F}^{-1}\big[B_{k,k_1,k_2}^{\mu,\nu,j_1,j_2}(t,\xi)\big](x), \tag{5.3}$$

$$\mathcal{F}^{-1}[B_{k,k_1,k_2}^{\mu,\nu,j_1,j_2}(t,\xi)](x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{ix\cdot\xi+it\Phi^{\mu,\nu}(\xi,\eta)}\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}^{\mu}}(t,\xi-\eta) \times \widehat{g_{k_2,j_2}^{\nu}}(\eta)\psi_k(\xi)d\eta d\xi. \tag{5.4}$$

From the linear decay estimates in Lemma 2.7, the bootstrap assumptions (3.1) and (4.49), and the estimate (4.48) in Lemma 4.2, we obtain the following estimates for any $t \in [2^{m-1}, 2^m] \subset [0, T']$,

$$\|g_{k,j}(t)\|_{L^2} \leq \|\varphi_j^k(x)g_k(t)\|_{L^2} \leq C\min\{2^{-j-(1+\alpha)k-8k_+}, 2^{-2j-2k+\tilde{\delta}m}\}\epsilon_1,$$
$$\|e^{-it\Lambda}g_k(t)\|_{L^\infty} \leq C\min\{2^{-m-(1+\alpha)k-6k_+}, 2^{-m+\tilde{\delta}m-k}\}\epsilon_1,$$
$$\|g_k(t)\|_{L^2} \leq C2^{-(N_0-10)k_++\delta m}\epsilon_0,$$

where C is some absolute constant.

Based on the possible size of j, we separate into two cases as follow.

Case 1. – If $j \geq (1+\delta)\max\{m+k_1, -k_-\} + 2\tilde{\delta}m$. We first consider the case when $\min\{j_1, j_2\} \geq j - \delta j - \delta m$, the following estimate holds,

$$\sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m} \|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B_{k_1,j_1,k_2,j_2}^{\mu,\nu}(t,\xi)dt]\|_{B_{k,j}}$$
$$\leq \sup_{t\in[2^{m-1},2^m]} \sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m} C2^{(2+\alpha)k} \times 2^{m+j+10k_++2k_1}\|g_{k_1,j_1}(t)\|_{L^2}\|g_{k_2,j_2}(t)\|_{L^2}$$
$$\leq C2^{(2+\alpha)k+m+\tilde{\delta}m+10\delta m-(2-2\delta)j-(2-2\alpha)k_1-6k_{1,+}}\epsilon_0$$
$$\leq C2^{-2\delta m-2\delta j}\epsilon_0,$$

where C is some absolute constant.

Now we proceed to consider the case $\min\{j_1, j_2\} \le j - \delta j - \delta m$. Note that, when η is not very close to $\xi/2$ (space resonance set), e.g., $|\eta - \xi/2| \ge 2^{-10}|\xi|$, the following estimates hold,

$$|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)| = 2\big|\mu\lambda'(|\xi-\eta|^2)(\xi-\eta) - \nu\lambda'(|\eta|^2)\eta\big| \ge 2^{-10}|\xi|\big(|\xi-\eta| + |\eta| + 1\big)^{-1/2}, \tag{5.5}$$

$$|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)| + |\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)| \le 2^{10}\max\{|\xi|,|\eta|\}(|\xi| + |\eta| + 1)^{-1/2}, \tag{5.6}$$

where $\lambda(|x|) := \Lambda(\sqrt{|x|})$. Therefore, from (5.6), we know that the following estimate holds if $|x| \in [2^{j-2}, 2^{j+2}]$,

$$|\nabla_\xi\big(x\cdot\xi + t\Phi^{\mu,\nu}(\xi,\eta)\big)| = \big|x + t\nabla_\xi\Phi^{\mu,\nu}(\xi,\eta)\big| \in [2^{j-4}, 2^{j+4}]. \tag{5.7}$$

If $j_2 = \min\{j_1, j_2\}$, then we can do change of variables first to switch the role of $\xi - \eta$ and η. As a result, the following estimate holds if $|x| \in [2^{j-2}, 2^{j+2}]$,

$$|\nabla_\xi\big(x\cdot\xi + t\Phi^{\mu,\nu}(\xi,\xi-\eta)\big)| = \big|x + t\nabla_\xi\Phi^{\mu,\nu}(\xi,\xi-\eta)\big| \in [2^{j-4}, 2^{j+4}].$$

To sum up, in whichever case, by doing integration by parts in ξ once, we gain 2^{-j} by paying the price of at most $\max\{2^{\min\{j_1,j_2\}}, 2^{-k}\}$. Hence, the net gain of doing integration by parts in "ξ" once is at least $2^{-\delta m - \delta j}$. After doing this process many times, we can see rapidly decay.

Case 2. – If $j \le (1+\delta)\max\{m + k_1, -k_-\} + 2\tilde{\delta}m$. As j is bounded from above now, from (5.2), we have the following upper bound and lower bound for k and k_1,

$$-m/(1+\alpha/3) \le k \le k_1 \le 2\beta m, \quad j \le \max\{m + k_1, -k_-\} + 3\tilde{\delta}m, \quad \beta := 1/(N_0 - 50), \tag{5.8}$$

Hence, it would be sufficient to consider fixed k and k_1 inside the range (5.8), as there are at most m^3 cases to consider, which is only a logarithmic loss.

After doing integration by parts in η many times, we can rule out the case when $\max\{j_1, j_2\} \le m + k_- - 3\beta m$. It remains to consider the case when $\max\{j_1, j_2\} \ge m + k_- - 3\beta m$. From $L^2 - L^\infty$ type bilinear estimate in Lemma 2.2, the following estimate holds after putting the input with the maximum spatial concentration in L^2 and the other input in L^∞,

$$\begin{aligned}
\sum_{\max\{j_1,j_2\}\ge m+k_- - 3\beta m} &\|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{\mu,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}} \\
&\le C 2^{(1+\alpha)k + m + j + 2k_1 + 10k_+ - m - (1+\alpha)k_1} \\
&\quad \times \min\{2^{-m-k_- - (1+\alpha)k_1 + 6\beta m}, 2^{-2k_1 - 2(m+k_- - 3\beta m) + \tilde{\delta}m}\}\epsilon_1^2 \\
&\le C\min\{2^{\alpha k + 12k_+ + (1-2\alpha)k_1 + 10\beta m}, 2^{-(1-\alpha)k + 12k_+ - \alpha k_1 - m + 10\beta m}\}\epsilon_0 \\
&\le C 2^{-10\delta m}\epsilon_0,
\end{aligned} \tag{5.9}$$

where C is some absolute constant. To sum up, from the above estimate and the previous discussion, it is easy to see that the desired estimate (5.1) holds. □

The main result of the High-Low type interaction is summarized in the following lemma.

LEMMA 5.2. – *Under the bootstrap assumption* (4.49), *the following estimate holds for any* $\mu \in \{+,-\}$, *and any* $t_1, t_2 \in [2^{m-1}, 2^m]$,

$$\sum_{k\in\mathbb{Z}}\sum_{j\geq -k_-}\sum_{k_1,k_2\in\mathbb{Z},k_2\leq k_1-10}\|\sum_{\nu\in\{+,-\}}\mathscr{F}^{-1}[\int_{t_1}^{t_2}B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)dt]\|_{B_{k,j}}\leq C2^{-\delta m}\epsilon_0, \tag{5.10}$$

where C *is some absolute constant.*

Proof. – Recall (4.42). Note that $\mu = +$ for the case we are considering. Recall (4.14) and (4.15). Motivated from the improved estimate (4.15), we split the symbol "$\tilde{q}_{+,\nu}(\xi,\eta)$" into two parts as follows,

$$\tilde{q}_{+,\nu}(\xi-\eta,\eta) = q^1_{+,\nu}(\xi-\eta,\eta) + q^2_{+,\nu}(\xi-\eta,\eta),$$

$$q^1_{+,\nu}(\xi-\eta,\eta) = c(\xi),\quad q^2_{+,\nu}(\xi-\eta,\eta) = q_{+,\nu}(\xi-\eta,\eta) - c(\xi). \tag{5.11}$$

Hence, motivated from the above decomposition of the symbol $\tilde{q}_{+,\nu}(\xi-\eta,\eta)$, we do decomposition as follows,

$$\sum_{\nu\in\{+,-\}}\int_{t_1}^{t_2}B^{+,\nu}_{k,k_1,k_2}(t,\xi)dt = \sum_{i=1,2}I^i_{k,k_1,k_2},$$

$$I^i_{k,k_1,k_2} = \sum_{\nu\in\{+,-\}}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}e^{it\Phi^{+,\nu}(\xi,\eta)}q^i_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\psi_k(\xi)d\eta dt,\ i=1,2.$$

Recall (5.11). Note that $q^1_{+,\nu}(\xi-\eta,\eta)$ doesn't depend on the sign "ν". Hence, we have

$$I^1_{k,k_1,k_2} = 2\int_{t_1}^{t_2}\int_{\mathbb{R}^2}e^{it(\Lambda(|\xi|)-\Lambda(|\xi-\eta|)}c(\xi)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{\mathrm{Re}(v)}(t,\eta)\psi_{k_2}(\eta)\psi_k(\xi)d\eta dt.$$

From (4.20) and the estimate (5.15) in Lemma 5.3, the following estimate holds after using the volume of the support of "η,"

$$\begin{aligned}
\|I^1_{k_1,k_2}\|_{B_{k,j}} &\leq \sup_{t\in[2^{m-1},2^m]}C_1 2^{(3+\alpha)k+m+j+10k_+}\|g_{k_1}(t)\|_{L^2}2^{2k_2}\|\widehat{\mathrm{Re}(v)}(t,\xi)\psi_{k_2}(\xi)\|_{L^\infty_\xi}\\
&\leq C_2 2^{(3+\alpha)k+m+\delta m+j+2k_2-(N_0-30)k_+}(\|\widehat{h}(t,\xi)\psi_{k_2}(\xi)\|_{L^\infty_\xi}+\|u\|^2_{H^{10}}+\|u\|^3_{H^{10}}+\|u\|^4_{H^{10}})\\
&\leq C_3\big(2^{(3+\alpha)k+2m+10\delta m+j+3k_2-(N_0-30)k_+}\epsilon_0 + 2^{(3+\alpha)k+3m+10\delta m+j+4k_2-(N_0-30)k_+}\epsilon_0\big),
\end{aligned}\tag{5.12}$$

where C_1, C_2, and C_3 are some absolute constants.

Now we proceed to estimate $I^2_{k_1,k_2}$. Recall (5.11) and (4.15). From the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2 and $L^\infty \to L^2$ type Sobolev embedding, we have

$$\begin{aligned}
\|I^2_{k_1,k_2}\|_{B_{k,j}} &\leq \sup_{t\in[2^{m-1},2^m]}C2^{(2+\alpha)k+m+j+k_2+k_1+10k_+}\|g_{k_1}(t)\|_{L^2}\|e^{it\Lambda}g_{k_2}(t)\|_{L^\infty}\\
&\leq C2^{(3+\alpha)k-(N_0-10)k_++m+j+2k_2+2\delta m}\epsilon_0,
\end{aligned}\tag{5.13}$$

where C is some absolute constant. To sum up, from (5.12) and (5.13), we can rule out the case when $k_2 \leq -(1+5\delta)\max\{(m+j)/2, (3m+j)/4\}$ or $k \geq 4(m+j)/(N_0-40)$. Now, we only need to consider the case when k_2 is restricted in the following range,

$$-(1+5\delta)\max\{(m+j)/2,(3m+j)/4\} \leq k_2 \leq k \leq (3m+j)/(N_0-40). \tag{5.14}$$

Similar to the idea used in the proof of Lemma 5.1, we also separate into two cases based on the size of "j" as follows.

Case 1. – If $j \geq (1+\delta)\max\{m+k,-k_-\}+10\delta m$. We first consider the case when $\min\{j_1,j_2\} \leq j-\delta j-\delta m$. Same as we considered in the High × High type interaction, we can also do integration by parts in "ξ" many times to see rapidly decay. Now, we proceed to consider the case when $\min\{j_1,j_2\} \geq j-\delta j-\delta m$. From L^2-L^∞ type bilinear estimate and $L^\infty \to L^2$ type Sobolev embedding, we have

$$\begin{aligned}\|I^1_{k_1,k_2}\|_{B_{k,j}} \leq & \sup_{t\in[2^{m-1},2^m]} \sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m} C2^{(1+\alpha)k+10k_++m+j+2k_1+k_2} \\ & \|g_{k_1,j_1}(t)\|_{L^2}\|g_{k_2,j_2}(t)\|_{L^2} \\ & \leq C2^{(1+\alpha)k+k_2+(1+50\beta)m-(1-50\beta)j}2^{-j/2-k_2/2}\epsilon_1^2 \leq C2^{-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant.

Case 2. – If $j \leq (1+\delta)\max\{m+k,-k_-\}+10\delta m$. For this case, whether j_1 is less than j_2 makes a difference. Hence, we separate into two cases based on whether j_1 is smaller than j_2 as follows.

If $j_1 \leq j_2$. – For this case, we don't need to do change of coordinates to switch the role between $\xi-\eta$ and η. Note that $|\nabla_\xi \Phi^{+,\nu}(\xi,\eta)| \leq C|\eta|$ holds for some absolute constant. Since this upper bound is better than the one used in the rough estimate, which leads to expect that the upper bound of "j" can be improved. More precisely, we can rule out the case when $j \geq \max\{m+k_2,-k_-\}+100\beta m$ and $j_1 \leq j-\delta m$ by doing integration by parts in ξ many times. If $j \geq \max\{m+k_2,-k_-\}+100\beta m$ and $j-\delta m \leq j_1 \leq j_2$, then the following estimate holds after using the L^2-L^∞ type bilinear estimate and $L^\infty \to L^2$ type Sobolev embedding,

$$\begin{aligned}&\sum_{j-\delta m\leq j_1\leq j_2} \|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{+,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}} \\ &\leq \sup_{t\in[2^{m-1},2^m]} \sum_{j-\delta m\leq j_1\leq j_2} C2^{(1+\alpha)k+10k_++m+j+2k_1}\|g_{k_1,j_1}(t)\|_{L^2}2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2} \\ &\leq C2^{(1+\alpha)k+k_2+(1+50\beta)m-(1-50\beta)j}2^{-25\beta j-25\beta k_2}\epsilon_1^2 \\ &\leq C2^{-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant.

It remains to consider the case when $j \leq \max\{m+k_2,-k_-\}+100\beta m$. If moreover $k_-+k_2 \leq -m+\beta m$, it is easy to see our desired estimate holds from (5.12) and (5.13). Hence, we only have to consider the case when $k_-+k_2 \geq -m+\beta m$. For this case, we have $j \leq m+k_2+100\beta m$. Recall (5.14), we know that $k_2 \geq -4m/5-30\beta m$.

Same as in the decomposition (5.3), we also do spatial localizations for two inputs. After doing integration by parts in "η" many times, we can rule out the case when $j_2 \leq m+k_{1,-}-10\delta m$. Therefore, it remains to consider the case when $j_2 \geq m+k_{1,-}-10\delta m$.

After putting g_{k_2,j_2} in L^2 and putting g_{k_1,j_1} in L^∞, we have

$$\begin{aligned}\sum_{j_2\geq \max\{m+k_{1,-}-10\delta m, j_1\}} &\|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{+,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}} \\ &\leq \sum_{j_2\geq \max\{m+k_{1,-}-10\delta m, j_1\}} C2^{(1+\alpha)k+10k_+} \\ &\quad \times 2^{2k_1+m+j} \sup_{t\in[2^m,2^{m+1}]} \|e^{-it\Lambda} g_{k_1,j_1}(t)\|_{L^\infty}\|g_{k_2,j_2}(t)\|_{L^2} \\ &\leq C2^{-m-k_2+150\beta m}\epsilon_1^2 \leq C2^{-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant.

If $-k_2 \leq j_2 \leq j_1$. – We first consider the case when $k_1+k_2 \leq -4m/5$. From the $L^2 - L^\infty$ type bilinear estimate and $L^\infty \to L^2$ type Sobolev embedding, the following estimate holds for some absolute constant C,

$$\begin{aligned}\sum_{j_2\leq j_1} &\|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{+,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}} \\ &\leq \sup_{t\in[2^{m-1},2^m]} \sum_{j_2\leq j_1} C2^{(1+\alpha)k+10k_+ + m+j+2k_1}\|g_{k_1,j_1}(t)\|_{L^2}2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2} \\ &\leq \sum_{-k_2\leq j_1} C2^{2m+(4+\alpha)k_1+k_2}2^{-2k_1-2j_1+50\beta m}\epsilon_1^2 \\ &\leq C2^{(2+\alpha)k+3k_2+2m+50\beta m}\epsilon_1^2 \\ &\leq C2^{-\beta m}\epsilon_0.\end{aligned}$$

Lastly, it remains to consider the case when $k_1+k_2 \geq -4m/5$. For this case, we do integration by parts in η many times to rule out the case when $j_1 \leq m+k_{1,-}-10\delta m$. For the case when $j_1 \geq m+k_{1,-}-10\delta m$, the following estimate holds from the $L^2 - L^\infty$ type bilinear estimate,

$$\begin{aligned}&\sum_{j_1\geq \max\{j_2, m+k_{1,-}-10\delta m\}} \left\|\mathcal{F}^{-1}\left[\int_{t_1}^{t_2} B^{+,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt\right]\right\|_{B_{k,j}} \\ &\leq \sum_{j_1\geq \max\{j_2, m+k_{1,-}-10\delta m\}} C2^{(1+\alpha)k+10k_+}2^{m+j+2k_1} \sup_{t\in[2^{m-1},2^m]} \|g_{k_1,j_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty} \\ &\leq C2^{-m-(1+\alpha)k_2+50\beta m}\epsilon_1^2 \leq C2^{-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant. Hence finishing the proof of the desired estimate (5.10). □

LEMMA 5.3. – *Under the bootstrap assumption* (3.1), *the following estimate holds for* $t \in [2^{m-1}, 2^m] \subset [0,T]$, $m \in \mathbb{N}$ *and* $k \in \mathbb{Z}, k \leq 0$,

$$\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} \leq C2^{2\delta m}\big(2^{2k+2m} + 2^{k+m}\big)\epsilon_0, \tag{5.15}$$

where C *is some absolute constant.*

Proof. – Recall (4.6). It is easy to see the following estimate holds for any $t \in [2^{m-1}, 2^m]$ and $k \leq 0$,

$$\|\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} \leq \epsilon_0 + C_1 \int_0^t \|f(s)\|_{H^{10}}^2 ds \leq C_2 2^{m+2\delta m}\epsilon_0, \tag{5.16}$$

where C_1 and C_2 are some absolute constants. Recall the equation satisfied by the height function "$h(t)$" in (1.3) and the Taylor expansion of the Dirichlet-Neumann operator in (1.6), we have

$$\partial_t \widehat{h}(t,\xi) = |\xi|\tanh(|\xi|)\widehat{\psi}(t,\xi) + \mathcal{F}[\Lambda_2[G(h)\psi]](\xi) + \mathcal{F}[\Lambda_{\geq 3}[G(h)\psi]](\xi).$$

Hence, from $L^2 - L^2$ type bilinear estimate (2.5) in Lemma 2.2 and the estimate (5.16), the following estimate holds for any $k \leq 0$,

$$\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} \leq \epsilon_0 + C_1\Big(\int_0^t 2^{2k}\|\widehat{\psi}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi} ds + \int_0^t 2^k \|h(s)\|_{H^{10}}\|\psi(s)\|_{H^{10}} ds\Big)$$

$$\leq \epsilon_0 + C_2\Big(\int_0^t 2^{2k}\|\widehat{f}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi} ds + \int_0^t 2^k\|f(s)\|_{H^{10}}^2 ds\Big) \leq C_3 2^{2\delta m}\big(2^{2k+2m} + 2^{k+m}\big)\epsilon_0, \tag{5.17}$$

where C_1, C_2, and C_3 are some absolute constants. Hence finishing the proof of the desired estimate (5.15). □

5.2. The Z_1 estimates of cubic terms and quartic terms.

The main goal of this subsection is to prove the following proposition,

PROPOSITION 5.4. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant C and any $t \in [2^{m-1}, 2^m]$,*

$$\sum_{k\in\mathbb{Z}}\sum_{j\geq -k_-}\Big[\sum_{k_3\leq k_2\leq k_1} \|\mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)]\|_{B_{k,j}} + \sum_{k_4\leq k_3\leq k_2\leq k_1} \|\mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)]\|_{B_{k,j}}\Big]$$

$$\leq C 2^{-m-\beta m}\epsilon_0, \tag{5.18}$$

where $T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$ and $K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)$ are defined in (4.37) *and* (4.38) *respectively.*

Proof. – Same as the strategy used in the estimate of quadratic terms, we can do integration by parts in "ξ" many times to rule out the case when $j \geq (1+\delta)\max\{m+k_1, -k_-\} + 2\tilde{\delta}m$. Hence, in the rest of this section, we restrict ourself to the case when

$$j \leq (1+\delta)\max\{m+k_1, -k_-\} + 2\tilde{\delta}m.$$

From the $L^2 - L^\infty - L^\infty$ type trilinear estimate in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\|\mathscr{F}^{-1}[T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)]\|_{B_{k,j}}\\
&\quad\le C2^{(1+\alpha)k+j+2k_1+2k_{1,+}+10k_+}\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}\|g_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}\\
&\quad\le C\min\{2^{(1+\alpha)k+2k_1+k_3+20\beta m}, 2^{(1+\alpha)k+3k_1-(N_0-30)k_{1,+}+k_3+m+\beta m}\}\epsilon_0,
\end{aligned}\tag{5.19}$$

$$\begin{aligned}
&\|\mathscr{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)]\|_{B_{k,j}}\\
&\quad\le C2^{(1+\alpha)k+10k_++j+2k_1+2k_{1,+}}\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}\|e^{-it\Lambda}g_{k_2}\|_{L^2}\|g_{k_3}(t)\|_{L^2}\\
&\qquad\times\|e^{-it\Lambda}g_{k_4}(t)\|_{L^\infty}\le C2^{(1+\alpha)k+k_4+20\beta m}\min\{2^{2k_1-m/2}, 2^{3k_1-(N_0-30)k_{1,+}+m/2}\}\epsilon_0.
\end{aligned}\tag{5.20}$$

From the rough estimate (5.19), we can rule out the case when $k_3 \le -m - 30\beta m$, or $k_1 \ge 2\beta m$ or $k \le -m/(1+\alpha/2)$ for the cubic terms. From the rough estimate (5.20), we can rule out the case when $k_4 \le -m/2 - 30\beta m$ or $k_1 \ge 2\beta m$ or $k \le -m/(2+\alpha)$ for the quartic terms.

Therefore, for the cubic terms, it would be sufficient to obtain the following desired estimate

$$\sup_{t\in[2^{m-1},2^m]}\|\mathscr{F}^{-1}[T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)](x)\|_{B_{k,j}} \le C2^{-m-\beta m}\epsilon_0,\tag{5.21}$$

where integers k, k_1, k_2, k_3 are fixed inside the following range

$$\text{(Cubic terms)}\quad -m-30\beta m \le k_3 \le k_2 \le k_1 \le 2\beta m,\quad -m/(1+\alpha/2) \le k \le 2\beta m.\tag{5.22}$$

For the quartic terms, it would be sufficient to obtain the following estimate,

$$\sup_{t\in[2^{m-1},2^m]}\|\mathscr{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)](x)\|_{B_{k,j}} \le C2^{-m-\beta m}\epsilon_0,\tag{5.23}$$

where integers k, k_1, k_2, k_3, k_4 are fixed inside the following range,

$$\text{(Quartic terms)}\quad -m/2-30\beta m \le k_4 \le k_3 \le k_2 \le k_1 \le 2\beta m,\quad -m/(2+\alpha) \le k \le 2\beta m.\tag{5.24}$$

From the results in Lemma 5.5, Lemma 5.6, and Lemma 5.7, we know that the desired estimates (5.21) and (5.23)) indeed hold. Hence finishing the desired estimate (5.18. □

LEMMA 5.5. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k_2 \le k_1 - 10$*, the desired estimate* (5.21) *for the cubic terms holds for fixed* k, k_1, k_2*, and* k_3 *inside the range listed in* (5.22) *and the desired estimate* (5.23) *holds for fixed* k, k_1, k_2, k_3 *and* k_4 *inside the range listed in* (5.24)*.*

Proof. – Recall the normal form transformation we did in Subsection 4.1. Note that the case when "$\tau = -$" is removed by the normal form transformation when $k_2 \le k_1 - 10$. Hence, we can restrict ourself to the case "$\tau = +$". Recall (4.8). Note that the following estimates hold for the case we are considering,

$$
\begin{aligned}
|\nabla_\xi \Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)| &= |\nabla_\xi \Phi^{+,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma)| = \big|\Lambda'(|\xi|)\frac{\xi}{|\xi|} - \Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}\big| \\
&\leq 2\max\{2^{k_1}\angle(\xi,\xi-\eta), |\xi|-|\xi-\eta|\} \leq 4|\eta| \leq 2^{k_2+3}.
\end{aligned} \tag{5.25}
$$

$$
\begin{aligned}
2^{k_1-k_{1,+}/2-10} &\leq |\nabla_\eta \Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)| = \big|\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|} + \kappa\Lambda'(|\eta-\sigma|)\frac{\eta-\sigma}{|\eta-\sigma|}\big| \\
&\leq 2^{k_1-k_{1,+}/2+10}.
\end{aligned} \tag{5.26}
$$

After doing spatial localizations for the inputs $\widehat{g_{k_1}}(\cdot)$ and $\widehat{g_{k_2}}(\cdot)$, we have the decomposition as follows,

$$
\begin{aligned}
T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi) &= \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi), \\
T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^{\tau}_{k_1,j_1}}(t,\xi-\eta) \\
&\quad \times \widehat{g^{\kappa}_{k_2,j_2}}(t,\eta-\sigma)\widehat{g^{\iota}_{k_3}}(t,\sigma)d\sigma d\eta, \\
K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi) &= \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k_1,j_1,k_2,j_2}(t,\xi), \\
K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k_1,j_1,k_2,j_2}(t,\xi) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa) \\
&\quad \times \widehat{g^{\mu_1}_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\mu_2}_{k_2,j_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)d\kappa d\sigma d\eta.
\end{aligned} \tag{5.27}
$$

Based on the possible size of j, we separate into two cases as follows.

If $j \geq \max\{m+k_2, -k_{1,-}\} + \beta m$. – Recall (5.25). By doing integration by parts in "ξ" many times, we can rule out the case $j_1 \leq j - \delta m$. If $j_1 \geq j - \delta m$, then from $L^2 - L^\infty - L^\infty$ type multilinear estimate in Lemma 2.2, the following estimates hold,

$$
\begin{aligned}
&\|\sum_{j_1\geq j-\delta m} \mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi)]\|_{B_{k,j}} \\
&\quad \leq \sum_{j_1\geq j-\delta m} C2^{(1+\alpha)k+j+2k_1+2k_{1,+}+10k_+}2^{k_2}\|g_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\|g_{k_1,j_1}(t)\|_{L^2} \\
&\quad \leq C2^{-m/2+30\beta m}2^{k_2-j}\epsilon_0 \leq C2^{-3m/2+40\beta m}\epsilon_0. \\
&\|\sum_{j_1\geq j-\delta m} \mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k_1,j_1,k_2,j_2}(t,\xi)]\|_{B_{k,j}} \\
&\quad \leq \sum_{j_1\geq j-\delta m} C2^{(1+\alpha)k+j+2k_1+2k_{1,+}+10k_+}2^{k_2}\|g_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty} \\
&\quad\quad \times \|e^{-it\Lambda}g_{k_4}(t)\|_{L^\infty}\|g_{k_1,j_1}(t)\|_{L^2} \\
&\quad \leq C2^{-2m+40\beta m}\epsilon_0.
\end{aligned}
$$

If $j \le \max\{m+k_2, -k_{1,-}\}+\beta m$. – From the $L^2-L^\infty-L^\infty-L^\infty$ type multilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\|\mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)]\|_{B_{k,j}} \\
&\quad\le C2^{(1+\alpha)k+10k_+ +j+2k_1+2k_{1,+}}\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}\|g_{k_4}\|_{L^2} \\
&\quad\le C2^{-3m/2+40\beta m}\epsilon_0.
\end{aligned}$$

Hence finishing the proof of the desired estimate (5.23) for the quartic terms.

Now we proceed to estimate the cubic terms "$T^{+,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$". If moreover $k_1+k_2\le -m/2-12\beta m$, then the following estimate holds from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and $L^\infty\to L^2$ type Sobolev embedding,

$$\begin{aligned}
&\|\mathcal{F}^{-1}[T^{+,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)]\|_{B_{k,j}} \\
&\quad\le C2^{(1+\alpha)k+10k_+ +j+2k_1+2k_{1,+}}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}2^{k_2}\|g_{k_2}(t)\|_{L^2}\|g_{k_3}(t)\|_{L^2} \\
&\quad\le C2^{2k_1+2k_2+20\beta m}\epsilon_0+2^{k_1+2k_2+20\beta m}\epsilon_0\le C2^{-m-\beta m}\epsilon_0,
\end{aligned}$$

where C is some absolute constant. If $k_1+k_2\ge -m/2-12\beta m$. Recall (5.26). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\le m+k_{1,-}-\beta m$. For the case when $\max\{j_1,j_2\}\ \ge\ m+k_{1,-}-\beta m$, the following estimate holds from $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2,

$$\begin{aligned}
&\sum_{\max\{j_1,j_2\}\ge m+k_{1,-}-\beta m}\|\mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi)]\|_{B_{k,j}} \\
&\le \sum_{j_1\ge\max\{m+k_{1,-}-\beta m,j_2\}} C2^{(1+\alpha)k+10k_+ +2k_1+j}\|g_{k_1,j_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty} \\
&+ \sum_{j_2\ge\max\{m+k_{1,-}-\beta m,j_1\}} C2^{(1+\alpha)k+10k_+ +j+2k_1}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty} \\
&\le C2^{-5m/2+50\beta m-k_2}\epsilon_0\le C2^{-m-\beta m}\epsilon_0,
\end{aligned} \tag{5.28}$$

where C is some absolute constant. Hence finishing the proof of desired estimates (5.21) and (5.23) for the case when $k_2\le k_1-10$. □

LEMMA 5.6. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k_1-10\le k_2\le k_1+1$ *and* $k_2\le k_3-10\le k_2+1$*, the desired estimate* (5.21) *for the cubic terms holds for fixed* k,k_1,k_2*, and* k_3 *inside the range listed in* (5.22) *and the desired estimate* (5.23) *holds for fixed* k,k_1,k_2,k_3 *and* k_4 *inside the range listed in* (5.24).

Proof. – From $L^2-L^\infty-L^\infty-L^\infty$ type mutilinear estimate, we have

$$\begin{aligned}
\|\mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)]\|_{B_{k,j}} &\le C2^{(1+\alpha)k+10k_+ +j+2k_1}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} \\
&\quad\times\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\|g_{k_4}(t)\|_{L^2} \\
&\le C2^{-3m/2+40\beta m}\epsilon_0.
\end{aligned} \tag{5.29}$$

Hence finishing the proof of the quartic terms. Now, it remains to estimate the cubic terms "$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$". After putting g_{k_3} in L^2 and the other two inputs in L^∞, from the

$L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 when $k \leq -2\beta m$, the following estimate holds for some absolute constant C,

$$\begin{aligned}\|\mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)]\|_{B_{k,j}} &\leq C2^{(1+\alpha)k+j+2k_1+2k_{1,+}}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^2}\|g_{k_3}(t)\|_{L^2}\\ &\leq C\max\{2^{\alpha k-2m+2\beta m}, 2^{(1+\alpha)k-m+\beta m}\}\epsilon_1^3 \leq C2^{-m-\beta m}\epsilon_0.\end{aligned}$$

Hence, it remains to consider the case when $k \geq -2\beta m$. Recall the normal form transformation we did in Subsection 4.1. Note that the case when η is close to $\xi/2$ is removed, see (4.30). Hence, the following estimate always holds for the case we are considering,

$$|\nabla_\eta \Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)| \geq 2^{k-k_{1,+}/2-10}. \tag{5.30}$$

From the above estimate, after doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\} \leq m+k_- - 3\beta m$. Hence, we only have to consider the case when $\max\{j_1,j_2\} \geq m+k_- - 3\beta m$. From the $L^2 - L^\infty - L^\infty$ type estimate (2.6) in Lemma 2.2, the following estimate holds,

$$\begin{aligned}&\sum_{\max\{j_1,j_2\}\geq m+k_--3\beta m}\|\mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi)]\|_{B_{k,j}}\\ &\leq \sum_{\max\{j_1,j_2\}\geq m+k_--3\beta m} C2^{(1+\alpha)k+10k_++j+2k_1}\|g_{k_1,j_1}(t)\|_{L^2}2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\\ &\leq C2^{-3m/2+50\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant. Hence finishing the proof. □

LEMMA 5.7. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k_1 - 10 \leq k_2 \leq k_1 + 1$ *and* $k_2 \leq k_3 - 10 \leq k_2 + 1$, *the desired estimate* (5.21) *for the cubic terms holds for fixed* k, k_1, k_2, *and* k_3 *inside the range listed in* (5.22) *and the desired estimate* (5.23) *holds for fixed* k, k_1, k_2, k_3 *and* k_4 *inside the range listed in* (5.24).

Proof. – Note that, because the size of "k_3" plays little role in (5.29), the estimate (5.29) still holds for the quartic terms. Hence, we only have to estimate the cubic term "$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$". Define

$$\mathcal{S}_1 := \{(+,-,-),(-,+,+)\}, \qquad \mathcal{S}_2 := (+,-,+),(-,+,-)\}, \tag{5.31}$$

$$\mathcal{S}_3 := \{(+,+,-),(-,-,+)\}, \qquad \mathcal{S}_4 := \{(+,+,+),(-,-,-)\}. \tag{5.32}$$

Recall (4.8). Note that the space resonance in both "η" and "σ" set is given as follows,

$$\begin{aligned}\mathcal{R}_{\tau,\kappa,\iota} &:= \{(\xi,\eta,\sigma) : \nabla_\eta\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma) = \nabla_\sigma\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma) = 0\}\\ &= \{(\xi,\eta,\sigma) : \xi = \big((1+\tau\kappa)(1+\kappa\iota)-\tau\kappa\big)\sigma, \eta = (1+\kappa\iota)\sigma\}, \quad \tau,\kappa,\iota \in \{+,-\}.\end{aligned}$$

More specifically, we have

$$\begin{aligned}&\mathcal{R}_{\tau,\kappa,\iota} = \{(\xi,\eta,\sigma) : \xi = \sigma, \eta = 2\sigma\}, \quad (\xi-\eta,\eta-\sigma,\sigma)|_{\mathcal{R}_{\tau,\kappa,\iota}} = (-\xi,\xi,\xi), \ (\tau,\kappa,\iota) \in \mathcal{S}_1,\\ &\mathcal{R}_{\tau,\kappa,\iota} = \{(\xi,\eta,\sigma) : \xi = \sigma, \eta = 0\}, \quad (\xi-\eta,\eta-\sigma,\sigma)|_{\mathcal{R}_{\tau,\kappa,\iota}} = (\xi,-\xi,\xi), \ (\tau,\kappa,\iota) \in \mathcal{S}_2,\\ &\mathcal{R}_{\tau,\kappa,\iota} = \{(\xi,\eta,\sigma) : \xi = -\sigma, \eta = 0\}, \quad (\xi-\eta,\eta-\sigma,\sigma)|_{\mathcal{R}_{\tau,\kappa,\iota}} = (\xi,\xi,-\xi), \ (\tau,\kappa,\iota) \in \mathcal{S}_3,\\ &\mathcal{R}_{\tau,\kappa,\iota} = \{(\xi,\eta,\sigma) : \xi = 3\sigma, \eta = 2\sigma\}, \quad (\xi-\eta,\eta-\sigma,\sigma)|_{\mathcal{R}_{\tau,\kappa,\iota}} = (\xi/3,\xi/3,\xi/3), \ (\tau,\kappa,\iota) \in \mathcal{S}_4.\end{aligned}$$

When $(\tau,\kappa,\iota)\in\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{S}_3$. – Note that, after changing of variables, those three cases are symmetric. Hence, it would be sufficient to estimate the case when $(\tau,\kappa,\iota)\in\mathcal{S}_1$ in details.

We first do change of variables and then localize around the space resonance set with a well chosen threshold. As a result, we can decompose the cubic term $T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$ as follows,

(5.33)
$$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)=\sum_{l_1,l_2\geq\bar{l}_\tau}C^{\tau,l_1,l_2}(t,\xi),\quad C^{\tau,l_1,l_2}(t,\xi)=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}}C^{\tau,l_1,l_2}_{j_1,j_2}(t,\xi),$$

(5.34)
$$C^{\tau,l_1,l_2}_{j_1,j_2}(t,\xi):=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi,2\xi+\eta+\sigma,\xi+\sigma)\widehat{g^{\tau}_{k_1,j_1}}(t,-\xi-\eta-\sigma)$$
$$\times\widehat{g^{\kappa}_{k_2,j_2}}(t,\xi+\eta)\widehat{g^{\iota}_{k_3}}(t,\xi+\sigma)\varphi_{l_1;\bar{l}_\tau}(\eta)\varphi_{l_2;\bar{l}_\tau}(\sigma)d\sigma d\eta,$$

where the phase $\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is defined as follows,

$$\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma):=\Lambda(|\xi|)-\tau\Lambda(|\xi+\eta+\sigma|)-\kappa\Lambda(|\xi+\eta|)-\iota\Lambda(|\xi+\sigma|),\quad(\tau,\kappa,\iota)\in\mathcal{S}_1,\tag{5.35}$$

the thresholds $\bar{l}_-:=-2m/5-10\beta m$ and $\bar{l}_+:=k_--10$ and the cutoff function $\varphi_{l;\bar{l}}(\cdot)$ with the threshold $\bar{l}$ is defined as follows,

$$\varphi_{l;\bar{l}}(x):=\begin{cases}\psi_{\leq\bar{l}}(|x|) & \text{if } l=\bar{l}\\ \psi_l(|x|) & \text{if } l>\bar{l}.\end{cases}\tag{5.36}$$

If $\tau=+$*, i.e.,* $(\tau,\kappa,\iota)=(+,-,-)$. – Recall the normal form transformation that we did in Subsection 4.1 , see (4.20) and (4.30). For the case we are considering, i.e., $(\tau,\kappa,\iota)\in\widetilde{\mathcal{S}}$, we already canceled out the case when $\max\{l_1,l_2\}=\bar{l}_+$. Hence it would be sufficient to consider the case when $\max\{l_1,l_2\}>\bar{l}_-$. By the symmetry between inputs, without loss of generality, we assume that $l_2=\max\{l_1,l_2\}>\bar{l}_+:=k_--10$. For this case, we take the advantage of the fact that $\nabla_\eta\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is relatively big, i.e., we are away from the space resonance in "η" set. More precisely, we have

$$\big|\nabla_\eta\widetilde{\Phi}^{+,-,-}(\xi,\eta,\sigma)\big|=\big|\Lambda'(|\xi+\eta+\sigma|)\frac{\xi+\eta+\sigma}{|\xi+\eta+\sigma|}-\Lambda'(|\xi+\eta|)\frac{\xi+\eta}{|\xi+\eta|}\big|\geq 2^{l_2-10}.\tag{5.37}$$

Hence, we can do integration by parts in "η" many times to rule out the case when $\max\{j_1,j_2\}\leq m+k_--\beta m$. From the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and the $L^\infty\to L^2$ type Sobolev embedding, the following estimate holds,

$$\sum_{\max\{j_1,j_2\}\geq m+k_--\beta m}\|\mathcal{F}^{-1}[C^{+,l_1,l_2}_{j_1,j_2}(t,\xi)](x)\|_{B_{k,j}}\leq\sum_{\max\{j_1,j_2\}\geq m+k_--\beta m}C2^{(1+\alpha)k+10k_++j+2k_1}$$
$$\times\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\|g_{k_2,j_2}(t)\|_{L^2}2^{k_2}\|g_{k_1,j_1}(t)\|_{L^2}\leq C2^{-3m/2+40\beta m}\epsilon_0,$$

where C is some absolute constant.

If $\tau=-$*, i.e.,* $(\tau,\kappa,\iota)=(-,+,+)$. – By the symmetry between l_1 and l_2, without loss of generality, we assume that $l_2=\max\{l_1,l_2\}$. Recall (5.35). We have

$$\big|\nabla_\xi\widetilde{\Phi}^{-,+,+}(\xi,\eta,\sigma)\big|$$
$$=\Big|\Lambda'(|\xi|)\frac{\xi}{|\xi|}+\Lambda'(|\xi+\eta|)\frac{\xi+\eta+\sigma}{|\xi+\eta+\sigma|}-\Lambda'(|\xi+\eta|)\frac{\xi+\eta}{|\xi+\eta|}-\Lambda'(|\xi+\sigma|)\frac{\xi+\sigma}{|\xi+\sigma|}\Big|.$$

From the above equality, we know that the following estimate holds,

$$\big|\nabla_\xi \widetilde{\Phi}^{-,+,+}(\xi,\eta,\sigma)\big|\varphi_{l_1;\bar{l}_\tau}(\eta)\varphi_{l_2;\bar{l}_\tau}(\sigma) \leq 2^{l_2+10}. \tag{5.38}$$

Hence, we can first rule out the case when $j \geq m + l_2 + 2\beta m$ by doing integration by parts in "ξ" many times. From now on, it would be sufficient to consider the case when $j \leq m + l_2 + 2\beta m$.

We first consider the case when $l_2 = \bar{l}_- = -2m/5 - 10\beta m$. After using the volume of supports in "η" and "σ," the following estimate holds,

$$\begin{aligned}\|\mathscr{F}^{-1}[C^{-,\bar{l}_-,\bar{l}_-}(t,\xi)](x)\|_{B_{k,j}} &\leq C2^{(1+\alpha)k+10k_++j+2k_1}2^{4\bar{l}}\|g_{k_1}(t)\|_{L^2}\|g_{k_2}(t)\|_{L^1}\|g_{k_3}(t)\|_{L^1}\\ &\leq C2^{5\bar{l}+m+30\beta m}\epsilon_1^3 \leq C2^{-m-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant. Now, we proceed to consider the case when $l_2 > \bar{l}_- = -2m/5 - 10\beta m$. Note that the following estimate holds for the case we are considering,

$$\big|\nabla_\eta \widetilde{\Phi}^{-,+,+}(\xi,\eta,\sigma)\big| \geq 2^{l_2-k_+/2-10}. \tag{5.39}$$

Therefore, we can do integration by parts in η many times to rule out the case when $\max\{j_1,j_2\} \leq m + l_2 - 4\beta m$. From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C when $\max\{j_1,j_2\} \geq m + l_2 - 4\beta m$,

$$\begin{aligned}&\sum_{\max\{j_1,j_2\}\geq m+l_2-4\beta m}\|\mathscr{F}^{-1}[C^{-,l_1,l_2}_{j_1,j_2}(t,\xi)](x)\|_{B_{k,j}}\\ &\quad\leq C2^{(1+\alpha)k+10k_++j+2k_1}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\\ &\qquad\times\Big[\sum_{j_2\geq\max\{m+l_2-4\beta m,j_1\}}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\\ &\qquad+\sum_{j_1\geq\max\{m+l_2-4\beta m,j_2\}}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\\ &\qquad\times\|g_{k_1,j_1}(t)\|_{L^2}\Big] \leq C2^{-2m-l_2-m/2+40\beta m}\epsilon_0 \leq C2^{-m-\beta m}\epsilon_0.\end{aligned} \tag{5.40}$$

When $(\tau,\kappa,\iota) \in \mathscr{S}_4$. – Very similarly, we localize around the space resonance set "$(\xi/3,\xi/3,\xi/3)$" by doing change of variables for "$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3}(t,\xi)$" as follows,

$$\begin{aligned}T^{\tau,\kappa,\iota}_{k_1,k_2,k_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} &e^{it\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi,2\xi/3+\eta+\sigma,\xi/3+\sigma)\widehat{g^{\tau}_{k_1}}(t,\xi/3-\eta-\sigma)\\ &\times\widehat{g^{\kappa}_{k_2}}(t,\xi/3+\eta)\widehat{g^{\iota}_{k_3}}(t,\xi/3+\sigma)d\sigma d\eta,\end{aligned}$$

where the phase $\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is defined as follows,

$$\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma) := \Lambda(|\xi|)-\tau\Lambda(|\xi/3-\eta-\sigma|)-\kappa\Lambda(|\xi/3+\eta)-\iota\Lambda(|\xi/3+\sigma|), \quad (\tau,\kappa,\iota)\in\mathscr{S}_4.$$

Recall the normal form transformation that we did in Subsection 4.1. The symbol around a neighborhood of $(\xi/3,\xi/3,\xi/3)$ has been canceled, see (4.30) and (4.40). Hence, the following decomposition holds,

$$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi) = \sum_{i=1,2} T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;i}(t,\xi),$$

$$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;1}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2;1}(t,\xi),$$

$$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;2}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_3\geq -k_{3,-}} T^{\tau,\kappa,\iota}_{k_1,j_1,k_3,j_3;2}(t,\xi)$$

$$T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2;1}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)} \tilde{d}_{\tau,\kappa,\iota}(\xi, 2\xi/3+\eta+\sigma, \xi/3+\sigma)\widehat{g^{\tau}_{k_1,j_1}}(t,\xi/3-\eta-\sigma)$$
$$\times \widehat{g^{\kappa}_{k_2,j_2}}(t,\xi/3+\eta)\widehat{g^{\iota}_{k_3}}(t,\xi/3+\sigma)\psi_{\geq k-20}(2\eta+\sigma)\,d\sigma\,d\eta, \tag{5.41}$$

$$T^{\tau,\kappa,\iota}_{k_1,j_1,k_3,j_3;2}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)} \tilde{d}_{\tau,\kappa,\iota}(\xi, 2\xi/3+\eta+\sigma, \xi/3+\sigma)\psi_{\geq k-20}(2\sigma+\eta)$$
$$\times \psi_{\leq k-20}(2\eta+\sigma)\widehat{g^{\tau}_{k_1,j_1}}(t,\xi/3-\eta-\sigma)\widehat{g^{\kappa}_{k_2}}(t,\xi/3+\eta) \tag{5.42}$$
$$\times \widehat{g^{\iota}_{k_3,j_3}}(t,\xi/3+\sigma)\,d\sigma\,d\eta.$$

The estimates of "$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;1}(t,\xi)$" and "$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;2}(t,\xi)$" are very similar. For simplicity, we only estimate $T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;1}(t,\xi)$ in details here. Note that "$2\eta+\sigma$" is bounded from below by 2^{k-10} for the case we are considering, which implies that the size of $\nabla_\eta\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is bounded from blow by $2^{k-k_+/2-20}$. Therefore, after doing integration by parts many times in "η," we can rule out the case when $\max\{j_1,j_2\}\leq m+k_- - 2\beta m$. For the case when $\max\{j_1,j_2\}\geq m+k_- - 2\beta m$, a similar estimate as in (5.40) holds for some absolute constant C as follows,

$$\sum_{\max\{j_1,j_2\}\geq m+k_- -2\beta m} \|\mathscr{F}^{-1}[T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2;1}(t,\xi)]\|_{B_{k,j}} \leq \text{R.H.S. of (5.40)} \leq C2^{-m-\beta m}\epsilon_0.$$

Hence finishing the proof. □

6. The high order weighted norm estimate

In this section, our main goal is to prove (4.51) under the smallness assumption (4.49). The plan of this section is listed as follows. (i) In Subsection 6.1, we first classify different scenarios when estimating the left hand side of (4.51) and then show a key decomposition, e.g., (6.17), holds when the vector field $\hat{L}_\xi$ hits the phases $\Phi^{\mu,\nu}(\xi,\eta)$. (ii) In Subsection 6.2 and Subsection 6.3, we finish the Z_2-estimate of the quadratic terms for the High-High type interaction and the High-Low type interaction respectively; (iii) In Subsection 6.4, we finish the Z_2-estimate of the cubic terms; (iv) In Subsection 6.5, we finish the Z_2-estimate of the quartic terms. Therefore, combining the aforementioned estimates with the estimate (7.13) of quintic and higher order reminder terms "$\mathscr{R}_1$" in Lemma 7.4 in Section 7, we finish the high order weighted norm estimate.

6.1. The set-up of the Z_2-norm estimate

Define

(6.1) $\hat{\Omega}_\xi := -\xi^\perp \cdot \nabla_\xi, \quad d_\Omega := 0, \quad \xi_\Omega := -\xi^\perp, \quad \hat{L}_\xi := -\xi \cdot \nabla_\xi, \quad d_L := -2, \quad \xi_L := -\xi.$

(6.2)
$\chi_k^1 := \{(k_1, k_2) : |k_1 - k_2| \le 10, k \le k_1 + 10\}, \quad \chi_k^2 := \{(k_1, k_2) : k_2 \le k_1 - 10, |k_1 - k| \le 10\}.$

Recall that $L := x \cdot \nabla + 2$ and $\Omega := x^\perp \cdot \nabla$ and the Z_2 norm is defined in (1.23). We have

$$\hat{\Omega}_\xi \widehat{g}(t,\xi) = \widehat{\Omega g}(t,\xi), \quad \hat{L}_\xi \widehat{g}(t,\xi) = \widehat{L g}(t,\xi),$$

$$\|g(t)\|_{Z_2} \in \Big(\sum_{\Gamma_\xi^1, \Gamma_\xi^2 \in \{\hat{\Omega}_\xi, \hat{L}_\xi\}} \|\Gamma_\xi^1 \Gamma_\xi^2 \widehat{g}(t,\xi)\|_{L^2} + \|\Gamma_\xi^1 \widehat{g}(t,\xi)\|_{L^2} \Big)[c, C], \tag{6.3}$$

where c and C are some absolute constants.

Since the estimate of the second part of the right hand side of (6.3) is similar and also much easier than the first part, for simplicity, we only estimate the first part in details here. Therefore, to prove the desired estimate (4.51), it would be sufficient to prove the following desired estimate for any $\Gamma_\xi^1, \Gamma_\xi^2 \in \{\hat{L}_\xi, \hat{\Omega}_\xi\}$(correspondingly, $\Gamma^1, \Gamma^2 \in \{L, \Omega\}$) and any $t_1, t_2 \in [2^{m-1}, 2^m]$,

$$\Big| \mathrm{Re}\Big[\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \overline{\Gamma_\xi^1 \Gamma_\xi^2 \widehat{g}(t,\xi)} \Gamma_\xi^1 \Gamma_\xi^2 \partial_t \widehat{g}(t,\xi) d\xi dt \Big] \Big| \le C 2^{2\tilde{\delta} m} \epsilon_0^2, \tag{6.4}$$

where C is some absolute constant.

Recall (4.35). We first classify the quadratic terms. Recall (4.36). From the direct computations, we have the following identity for the quadratic terms,

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\Gamma_\xi^1 \Gamma_\xi^2 \widehat{g_k}(t,\xi)} \Gamma_\xi^1 \Gamma_\xi^2 B_{k,k_1,k_2}^{\mu,\nu}(t,\xi) d\xi dt \\
&= \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\Gamma_\xi^1 \Gamma_\xi^2 \widehat{g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} \Big[\Gamma_\xi^1 \Gamma_\xi^2 \big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta) \widehat{g_{k_1}^{\mu}}(t,\xi-\eta) \big) \widehat{g_{k_2}^{\nu}}(t,\eta) \\
&\quad + \sum_{l,n=\{1,2\}} it \big(\Gamma_\xi^l \Phi^{\mu,\nu}(\xi,\eta) \big) \Gamma_\xi^n \big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta) \widehat{g_{k_1}^{\mu}}(t,\xi-\eta) \big) \widehat{g_{k_2}^{\nu}}(t,\eta) \\
&\quad - t^2 \Gamma_\xi^1 \Phi^{\mu,\nu}(\xi,\eta) \Gamma_\xi^2 \Phi^{\mu,\nu}(\xi,\eta) \tilde{q}_{\mu,\nu}(\xi-\eta,\eta) \widehat{g_{k_1}^{\mu}}(t,\xi-\eta) \widehat{g_{k_2}^{\nu}}(t,\eta) \Big] d\eta d\xi dt.
\end{aligned} \tag{6.5}$$

To make the formulation (6.5) symmetric, we separate $\Gamma_\xi^i \widehat{g}(t,\xi-\eta)$, $i \in \{1,2\}$, into two parts as follows,

$$\Gamma_\xi^i \widehat{g}(t,\xi-\eta) = \Gamma_{\xi-\eta}^i \widehat{g}(t,\xi-\eta) - \Gamma_\eta^i \widehat{g}(t,\xi-\eta).$$

After applying the above decomposition to the equality (6.5), we do integration by parts in "η" in (6.5) to move the derivative in front of $\Gamma_\eta^i \widehat{g}(t,\xi-\eta)$ around, see (6.1). As a result, the following equality holds,

$$\mathrm{Re}\Big[\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \overline{\Gamma_\xi^1 \Gamma_\xi^2 \widehat{g}_k(t,\xi)} \Gamma_\xi^1 \Gamma_\xi^2 B_{k,k_1,k_2}^{\mu,\nu}(t,\xi) d\xi dt \Big] = \sum_{i=1,2,3,4} \mathrm{Re}[P_{k,k_1,k_2}^i], \tag{6.6}$$

where

$$P^1_{k,k_1,k_2} := \sum_{\{l,n\}=\{1,2\}} \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} it(\Gamma^l_\xi + \Gamma^l_\eta)\Phi^{\mu,\nu}(\xi,\eta)$$

$$\times \big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{g^\nu_{k_2}}(t,\eta)\widehat{\Gamma^n g^\mu_{k_1}}(t,\xi-\eta) + \widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2}}(t,\eta)\big)$$

$$+ (\Gamma^n_\xi + \Gamma^n_\eta + d_{\Gamma^n})\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\big] d\eta\, d\xi\, dt, \tag{6.7}$$

$$P^2_{k,k_1,k_2} := -\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} t^2(\Gamma^1_\xi + \Gamma^1_\eta)\Phi^{\mu,\nu}(\xi,\eta)(\Gamma^2_\xi + \Gamma^2_\eta)$$

$$\times \Phi^{\mu,\nu}(\xi,\eta)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt, \tag{6.8}$$

$$P^3_{k,k_1,k_2} := \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} \Big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma^1\Gamma^2 g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)$$

$$+ \widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{\Gamma^1\Gamma^2 g^\nu_{k_2}}(t,\eta)\big)$$

$$+ (\Gamma^1_\xi + \Gamma^1_\eta + d_{\Gamma^1})(\Gamma^2_\xi + \Gamma^2_\eta + d_{\Gamma^2})\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)$$

$$+ (\Gamma^l_\xi + \Gamma^l_\eta + d_{\Gamma^l})\tilde{q}_{\mu,\nu}(\xi-\eta,\eta) \tag{6.9}$$

$$\times \big(\widehat{\Gamma^n g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta) + \widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2}}(t,\eta)\big)\Big) d\eta\, d\xi\, dt,$$

$$P^4_{k,k_1,k_2} := \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} P^{4,j_1,j_2}_{k,k_1,k_2}, \quad P^{4,j_1,j_2}_{k,k_1,k_2} := \sum_{\{l,n\}=\{1,2\}} \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}$$

$$\times e^{it\Phi^{\mu,\nu}(\xi,\eta)} \tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{\Gamma^l g^\mu_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2,j_2}}(t,\eta)\, d\eta\, d\xi\, dt. \tag{6.10}$$

Now we reveal a subtle structure inside the symbol "$(\Gamma_\xi + \Gamma_\eta)\Phi^{\mu,\nu}(\xi,\eta)$," which appears in $P^i_{k,k_1,k_2}, i \in \{1,2\}$, see (6.7) and (6.8). Note that, the following equalities hold when $|\eta| \leq 2^{-10}|\xi|$ and $\mu = +$,

$$\big(\hat{L}_\xi + \hat{L}_\eta\big)\Phi^{\mu,\nu}(\xi,\eta) = -2\xi\cdot\big(\lambda'(|\xi|^2)\xi - \lambda'(|\xi-\eta|^2)(\xi-\eta)\big) - 2\eta\cdot\big(-\lambda'(|\xi-\eta|^2)(\eta-\xi)$$

$$-\nu\lambda'(|\eta|)\eta\big) = -4\big(\lambda'(|\xi|^2) + \lambda''(|\xi|^2)|\xi|^2\big)\xi\cdot\eta + O(|\eta|^2), \tag{6.11}$$

$$(\hat{\Omega}_\xi + \hat{\Omega}_\eta)\Phi^{\mu,\nu}(\xi,\eta) = -2\xi^\perp\cdot\big(\lambda'(|\xi|^2)\xi - \mu\lambda(|\xi-\eta|^2)(\xi-\eta)\big)$$

$$-2\eta^\perp\cdot\big(-\mu\lambda(|\xi-\eta|^2)(\eta-\xi) - \nu\lambda'(|\eta|^2)\eta\big) = -2\mu\lambda'(|\xi-\eta|^2)\big(\xi^\perp\cdot\eta + \eta^\perp\cdot\xi\big) = 0, \tag{6.12}$$

where $\lambda(|x|) := \Lambda(\sqrt{|x|})$. The following approximation holds when $|\xi|$ is very close to zero,

$$\Lambda(|\xi|) = |\xi|^2 - \frac{1}{6}|\xi|^4 + O(|\xi|^6), \quad \lambda(|\xi|) = |\xi| - \frac{1}{6}|\xi|^2 + O(|\xi|^3), \quad |\xi| \leq 2^{-10}. \tag{6.13}$$

Moreover, the following equalities hold when $|\xi| \leq 2^{-10}|\eta|$ and $\mu\nu = -$,

$$\big(\hat{L}_\xi + \hat{L}_\eta\big)\Phi^{\mu,\nu}(\xi,\eta) = -2\lambda'(|\xi|^2)|\xi|^2 + \mu 2\lambda'(|\xi-\eta|^2)\xi\cdot(\xi-\eta) + 2\mu\lambda'(|\xi-\eta|^2)\eta\cdot(\eta-\xi)$$

$$+ 2\nu\lambda'(|\eta|^2)|\eta|^2 = -4\mu\big(\lambda'(|\eta|^2) + \lambda''(|\eta|^2)|\eta|^2\big)\xi\cdot\eta + O(|\xi|^2), \tag{6.14}$$

$$(\hat{\Omega}_\xi + \hat{\Omega}_\eta)\Phi^{\mu,\nu}(\xi,\eta) = -2\mu\lambda'(|\xi-\eta|^2)\big(\xi^\perp\cdot\eta + \eta^\perp\cdot\xi\big) = 0. \tag{6.15}$$

Now, we show that similar decompositions also hold for the phase $\Phi^{\mu,\nu}(\xi,\eta)$ in two different scenarios so that we can link the symbol $(\Gamma_\xi + \Gamma_\eta)\Phi^{\mu,\nu}(\xi,\eta)$ with the phase $\Phi^{\mu,\nu}(\xi,\eta)$. Note that the following expansion holds when $|\eta| \le 2^{-10}|\xi|$ and $\mu = +$,

$$\Phi^{\mu,\nu}(\xi,\eta) = \lambda(|\xi|^2) - \lambda(|\xi|^2 - 2\xi\cdot\eta + |\eta|^2) - \nu\lambda(|\eta|^2) = 2\lambda'(|\xi|^2)\xi\cdot\eta + O(|\eta|^2). \tag{6.16}$$

Hence, from (6.16) and (6.11), the following identity holds when $|\eta| \le 2^{-10}|\xi|$ and $\mu = +$,

$$\big(\hat{L}_\xi + \hat{L}_\eta\big)\Phi^{\mu,\nu}(\xi,\eta) = \tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta) + O(|\eta|^2), \quad \tilde{c}(\xi) := -\frac{2\lambda''(|\xi|^2)|\xi|^2 + 2\lambda'(|\xi|^2)}{\lambda'(|\xi|^2)}. \tag{6.17}$$

Moreover, the following approximation holds for the phase $\Phi^{\mu,\nu}(\xi,\eta)$ when $|\xi| \le 2^{-10}|\eta|$ and $\mu\nu = -$,

$$\Phi^{\mu,\nu}(\xi,\eta) = \lambda(|\xi|^2) - \mu\big(\lambda(|\xi|^2 - 2\xi\cdot\eta + |\eta|^2) - \lambda(|\eta|^2)\big) = 2\mu\lambda'(|\eta|^2)\xi\cdot\eta + O(|\xi|^2). \tag{6.18}$$

Therefore, from (6.18) and (6.14), the following identity holds when $|\xi| \le 2^{-10}|\eta|$ and $\mu\nu = -$,

$$\big(\hat{L}_\xi + \hat{L}_\eta\big)\Phi^{\mu,\nu}(\xi,\eta) = \tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta) + O(|\xi|^2). \tag{6.19}$$

6.2. Z_2-norm estimate of the quadratic terms: if $|k_1 - k_2| \le 10$

Recall the decomposition (6.6). We know that the Z_2-norm estimate of the quadratic terms in the High-High type interaction follows from the estimate (6.20) in Lemma 6.1 and the estimate (6.22) in Lemma 6.2.

LEMMA 6.1. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant* C*,*

$$\Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^3_{k,k_1,k_2}\Big| + \Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^4_{k,k_1,k_2}\Big| \le C2^{2\tilde{\delta}m}\epsilon_0. \tag{6.20}$$

Proof. – Recall (6.6). From the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, we have

$$\begin{aligned}
&\Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^3_{k,k_1,k_2}\Big| \\
&\quad\le \sup_{t_1,t_2\in[2^{m-1},2^m]} \sum_{|k_1-k_2|\le 10} C2^{m+2k_1}\|P_{\le k_1+20}\Gamma^1\Gamma^2 g(t)\|_{L^2}\big(\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty} \\
&\quad\quad + \|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\big)\Big(\sum_{l,m\in\{1,2\}}\|\Gamma^1\Gamma^2 g_{k_m}(t)\|_{L^2} + \|\Gamma^l g_{k_m}\|_{L^2} + \|g_{k_m}(t)\|_{L^2}\Big) \qquad (6.21)\\
&\quad\le C2^{2\tilde{\delta}m}\epsilon_0^2,
\end{aligned}$$

where C is some absolute constant. The estimate of P^4_{k,k_1,k_2} is similar but slightly different. The spatial concentrations of inputs play a role. Note that, from the definition of Z_i-norms, $i\in\{1,2\}$ in (1.22) and (1.23) and the linear decay estimate (2.11) in Lemma 2.7, the following estimate holds for some absolute constant C,

$$\|\Gamma^l g_{k,j}\|_{L^2} \le C2^{-k-j+\tilde{\delta}m}\epsilon_1, \quad \|e^{-it\Lambda}\Gamma^l g_{k,j}\|_{L^\infty} \le C2^{-m+k+2j}\|\varphi^k_j(x)P_k g(t)\|_{L^2}.$$

After first doing spatial localizations for the inputs $g_{k_1}(t)$ and $g_{k_2}(t)$ and then put the input with smaller spatial concentration in L^∞ and the other input in L^2, the following estimate holds for some constants C_1 and C_2,

$$\begin{aligned}
&\Big|\sum_{|k_1-k_2|\leq 10, k\leq k_1+20} P^4_{k,k_1,k_2}\Big| \\
&\quad\leq \sup_{t_1,t_2\in[2^{m-1},2^m]} \sum_{|k_1-k_2|\leq 10}\sum_{\{l,m\}=\{1,2\}} C_1 2^{m+2k_1}\|P_{\leq k_1+20}\Gamma^1\Gamma^2 g(t)\|_{L^2} \\
&\qquad\times\Big(\sum_{j_1\geq j_2}\|\Gamma^l g_{k_1,j_1}\|_{L^2}\|e^{-it\Lambda}\Gamma^n g_{k_2,j_2}\|_{L^\infty}+\sum_{j_2\geq j_1}\|e^{-it\Lambda}\Gamma^l g_{k_1,j_1}\|_{L^\infty}\|\Gamma^n g_{k_2,j_2}\|_{L^2}\Big) \\
&\quad\leq \sum_{j_2} C_2 2^{2k_1+2j_2}\|\varphi^{k_2}_{j_2}(x)P_{k_2}g(t)\|_{L^2}\Big(\sum_{j_1\geq j_2}2^{2\tilde{\delta}m-j_1}\epsilon_1\Big) \\
&\qquad+\sum_{j_1} C_2 2^{2k_1+2j_1}\|\varphi^{k_1}_{j_1}(x)P_{k_1}g(t)\|_{L^2}\Big(\sum_{j_2\geq j_1}2^{2\tilde{\delta}m-j_2}\epsilon_1\Big) \\
&\quad\leq C_2 2^{2\tilde{\delta}m}\epsilon_0^2.
\end{aligned}$$

From the above estimate and the estimate (6.21), we know that the desired estimate (6.20) holds. □

LEMMA 6.2. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant* C*,*

$$\Big|\sum_{|k_1-k_2|\leq 10, k\leq k_1+20} P^1_{k,k_1,k_2}\Big| + \Big|\sum_{|k_1-k_2|\leq 10, k\leq k_1+20} P^2_{k,k_1,k_2}\Big| \leq C 2^{2\tilde{\delta}m}\epsilon_0. \tag{6.22}$$

Proof. – Note that, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&|P^1_{k,k_1,k_2}|+|P^2_{k,k_1,k_2}| \\
&\qquad\leq \sup_{t\in[2^{m-1},2^m]} C(2^{2m+k+3k_1}+2^{3m+2k+4k_1})\sum_{i,j=1,2}\big(\|g_{k_i}(t)\|_{L^2}+2^{k_1}\|\nabla_\xi\widehat{g}_{k_i}(t,\xi)\|_{L^2}\big) \\
&\qquad\times\|\Gamma^1\Gamma^2 g_k\|_{L^2}\|e^{-it\Lambda}g_{k_j}(t)\|_{L^\infty} \qquad (6.23)\\
&\qquad\leq C2^{\tilde{\delta}m+\delta m}(2^{m+k+k_{1,-}-15k_{1,+}}+2^{2m+2k+2k_{1,-}-14k_{1,+}})\epsilon_0^2. \qquad (6.24)
\end{aligned}$$

From the above rough estimate (6.23), we can rule out the case when $k+k_{1,-}\leq -m+\tilde{\delta}m/3$ or $k_1\geq m/5$. From now on, we restrict ourself to the case when $k+k_{1,-}\geq -m+\tilde{\delta}m/3$ and $k_1\leq m/5$.

Recall (6.7) and (6.15). We know that the integral inside P^1_{k,k_1,k_2} actually vanishes when $\Gamma^l=\hat{\Omega}_\xi$. Hence, we only need to consider the case when $\Gamma^l_\xi=\hat{L}_\xi$. Based on the possible size of k, we separate into two cases as follows.

Case 1: if $k \leq k_1 - 10$. – Recall the normal form transformation that we did in Subsection 4.1. For the case we are considering, we have $\mu\nu = -$. Recall (6.19). To take the advantage of this decomposition, we decompose P^1_{k,k_1,k_2} and P^2_{k,k_1,k_2} into two parts respectively as follows,

$$(6.25)\quad |P^1_{k,k_1,k_2}| \leq \sum_{\Gamma\in\{L,\Omega\}} |\Gamma^{1,1}_{k,k_1,,k_2}| + |\Gamma^{1,2}_{k,k_1,,k_2}|, \quad |P^2_{k,k_1,k_2}| \leq |\widetilde{P}^1_{k,k_1,k_2}| + |\widetilde{P}^2_{k,k_1,k_2}|,$$

where

$$\Gamma^{1,i}_{k,k_1,,k_2} := \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} it\tilde{q}^i_{\mu,\nu}(\xi-\eta,\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)$$

$$(6.26)\quad \times\widehat{g^{\nu}_{k_2}}(t,\eta)+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta)\big)+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big]d\eta d\xi dt,$$

$$(6.27)\quad \widetilde{P}^i_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{LLg_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} t^2\widehat{q}^i_{\mu,\nu}(\xi,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)d\eta d\xi dt,$$

where

$$(6.28)\quad \tilde{q}^1_{\mu,\nu}(\xi-\eta,\eta) = \tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta), \quad \tilde{q}^2_{\mu,\nu}(\xi-\eta,\eta) := (\hat{L}_\xi+\hat{L}_\eta)\Phi^{\mu,\nu}(\xi,\eta)-\tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta).$$

$$(6.29)\quad \widehat{q}^1_{\mu,\nu}(\xi,\eta) = \widehat{p}^1_{\mu,\nu}(\xi,\eta)\Phi^{\mu,\nu}(\xi,\eta), \quad \widehat{p}^1_{\mu,\nu}(\xi,\eta) := \tilde{q}_{\mu,\nu}(\xi-\eta,\eta)(\hat{L}_\xi+\hat{L}_\eta)\Phi^{\mu,\nu}(\xi,\eta)\tilde{c}(\xi-\eta),$$

$$(6.30)\quad \widehat{q}^2_{\mu,\nu}(\xi,\eta) = \tilde{q}_{\mu,\nu}(\xi-\eta,\eta)(\hat{L}_\xi+\hat{L}_\eta)\Phi^{\mu,\nu}(\xi,\eta)\big((\hat{L}_\xi+\hat{L}_\eta)\Phi^{\mu,\nu}(\xi,\eta)-\tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta)\big).$$

From the estimate (2.3) Lemma 2.1, the following estimates hold for some absolute constant C,

$$(6.31)\quad \begin{aligned} \|\tilde{q}^2_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \leq C2^{2k}, \quad \|\widehat{p}^1_{\mu,\nu}(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} &\leq C2^{k+3k_1}, \\ \|\widehat{q}^2_{\mu,\nu}(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} &\leq C2^{3k+3k_1}. \end{aligned}$$

After doing integration by parts in "η" once for $\Gamma^{1,2}_{k,k_1,,k_2}$ and doing integration by parts in "η" twice for $\widetilde{P}^2_{k,k_1,k_2}$, the following estimates hold for some absolute constants C_1 and C_2,

$$\sum_{k\leq k_1+20,|k_1-k_2|\leq 10} |\Gamma^{1,2}_{k,k_1,,k_2}| + \big|\widetilde{P}^2_{k,k_1,k_2}\big|$$
$$\leq \sup_{t\in[2^{m-1},2^m]} \sum_{k\leq k_1+20,|k_1-k_2|\leq 10} \Big[C_1\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} 2^{m+k+k_1+k_{1,+}}$$
$$\times \Big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla^i_\xi \widehat{g_{k_1}}(t,\xi)\|_{L^2} + 2^{ik_1}\|\nabla^i_\xi \widehat{g_{k_2}}(t,\xi)\|_{L^2}\Big)\Big(\sum_{i=1,2}\|e^{-it\Lambda}g_{k_i}\|_{L^\infty}\Big)$$
$$+ \sum_{j_1\geq j_2} C_1 2^{k+3k_1+k_{1,+}}$$
$$\times \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} 2^{2j_2}\|\varphi^{k_2}_{j_2}(x)P_{k_2}g(t)\|_{L^2} 2^{j_1}\|\varphi^{k_1}_{j_1}(x)P_{k_1}g(t)\|_{L^2}$$
$$+ \sum_{j_2\geq j_1} C_1 2^{k+3k_1+k_{1,+}}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}$$
$$\times 2^{2j_1}\|\varphi^{k_2}_{j_2}(x)P_{k_2}g(t)\|_{L^2} 2^{j_2}\|\varphi^{k_1}_{j_1}(x)P_{k_1}g(t)\|_{L^2}\Big] \leq C_2 2^{2\tilde{\delta}m}\epsilon_0^2. \tag{6.32}$$

To sum up, from the estimate (6.25), the estimate (6.32), the estimate (6.33) in Lemma 6.3 and the estimate (6.45) in Lemma 6.4, we finish the estimate of P^i_{k,k_1,k_2}, $i\in\{1,2\}$, for the case we are considering.

Case 2: If $k\geq k_1-10$ *and* $|k_1-k_2|\leq 10$. – For the case we are considering, the sizes of all frequencies are comparable, which implies that the estimate (6.32) also holds for P^1_{k,k_1,k_2} and P^2_{k,k_1,k_2} without decomposing the symbols of quadratic terms as in the estimate (6.25).

Hence finishing the proof of the desired estimate (6.22). □

LEMMA 6.3. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k+k_{1,-}\geq -m+\tilde{\delta}m/3$, $k\leq k_1-10$ *and* $k_1\leq m/5$, *the following estimate holds for some absolute constant* C,

$$|\Gamma^{1,1}_{k,k_1,k_2}| \leq C2^{9/5\tilde{\delta}m}\epsilon_0^2. \tag{6.33}$$

Proof. – Recall the associated symbol $\tilde{q}^1_{\mu,\nu}(\xi-\eta,\eta)$ of $\Gamma^{1,1}_{k,k_1,k_2}$ in (6.28). To take the advantage of smallness of symbol near the time resonance set, we do integration by parts in time once. As a result, we have

$$\Gamma^{1,1}_{k,k_1,k_2} = \sum_{i=1,2}\widetilde{\Gamma}^{1,i}_{k,k_1,k_2}, \quad \widetilde{\Gamma}^{1,1}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}} \widetilde{\Gamma}^{j_1,j_2,1,1}_{k,k_1,k_2}, \tag{6.34}$$

$$
\begin{aligned}
\widetilde{\Gamma}^{j_1,j_2,1,1}_{k,k_1,k_2} := &-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}e^{it\Phi^{\mu,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\\
&\times\big(\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\widehat{\Gamma g^{\mu}_{k_1,j_1}}(t,\xi-\eta)+\widehat{g^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t,\eta)\big)\\
&+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi,\eta)\widehat{g^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2e}}(t,\eta)\big]d\eta d\xi dt\\
&+\sum_{i=1,2}(-1)^i\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t_i,\xi)}e^{it_i\Phi^{\mu,\nu}(\xi,\eta)}t_i\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\\
&\times\big(\widehat{\Gamma g^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{g^{\nu}_{k_2 j_2}}(t_i,\eta)+\widehat{g^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t_i,\eta)\big)\\
&+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t_i,\eta)\big]d\eta d\xi,
\end{aligned}
\tag{6.35}
$$

$$
\begin{aligned}
\widetilde{\Gamma}^{1,2}_{k,k_1,k_2} = &-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}t\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\partial_t\big(\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\\
&\times\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta)\big)\big)\\
&+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\partial_t\big(\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big)\big]d\eta d\xi dt.
\end{aligned}
\tag{6.36}
$$

For $\widetilde{\Gamma}^{1,1}_{k,k_1,k_2}$, we do integration by parts in "η" once. As a result, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constants C ,

$$
\begin{aligned}
|\widetilde{\Gamma}^{1,1}_{k,k_1,k_2}| \le &\sup_{t\in[2^{m-1},2^m]} C2^{k_1,+}\|\Gamma^1\Gamma^2 g_k\|_{L^2}\\
&\times\Big[2^{-k+k_1}\big(\sum_{i=0,1,2}2^{ik_1}\|\nabla^i_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}+2^{ik_1}\|\nabla^i_\xi\widehat{g}_{k_2}(t,\xi)\|_{L^2}\big)\\
&\times\big(\sum_{i=1,2}\|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\big)+\sum_{j_1\ge j_2}2^{-k+3k_1+j_1}\|e^{-it\Lambda}\mathcal{F}^{-1}\big[\nabla_\xi\widehat{g}_{k_2,j_2}\big]\|_{L^\infty}\|g_{k_1,j_1}\|_{L^2}\\
&+\sum_{j_2\ge j_1}2^{-k+3k_1+j_2}\|e^{-it\Lambda}\mathcal{F}^{-1}\big[\nabla_\xi\widehat{g}_{k_1,j_1}\big]\|_{L^\infty}\|g_{k_2,j_2}\|_{L^2}\Big]\\
\le &C2^{-m-k-k_1+2\tilde{\delta}m+\delta m}\epsilon_0^2\le C2^{9\tilde{\delta}m/5}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2}_{k,k_1,k_2}$ in (6.36). Since "∂_t" can hit every input inside $\widetilde{\Gamma}^{1,2}_{k,k_1,k_2}$, which creates many terms. We put terms that have similar structures together and split $\widetilde{\Gamma}^{1,2}_{k,k_2,k_2}$ into five parts as follows,

$$
\widetilde{\Gamma}^{1,2}_{k,k_1,k_2} = \sum_{i=1,2,3,4,5}\widehat{\Gamma}^{i}_{k,k_1,k_2},
\tag{6.37}
$$

$$
\begin{aligned}
\widehat{\Gamma}^{1}_{k,k_1,k_2} = &-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}t\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\\
&\times\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta)\big)\\
&+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big]
\end{aligned}
$$

$$\times \overline{\big(\partial_t \widehat{\Gamma^1\Gamma^2 g_k}(t,\xi) - \sum_{\nu'\in\{+,-\}} \sum_{(k_1',k_2')\in\chi_k^2} \widetilde{B}^{+,\nu'}_{k,k_1',k_2'}(t,\xi)\big)} d\eta d\xi dt, \tag{6.38}$$

where $\widetilde{B}^{+,\nu'}_{k,k_1',k_2'}(t,\xi)$ is defined in (7.8),

$$\widehat{\Gamma}^2_{k,k_1,k_2} = \sum_{\nu'\in\{+,-\}} \sum_{(k_1',k_2')\in\chi_k^2} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{e^{it\Phi^{+,\nu'}(\xi,\kappa)}\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)}$$

$$\times te^{it\Phi^{\mu,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta) + \widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta)\big)$$

$$+ (\Gamma_\xi + \Gamma_\eta + d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big]d\kappa d\eta d\xi dt, \tag{6.39}$$

$$\widehat{\Gamma}^3_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} t\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)\partial_t\widehat{g^{\nu}_{k_2}}(t,\eta)$$

$$+\partial_t\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta) + \widehat{\Gamma\Lambda_{\geq 3}[\partial_t g^{\mu}_{k_1}]}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta) + \widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma\Lambda_{\geq 3}[\partial_t g^{\nu}_{k_2}]}(t,\eta)\big)$$

$$+ (\Gamma_\xi + \Gamma_\eta + d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\partial_t\big(\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big)\big]d\eta d\xi dt, \tag{6.40}$$

$$\widehat{\Gamma}^i_{k,k_1,k_2} = \sum_{k_1',k_2'\in\mathbb{Z}} \Gamma^{k_1',k_2';i-3}_{k,k_1,k_2}, \quad \Gamma^{k_1',,k_2';i-4}_{k,k_1,k_2} := \sum_{j_1'\geq -k'_{1,-},\, j_2'\geq -k'_{2,-}} \Gamma^{k_1',j_1',k_2',j_2';i-4}_{k,k_1,k_2}, \quad i\in\{4,5\}, \tag{6.41}$$

$$\Gamma^{k_1',j_1',k_2',j_2';1}_{k,k_1,k_2} := \sum_{\tau,\iota\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} t\tilde{c}(\xi-\eta)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$

$$\times\big(P_\mu[e^{it\Phi^{\tau,\iota}(\xi-\eta,\sigma)}\tilde{q}_{\tau,\iota}(\xi-\eta-\sigma,\sigma)\widehat{g^{\iota}_{k_2',j_2'}}(t,\sigma)\Gamma_{\xi-\eta}\widehat{g^{\tau}_{k_1',j_1'}}(t,\xi-\eta-\sigma)]\widehat{g^{\nu}_{k_2}}(t,\eta)$$

$$+ \widehat{g^{\mu}_{k_1}}(t,\xi-\eta)P_\nu[e^{it\Phi^{\tau,\iota}(\eta,\sigma)}\tilde{q}_{\tau,\iota}(\eta-\sigma,\sigma)\Gamma_\eta\widehat{g^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{g^{\iota}_{k_2',j_2'}}(t,\sigma)]\big)d\sigma d\eta d\xi dt, \tag{6.42}$$

$$\Gamma^{k_1',j_1',k_2',j_2';2}_{k,k_1,k_2} = \sum_{\tau,\iota\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} it^2\tilde{c}(\xi-\eta)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$

$$\times\big(P_\mu[e^{it\Phi^{\tau,\iota}(\xi-\eta,\sigma)}\Gamma_{\xi-\eta}\Phi^{\tau,\iota}(\xi-\eta,\sigma)\tilde{q}_{\tau,\iota}(\xi-\eta-\sigma,\sigma)\widehat{g^{\iota}_{k_2',j_2'}}(t,\sigma)\widehat{g^{\tau}_{k_1',j_1'}}(t,\xi-\eta-\sigma)]\widehat{g^{\nu}_{k_2}}(t,\eta)$$

$$+ \widehat{g^{\mu}_{k_1}}(t,\xi-\eta)P_\nu[e^{it\Phi^{\tau,\iota}(\eta,\sigma)}\Gamma_\eta\Phi^{\tau,\iota}(\eta,\sigma)\tilde{q}_{\tau,\iota}(\eta-\sigma,\sigma)\widehat{g^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{g^{\iota}_{k_2',j_2'}}(t,\sigma)]\big)d\sigma d\eta d\xi dt. \tag{6.43}$$

Recall (6.38). For $\widehat{\Gamma}^1_{k,k_2,k_2}$, we do integration by parts in "η" once. From (7.7) in Lemma 7.2 and the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some constant C,

$$|\widehat{\Gamma}^1_{k,k_1,k_2}| \leq \sup_{t\in[2^{m-1},2^m]} C2^{-k+5k_{1,+}+k_1}\big(2^{\tilde{\delta}m+\delta m} + 2^{3\tilde{\delta}m+k}\big)$$
$$\times\big[\big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla^i_\xi\widehat{g_{k_1}}(t,\xi)\|_{L^2}\big)\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}$$
$$+\big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla^i_\xi\widehat{g_{k_2}}(t,\xi)\|_{L^2}\big)\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}$$

$$
\begin{aligned}
&+ \sum_{j_1\geq j_2} 2^{-m+2j_2+j_1+k_1+k_2}\|\varphi_{j_1}^{k_1}(x)g_{k_1}(t)\|_{L^2}\|\varphi_{j_2}^{k_2}(x)g_{k_2}(t)\|_{L^2}\\
&+ \sum_{j_2\geq j_1} 2^{-m+2j_1+j_2+k_1+k_2}\|\varphi_{j_1}^{k_1}(x)g_{k_1}(t)\|_{L^2}\|\varphi_{j_2}^{k_2}(x)g_{k_2}(t)\|_{L^2}\big]\\
\leq{}& C2^{-m+2\tilde{\delta}m+\delta m-k-k_1}\epsilon_0^2 + 2^{-m+4\beta m}\epsilon_0^2 \leq C2^{9\tilde{\delta}m/5}\epsilon_0^2.
\end{aligned}
$$

Recall (6.39). For $\widehat{\Gamma}^2_{k,k_2,k_2}$, we do integration by parts in "η" once. Recall that $|k_1'-k|\leq 10$. The loss of 2^{-k} from integration by parts in "η" is compensated by the smallness of $2^{2k_1'}$ from the symbol $\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)$. As a result, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
|\widehat{\Gamma}^2_{k,k_1,k_2}| \leq{}& \sup_{t\in[2^{m-1},2^m]}\sum_{k_2'\leq k-10} C2^{m+k+k_1}\|\Gamma^1\Gamma^2 g_{k_1'}\|_{L^2}\|e^{-it\Lambda}g_{k_2'}(t)\|_{L^\infty}\\
&\times\Big(\sum_{i=0,1,2}2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_1}(t,\xi)\|_{L^2}+2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_2}(t,\xi)\|_{L^2}\Big)\\
&\times\Big(\sum_{i=1,2}2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g}_{k_i}]\|_{L^\infty}+\|e^{-it\Lambda}g_{k_i}\|_{L^\infty}\Big)\\
\leq{}& C2^{-m/2+\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widehat{\Gamma}^3_{k,k_2,k_2}$. Recall (6.40). From estimate (7.1) in Lemma 7.1, estimate (5.18) in Proposition 5.4, (7.13) in Lemma 7.4, and the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
|\widehat{\Gamma}^3_{k,k_1,k_2}| \leq{}& \sup_{t\in[2^{m-1},2^m]}\sum_{l=1,2}C2^{2m+(2-\alpha)k_1}\\
&\Big[\Big(\|\partial_t\widehat{g}_{k_l}(t,\xi)-\sum_{\mu,\nu\in\{+,-\}}\sum_{(k_1',k_2')\in\chi^1_{k_l}}B^{\mu,\nu}_{k_l,k_1',k_2'}(t,\xi)\|_{L^2}\|e^{-it\Lambda}\Gamma g_{k_{3-l}}(t)\|_{L^\infty}\\
&+\|\Gamma g_{k_{3-l}}(t)\|_{L^2}\sum_{(k_1',k_2')\in\chi^1_{k_l}}\|e^{-it\Lambda}\mathcal{F}^{-1}[B^{\mu,\nu}_{k_l,k_1',k_2'}(t,\xi)]\|_{L^\infty}\\
&+\|\Lambda_{\geq 3}[\partial_t g_{k_l}]\|_{Z_1}\|e^{-it\Lambda}g_{k_{3-l}}(t)\|_{L^\infty}\Big]\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\leq C2^{3\tilde{\delta}m/2}\epsilon_0^2.
\end{aligned}
$$

Lastly, we estimate $\widehat{\Gamma}^4_{k,k_1,k_2}$ and $\widehat{\Gamma}^5_{k,k_1,k_2}$. Recall (6.41). Based on the size of difference between k_1' and k_2' and the size of $k_{1,-}'+k_2$, we split into three cases as follows,

If $|k_1'-k_2'|\leq 10$. – For this case, we know that $\nabla_\sigma\Phi^{\tau,\iota}(.,.)$ is bounded from blow by $2^{k_{1,-}-k_{1,+}'}$. Hence, to take advantage of this fact, we do integration by parts in "σ" once for $\Gamma^{k_1',k_2';1}_{k,k_1,k_2}$ and do integration by parts in "σ" twice for $\Gamma^{k_1',k_2';2}_{k,k_1,k_2}$. As a result, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{|k_1'-k_2'|\leq 10,|k_1-k_2|\leq 10}\sum_{i=1,2}|\Gamma^{k_1',k_2';i}_{k,k_1,k_2}| \leq{}& \sup_{t\in[2^{m-1},2^m]}\sum_{|k_1'-k_2'|\leq 10,|k_1-k_2|\leq 10}C2^{m+k_1+k_1'+2k_{1,+}'}\\
&\times\Big(\sum_{i=0,1,2}2^{ik_1'}\|\nabla_\xi^i\widehat{g_{k_1'}}(t,\xi)\|_{L^2}+2^{ik_1'}\|\nabla_\xi^i\widehat{g_{k_2'}}(t,\xi)\|_{L^2}\Big)
\end{aligned}
$$

$$\times \Big(\sum_{i=1,2} 2^{k_1} \| e^{-it\Lambda} \mathscr{F}^{-1}[\nabla_\xi \widehat{g}_{k_i'}] \|_{L^\infty} + \| e^{-it\Lambda} g_{k_i'} \|_{L^\infty} \Big)$$
$$\times \Big(\sum_{i=1,2} \| e^{-it\Lambda} g_{k_i} \|_{L^\infty} \Big) \| \Gamma^1 \Gamma^2 g_k(t) \|_{L^2} \le C 2^{-\beta m} \epsilon_0^2.$$

If $k_2' \le k_1' - 10$ *and* $k_{1,-}' + k_2' \le -19m/20$. – Note that $|k_1' - k_1| \le 10$. For this case, we use the same strategy that we used in the estimates (5.12) and (5.13). From the estimate (5.15) in Lemma 5.3, we know that the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{k_2' \le k_1 - 10, k_2' + k_{1,-} \le -9m/10} \sum_{i=1,2} |\Gamma^{k_1', k_2'; i}_{k, k_1, k_2}| \\
&\le \sup_{t \in [2^{m-1}, 2^m]} \sum_{k_2' \le k_1 - 10, k_2' + k_{1,-} \le -9m/10} C \big(2^{3k_2'} \| \widehat{g}_{k_2'}(t,\xi) \|_{L^\infty_\xi} \\
&\quad + 2^{k_1' + 2k_2'} \| \widehat{\mathrm{Re}[v]}(t,\xi) \psi_{k_2'}(\xi) \|_{L^\infty_\xi} \big) \Big(\sum_{i=1,2} \| e^{-it\Lambda} g_{k_i} \|_{L^\infty} \Big) \\
&\quad \times \big(2^{2m + 2k_1 + k_1'} \| \Gamma g_{k_1'} \|_{L^2} + 2^{3m + 2k_1 + 2k_1' + k_2'} \| g_{k_1'}(t) \|_{L^2} \big) \| \Gamma^1 \Gamma^2 g_k(t) \|_{L^2} \\
&\le \sum_{k_2' \le k_1 - 10, k_2' + k_{1,-} \le -9m/10} C 2^{3\tilde{\delta} m + 2m + 2k_1 + 3k_2'} (1 + 2^{2m + 2k_{1,-} + 2k_2'}) \epsilon_0^2 \\
&\le C 2^{-\beta m} \epsilon_0^2.
\end{aligned}$$

If $k_2' \le k_1' - 10$ *and* $k_{1,-} + k_2' \ge -19m/20$. – We first do integration by parts in "σ" many times to rule out the case when $\max\{j_1', j_2'\} \le m + k_{1,-} - \beta m$. If $\max\{j_1', j_2'\} \ge m + k_{1,-} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{i=1,2} \sum_{\max\{j_1', j_2'\} \ge m + k_{1,-} - \beta m} |\Gamma^{k_1', j_1', k_2', j_2'; i}_{k, k_1, k_2}| \\
&\le \sup_{t \in [2^{m-1}, 2^m]} C \Big[\sum_{j_1' \ge \max\{j_2', m + k_{1,-} - \beta m\}} \big(2^{m + j_1' + 5k_1} + 2^{2m + 5k_1 + k_2'} \big) \\
&\quad \times \| g_{k_1', j_1'}(t) \|_{L^2} \| g_{k_2', j_2'}(t) \|_{L^1} \Big(\sum_{i=1,2} \| e^{-it\Lambda} g_{k_i} \|_{L^\infty} \Big) \\
&\quad + \sum_{j_2' \ge \max\{j_1', m + k_{1,-} - \beta m\}} \times \big(2^{m + j_2' + 5k_1} + 2^{2m + 5k_1 + k_2'} \big) \| g_{k_2', j_2'}(t) \|_{L^2} \| g_{k_1', j_1'}(t) \|_{L^1} \\
&\quad \times \Big(\sum_{i=1,2} \| e^{-it\Lambda} g_{k_i} \|_{L^\infty} \Big) \Big] \| \Gamma^1 \Gamma^2 g_k(t) \|_{L^2} \le C 2^{-m - k_2' + 10\beta m} \epsilon_0^2
\end{aligned}$$
$$\le C 2^{-\beta m} \epsilon_0^2. \tag{6.44}$$

Hence finishing the proof. □

LEMMA 6.4. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k + k_{1,-} \ge -m + \tilde{\delta} m/3$, $k \le k_1 - 10$, *and* $k_1 \le m/5$, *the following estimate holds for some absolute constant* C,

$$|\widetilde{P}^1_{k, k_1, k_2}| \le C 2^{9/5 \tilde{\delta} m} \epsilon_0^2. \tag{6.45}$$

Proof. – Recall (6.27) and its associated symbol in (6.29). To take the advantage of the small symbol near the time resonance set, for $\widetilde{P}^1_{k,k_1,k_2}$, we do integration by parts in time once. As a result, we have

$$\widetilde{P}^1_{k,k_1,k_2} = \sum_{i=1,2,3,4,5} \widehat{P}^i_{k,k_1,k_2},$$

$$\widehat{P}^1_{k,k_1,k_2} = \sum_{i=1,2}(-1)^i \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{LLg_k}(t_i,\xi)} e^{it_i\Phi^{\mu,\nu}(\xi,\eta)} i t_i^2 \widehat{p}^1_{\mu,\nu}(\xi,\eta) \times \widehat{g^\nu_{k_2}}(t_i,\eta)\widehat{g^\mu_{k_1}}(t_i,\xi-\eta)d\eta d\xi$$

$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{LLg_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} i2t\widehat{p}^1_{\mu,\nu}(\xi,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)d\eta d\xi dt. \tag{6.46}$$

$$\widehat{P}^2_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\big(\partial_t\widehat{LLg_k}(t,\xi) - \sum_{\nu\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2} \widetilde{B}^{+,\nu}_{k,k_1',k_2'}(t,\xi)\big)}$$

$$\times e^{it\Phi^{\mu,\nu}(\xi,\eta)} i t^2 \widehat{p}^1_{\mu,\nu}(\xi,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)d\eta d\xi dt, \tag{6.47}$$

$$\widehat{P}^3_{k,k_1,k_2} := \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} \widehat{P}^{3,j_1,j_2}_{k,k_1,k_2}, \tag{6.48}$$

$$\widehat{P}^{3,j_1,j_2}_{k,k_1,k_2} = \sum_{\nu'\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)} i t^2 e^{-it\Phi^{+,\nu'}(\xi,\kappa)}$$
$$\times \overline{\widehat{LLg_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)} \widehat{p}^1_{\mu,\nu}(\xi,\eta)$$
$$\times \widehat{g^\mu_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)d\kappa d\eta d\xi dt, \tag{6.49}$$

$$\widehat{P}^4_{k,k_1,k_2} := -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)} i t^2 \widehat{p}^1_{\mu,\nu}(\xi,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}$$

$$\times \big(\widehat{\Lambda_{\geq 3}[\partial_t g^\mu_{k_1}]}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta) + \widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{\Lambda_{\geq 3}[\partial_t g^\nu_{k_2}]}(t,\eta)\big)d\eta d\xi dt, \tag{6.50}$$

$$\widehat{P}^5_{k,k_1,k_2} = \sum_{k_1',k_2'\in\mathbb{Z}} \widehat{P'}^{k_1',k_2'}_{k,k_1,k_2}, \tag{6.51}$$

$$\widehat{P'}^{k_1',k_2'}_{k,k_1,k_2} = \sum_{j_1'\geq -k_{1,-}', j_2'\geq -k_{2,-}'} \widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2},$$

$$\widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2} := \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} i t^2 \widehat{p}^1_{\mu,\nu}(\xi,\eta) \tag{6.52}$$
$$\times \big[P_\mu[e^{it\Phi^{\mu',\nu'}(\xi-\eta,\sigma)}\tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)]$$

$$\times \psi_{k_1}(\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta) + \widehat{g^\mu_{k_1}}(t,\xi-\eta)P_\nu \tag{6.53}$$

$$\times [e^{it\Phi^{\mu',\nu'}(\eta,\sigma)}\tilde{q}_{\mu',\nu'}(\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k'_1,j'_1}}(t,\eta-\sigma)\widehat{g^{\nu'}_{k'_2,j'_2}}(t,\sigma)]\psi_{k_2}(\eta)\big]d\sigma d\eta d\xi dt.$$

Recall (6.46) and (6.47). For $\widehat{P}^1_{k,k_1,k_2}$ and $\widehat{P}^2_{k,k_1,k_2}$, we do integration by parts in "η" twice. As a result, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2 and estimate (7.7) in Lemma 7.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{i=1,2}|\widehat{P}^i_{k,k_1,k_2}|\\
&\quad\le \sup_{t\in[2^{m-1},2^m]} C2^{k_1+6k_{1,+}+\delta m}\big(2^{\tilde{\delta}m-k}+2^{3\tilde{\delta}m}\big)\\
&\qquad\times\Big[\Big(\sum_{i=0,1,2,j=1,2}2^{ik_1}\|\nabla^i_\xi\widehat{g}_{k_j}(t,\xi)\|_{L^2}\Big)\Big(\sum_{i=1,2}\|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\Big)\\
&\qquad\quad+\sum_{j_1\ge j_2}2^{2k_1+j_1}\|e^{-it\Lambda}\mathcal{F}^{-1}\big[\nabla_\xi\widehat{g}_{k_2,j_2}\big]\|_{L^\infty}\|g_{k_1,j_1}(t)\|_{L^2}\\
&\qquad\quad+\sum_{j_2\ge j_1}2^{2k_1+j_2}\|e^{-it\Lambda}\mathcal{F}^{-1}\big[\nabla_\xi\widehat{g}_{k_1,j_1}\big]\|_{L^\infty}\|g_{k_2,j_2}(t)\|_{L^2}\Big]\\
&\quad\le C2^{-m-k-k_1+2\tilde{\delta}m+\delta m}\epsilon_0^2+C2^{-m+4\beta m}\epsilon_0^2\le C2^{9\tilde{\delta}m/5}\epsilon_0^2.
\end{aligned}$$

Now, we proceed to estimate $\widehat{P}^3_{k,k_1,k_2}$. Recall (6.49). Note that $(k'_1,k'_2)\in\chi^2_k$, i.e., $|k'_1-k|\le 10$. Hence the symbol $\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)$ contributes the smallness of "2^{2k}". By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\le m+k_--k_{1,+}-\beta m$. If $\max\{j_1,j_2\}\ge m+k_--k_{1,+}-\beta m$, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{\substack{k'_2\le k'_1-10\\ \max\{j_1,j_2\}\ge m+k_--k_{1,+}-\beta m}}|\widehat{P'}^3_{k,k_1,j_1,k_2,j_2}|\\
&\quad\le\sup_{t\in[2^{m-1},2^m]}\sum_{\substack{k'_2\le k'_1-10\\ \max\{j_1,j_2\}\ge m+k_--k_{1,+}-\beta m}}C2^{3m+3k+3k_1}\\
&\qquad\times\Big(\sum_{j_1\ge\max\{j_2,m+k_--k_{1,+}-\beta m\}}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\|g_{k_1,j_1}(t)\|_{L^2}\\
&\qquad\quad+\sum_{j_2\ge\max\{j_1,m+k_--k_{1,+}-\beta m\}}\|g_{k_2,j_2}(t)\|_{L^2}\\
&\qquad\times\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\Big)\|LLg_{k'_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k'_2}(t)\|_{L^\infty}\\
&\quad\le C2^{-m/2+10\beta m}\epsilon_0^2.
\end{aligned}$$

Now, we proceed to estimate $\widehat{P}^4_{k,k_1,k_2}$. Recall (6.50) and the estimate of symbol "$\widehat{p}^1_{\mu,\nu}(\xi,\eta)$" in (6.31). For this case, we do integration by parts in "η" once. As a result, from estimate (5.18) in Proposition (5.4), estimate (7.13) in Lemma (7.4), and L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$|\widehat{P}^4_{k,k_1,k_2}|\le\sup_{t\in[2^{m-1},2^m]}C2^{2m+(2-\alpha)k_1+k_{1,+}}\|\Gamma^1\Gamma^2g_k(t)\|_{L^2}$$

$$\times \big(\|\Lambda_{\geq 3}[\partial_t g_{k_1}]\|_{Z_1}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} + \|\Lambda_{\geq 3}[\partial_t g_{k_2}]\|_{Z_1}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\big)$$
$$\leq C2^{-\beta m+2\tilde{\delta}m}\epsilon_0^2 \leq C2^{-\delta m}\epsilon_0^2.$$

Lastly, we proceed to estimate $\widehat{P}^5_{k,k_1,k_2}$. Recall (6.51) and (6.53). We first consider the case when $|k_1'-k_2'|\leq 10$. By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1',j_2'\}\leq m+k_{1,-}-k_{1,+}'-\beta m$. If $\max\{j_1',j_2'\}\geq m+k_{1,-}-k_{1,+}'-\beta m$, after using the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\sum_{\substack{|k_1'-k_2'|\leq 10, k_1\leq k_1'+10\\ \max\{j_1',j_2'\}\geq m+k_{1,-}-k_{1,+}'-\beta m}} |\widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2}|$$
$$\leq \sup_{t\in[2^{m-1},2^m]} \sum_{\substack{i=1,2\\ |k_1'-k_2'|\leq 10, k_1\leq k_1'+10}} C2^{3m+k+3k_1+2k_1'}$$
$$\times \Big(\sum_{j_1'\geq \max\{j_2', m+k_{1,-}-k_{1,+}'-\beta m\}} \|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}\|_{L^\infty}$$
$$+ \sum_{j_2'\geq \max\{j_1', m+k_{1,-}-k_{1,+}'-\beta m\}} \|g_{k_2',j_2'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1',j_1'}\|_{L^\infty}\Big)$$
$$\times \|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\|LLg_k(t)\|_{L^2}$$
$$\leq C2^{-m+10\beta m}\epsilon_0^2.$$

It remains to consider the case when $k_2'\leq k_1'-10$. We split it into four cases based on the size of $k_1'+k_2'$ and whether k is greater than k_2' as follows.

If $k_{1,-}'+k_2'\leq -19m/20$ and $k\leq k_2'+20$. – By using the same strategy that we used in the estimates (5.12) and (5.13), from estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\sum_{k_2'\leq k_1'-10, |k_1-k_1'|\leq 20} |\widehat{P'}^{k_1',k_2'}_{k,k_1,k_2}|$$
$$\leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\leq k_1'-10,|k_1-k_1'|\leq 20} C2^{3m+k+4k_1}\|LLg_k(t)\|_{L^2}\|g_{k_1'}(t)\|_{L^2}$$
$$\times \big(\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty} + \|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\big)$$
$$\times \big(2^{k_1'+2k_2'}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2'}(\xi)\|_{L^\infty_\xi} + 2^{3k_2'}\|\widehat{g}_{k_2'}(t,\xi)\|_{L^\infty_\xi}\big)$$
$$\leq C2^{3m+2\tilde{\delta}m+4k_2'+3k_1-15k_{1,+}}(1+2^{m+k_1+k_2'}) \leq C2^{-\beta m}\epsilon_0^2.$$

If $k_{1,-}'+k_2'\leq -19m/20$ and $k\geq k_2'+20$. – For the case we are considering, we have $|\sigma|\leq 2^{-5}|\xi|\leq 2^{-10}|\eta|$. Hence, the following estimate holds,

$$|\nabla_\eta\big(\Phi^{\mu,\nu}(\xi,\eta)+\nu(\Phi^{\mu',\nu'}(\eta,\sigma))\big)| + |\nabla_\eta\big(\Phi^{\mu,\nu}(\xi,\eta)+\mu(\Phi^{\mu',\nu'}(\xi-\eta,\sigma))\big)|$$
$$\geq 2^{-10}|\xi-\sigma|(1+|\eta|)^{-1/2} \geq 2^{k-k_{1,+}/2-20}. \tag{6.54}$$

To take advantage of this fact, we do integration by parts in "η" once. As a result, from estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
|\widehat{P'}^{k_1',k_2'}_{k,k_1,k_2}| \le \sup_{t\in[2^{m-1},2^m]} & C2^{2m+3k_1+k_{1,+}} \\
& \times \Big(\sum_{i=1,2} \|e^{-it\Lambda} g_{k_i}(t)\|_{L^\infty} + 2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi \widehat{g_{k_i}}(t,\xi)]\|_{L^\infty}\Big) \\
& \times \Big(\sum_{i=1,2} \|g_{k_1'}(t)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_i}(t,\xi)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_1'}(t,\xi)\|_{L^2}\Big) \\
& \times \big(2^{k_1'+2k_2'}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2'}(\xi)\|_{L^\infty_\xi} + 2^{3k_2'}\|\widehat{g}_{k_2'}(t,\xi)\|_{L^\infty_\xi}\big)\|LLg_k(t)\|_{L^2} \\
\le{}& C2^{-\beta m}\epsilon_0^2.
\end{aligned}
$$

If $k_{1,-}'+k_2' \ge -19m/20$ *and* $k \le k_2'+20$. – By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1', j_2'\} \le m+k_{1,-}-\beta m$. If $\max\{j_1', j_2'\} \ge m+k_{1,-}-\beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{\max\{j_1',j_2'\}\ge m+k_{1,-}-\beta m} & |\widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2}| \\
\le \sup_{t\in[2^{m-1},2^m]} & C2^{3m+k+3k_1+2k_1'}\Big(\sum_{i=1,2}\|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\Big) \\
\times \Big(& \sum_{j_1'\ge \max\{j_2', m+k_{1,-}-\beta m\}} \|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}(t)\|_{L^\infty} \\
& + \sum_{j_2'\ge \max\{j_1', m+k_{1,-}-\beta m\}} \|g_{k_2',j_2'}(t)\|_{L^2}\times\|e^{-it\Lambda}g_{k_1',j_1'}(t)\|_{L^\infty}\Big)\|LLg_k(t)\|_{L^2} \\
\le C2^{-m-k_2'+10\beta m}&\epsilon_0^2 \le C2^{-\beta m}\epsilon_0^2.
\end{aligned}
$$

If $k_{1,-}'+k_2' \ge -19m/20$ *and* $k \ge k_2'+20$. – By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1', j_2'\} \le m+k_{1,-}-\beta m$. Now, it remains to consider the case when $\max\{j_1', j_2'\} \ge m+k_{1,-}-\beta m$. As $k \ge k_2'+20$, it is easy to see that the estimate (6.54) still holds. For this case, we do integration by parts in "η" once. As a result, from the $L^2 - L^\infty - L^\infty$ type estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{\max\{j_1',j_2'\}\ge m+k_{1,-}-\beta m} & |\widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2}| \\
\le C2^{2m+4k_1} & \Big(\sum_{i=1,2} 2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi\widehat{g_{k_i}}(t,\xi)]\|_{L^\infty} + \|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\Big) \\
\times\Big(& \sum_{j_1'\ge\max\{j_2', m+k_{1,-}-\beta m\}} 2^{j_1'}\|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}(t)\|_{L^\infty} \\
& + \sum_{j_2'\ge\max\{j_1', m+k_{1,-}-\beta m\}} 2^{-m+2j_1'}\|g_{k_1',j_1'}(t)\|_{L^2}\|g_{k_2',j_2'}\|_{L^2}\Big)\|LLg_k(t)\|_{L^2} \\
\le C2^{-m/2+10\beta m}&\epsilon_0^2 + C2^{-m-k_2'+10\beta m}\epsilon_0^2 \le C2^{-\beta m}\epsilon_0^2.
\end{aligned}
$$

Hence finishing the proof. □

6.3. Z_2-norm estimate of the quadratic terms: if $k_2 \le k_1 - 10$.

Note that, for the case we are considering, we have $\mu = +$ (see (4.42)). To simplify the problem, we first rule out the very high frequency case and very low frequency case.

We first consider the case when $k_1 + k_2 \le -19m/20$. By using the same strategy that we used in the estimates of (5.12) and (5.13), from estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)}\Gamma_\xi^1\Gamma_\xi^2 B^{+,\nu}_{k,k_1,k_2}(t,\xi)d\xi dt\Big| \\
&\quad \le \sup_{t\in[2^{m-1},2^m]} C\|\Gamma_1\Gamma_2 g_k(t)\|_{L^2}\Big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla_\xi^i \widehat{g}_{k_1}(t,\xi)\|_{L^2}\Big)2^{m+k_1}\big(1+2^{2m+2k_2+2k_1}\big) \\
&\qquad \times \min\big\{2^{k_1+2k_2}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2}(\xi)\|_{L^\infty_\xi} + 2^{3k_2}\|\widehat{g}_{k_2}(t,\xi)\|_{L^\infty_\xi}, 2^{k_1+k_2}\|g_{k_2}(t)\|_{L^2}\big\} \\
&\quad \le C2^{3\tilde{\delta}m}\min\{2^{m+2k_1+k_2}(1+2^{2m+2k_1+2k_2}), 2^{2m+k_1+3k_2}(1+2^{3m+3k_1+3k_2})\} \\
&\quad \le C2^{-\beta m}\epsilon_0^2.
\end{aligned}$$

Next, we consider the case when k_1 is relatively big. More precisely, we consider the case when $k_1 \ge 5\beta m$ and $k_1 + k_2 \ge -19m/20$. Recall (6.5). Note that $\Gamma_\xi\widehat{g_{k_1}}(t,\xi-\eta) = -\xi_\Gamma\cdot\nabla_\eta\widehat{g_{k_1}}(t,\xi-\eta)$. When Γ_ξ hits $\widehat{g_{k_1}}(t,\xi-\eta)$, we do integration by parts in "η "to move around the derivative ∇_η in front of $\widehat{g_{k_1}}(t,\xi-\eta)$. As a result, from the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{k_1\ge 5\beta m, k_2\ge -m-k_1}\Big|\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)}\Gamma_\xi^1\Gamma_\xi^2 B^{+,\nu}_{k,k_1,k_2}(t,\xi)d\xi dt\Big| \\
&\quad \le \sum_{k_1\ge 5\beta m, k_2\ge -m-k_1}\sup_{t\in[2^{m-1},2^m]} C\|\Gamma_1\Gamma_2 g_k(t)\|_{L^2}\|g_{k_1}(t)\|_{L^2}2^{k_2} \\
&\qquad \times\big(2^{-2k_2}\|g_{k_2}(t)\|_{L^2} + 2^{-k_2}\|\nabla_\xi\widehat{g}_{k_2}(t)\|_{L^2} + \|\nabla_\xi^2\widehat{g}_{k_2}(t)\|_{L^2}\big) \\
&\quad \le \sum_{k_1\ge 5\beta m, k_2\ge -m-k_1} C2^{3m+\beta m+4k_1-k_2-(N_0-30)k_{1,+}}\epsilon_0 \le C2^{-\beta m}\epsilon_0. \qquad (6.55)
\end{aligned}$$

Hence, for the rest of this subsection, we restrict ourself to the case when $k_1+k_2 \ge -19m/20$ and $k_1 \le 5\beta m$. Recall the decomposition (6.6). We know that the desired estimate for the remaining cases follows from the estimate (6.56) in Lemma 6.5, the estimate (6.71) in Lemma 6.6, and the estimate (6.93) in Lemma 6.8. Hence finishing the Z_2-norm estimate of the quadratic terms for the High× Low type interaction.

LEMMA 6.5. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant C,*

(6.56)
$$\sum_{k_1,k_2\in\mathbb{Z}, |k-k_1|\le 10, k_2\le k_1-10, k_1+k_2\ge -19m/20, k_1\le 5\beta m} |\mathrm{Re}[P^3_{k,k_1,k_2}]| + |P^4_{k,k_1,k_2}| \le C2^{2\tilde{\delta}m}\epsilon_0^2.$$

Proof. – We first estimate P^4_{k,k_1,k_2}. Recall (6.10). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\le m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\ge m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\sum_{\max\{j_1,j_2\}\ge m+k_{1,-}-\beta m}|P^{4,j_1,j_2}_{k,k_1,k_2}|$$
$$\le \sup_{t\in[2^{m-1},2^m]} C2^{m+2k_1}\Big[\sum_{j_1\ge\max\{j_2,m+k_{1,-}-\beta m\}} 2^{-m+k_1+j_1+k_2+2j_2}$$
$$\times\|\varphi^{k_1}_{j_1}(x)g_{k_1}(t)\|_{L^2}\|\varphi^{k_2}_{j_2}(x)g_{k_2}(t)\|_{L^2}$$
$$+\sum_{j_2\ge\max\{j_1,m+k_{1,-}-\beta m\}} 2^{-m+k_1+2j_1+k_2+j_2}\|\varphi^{k_2}_{j_2}(x)g_{k_2}(t)\|_{L^2}\Big]$$
$$\times\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\le C2^{-m-k_2+20\beta m}\epsilon_0^2\le C2^{-\beta m}\epsilon_0^2. \tag{6.57}$$

It remains to estimate P^3_{k,k_1,k_2}, we decompose it into three parts as follows,

$$P^3_{k,k_1,k_2}=\sum_{i=1,2,3}Q^i_{k,k_1,k_2}, \tag{6.58}$$

$$Q^1_{k,k_1,k_2}=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde q_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt, \tag{6.59}$$

$$Q^2_{k,k_1,k_2}=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde q_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{\Gamma^1\Gamma^2 g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt, \tag{6.60}$$

$$Q^3_{k,k_1,k_2}=\sum_{j_1\ge -k_{1,-},\,j_2\ge -k_{2,-}}Q^{j_1,j_2,3}_{k,k_1,k_2},\quad Q^{j_1,j_2,3}_{k,k_1,k_2}:=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}e^{it\Phi^{+,\nu}(\xi,\eta)}$$
$$\times\Big[\sum_{\{l,n\}=\{1,2\}}(\Gamma^l_\xi+\Gamma^l_\eta+d_{\Gamma^l})\tilde q_{+,\nu}(\xi-\eta,\eta)$$
$$\times\big(\widehat{\Gamma^n g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)+\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2,j_2}}(t,\eta)\big)$$
$$+(\Gamma^1_\xi+\Gamma^1_\eta+d_{\Gamma^1})(\Gamma^2_\xi+\Gamma^2_\eta+d_{\Gamma^2})\tilde q_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)\Big]d\eta\, d\xi\, dt. \tag{6.61}$$

We first estimate Q^1_{k,k_1,k_2}. Note that after switching the role of ξ and $\xi-\eta$ inside Q^1_{k,k_1,k_2}, we have

$$\mathrm{Re}\big[Q^1_{k,k_1,k_2}\big]=\mathrm{Re}\big[\widetilde{Q}^1_{k,k_1,k_2}\big],$$
$$\widetilde{Q}^1_{k,k_1,k_2}:=\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}e^{it\Phi^{+,\nu}(\xi,\eta)}p^{+,\nu}_{k,k_1}(\xi-\eta,\eta)\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt,$$

where

$$p^{+,\nu}_{k,k_1}(\xi-\eta,\eta)=\tilde q_{+,\nu}(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)+\overline{\tilde q_{+,-\nu}(\xi,-\eta)}\psi_k(\xi-\eta)\psi_{k_1}(\xi). \tag{6.62}$$

From (4.10) and (4.39), we have

$$p^{+,\nu}_{k,k_1}(\xi-\eta,\eta) = p^{+,\nu,1}_{k,k_1}(\xi-\eta,\eta) + p^{+,\nu,2}_{k,k_1}(\xi-\eta,\eta) = O(1)\xi\cdot\eta + O(|\eta|^2).$$

Recall (6.16). From the above decomposition, we can decompose $p_{+,\nu}(\xi-\eta,\eta)$ into two parts as follows,

$$p^{+,\nu}_{k,k_1}(\xi-\eta,\eta) = \sum_{i=1,2} \tilde{p}^{+,\nu,i}_{k,k_1}(\xi-\eta,\eta), \quad \tilde{p}^{+,\nu,1}_{k,k_1}(\xi-\eta,\eta) = \frac{-i}{2} a_{k,k_1}(\xi)\Phi^{+,\nu}(\xi,\eta), \tag{6.63}$$

where uniquely determined symbols $a_{k,k_1}(\xi)$ and $\tilde{p}^{+,\nu,2}_{k,k_1}(\xi-\eta,\eta)$ satisfy the following estimate for some absolute constant C,

$$\|\tilde{p}^{+,\nu,2}_{k,k_1}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \leq C 2^{2k_2}, \ \|a_{k,k_1}(\xi)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \leq C. \tag{6.64}$$

Correspondingly, we decompose $\widetilde{Q}^1_{k,k_1,k_2}$ into two parts as follows,

$$\widetilde{Q}^1_{k,k_1,k_2} = \sum_{i=1,2} \widetilde{Q}^{1;i}_{k,k_1,k_2},$$
$$\widetilde{Q}^{1;i}_{k,k_1,k_2} := \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}$$
$$\times \tilde{p}^{+,\nu,i}_{k,k_1}(\xi-\eta,\eta)\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt, \quad i\in\{1,2\}.$$

From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds,

$$\sum_{|k-k_1|\leq 10, k_2\leq k_1-10} |\widetilde{Q}^{1;2}_{k,k_1,k_2}|$$
$$\leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k-k_1|\leq 10, k_2\leq k_1-10} C 2^{2k_2}\|\Gamma^1\Gamma^2 g_{k_1}(t)\ \|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}$$
$$\times\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \leq C 2^{2\tilde{\delta}m}\epsilon_0^2, \tag{6.65}$$

where C is some absolute constant. For $\widetilde{Q}^{1;1}_{k,k_1,k_2}$, we do integration by parts in time. As a result, we have

$$\widetilde{Q}^{1;1}_{k,k_1,k_2} = \sum_{i=1,2} \widehat{Q}^{1;i}_{k,k_1,k_2},$$
$$\widehat{Q}^{1;1}_{k,k_1,k_2} := \sum_{i=1,2}(-1)^{i-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g}(t_i,\xi)} e^{it_i\Phi^{+,\nu}(\xi,\eta)}\widehat{\Gamma^1\Gamma^2 g}(t_i,\xi-\eta)\frac{a_{k,k_1}(\xi)}{2}\widehat{g^\nu_{k_2}}(t_i,\eta)\,d\eta\, d\xi$$
$$+\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\frac{a_{k,k_1}(\xi)}{2}\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\partial_t\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt,$$
$$\widehat{Q}^{1;2}_{k,k_1,k_2} := \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} a_{k,k_1}(\xi)\partial_t\big(\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\big)\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt.$$

From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$|\widehat{Q}^{1;1}_{k,k_1,k_2}| \leq \sup_{t\in[2^{m-1},2^m]} C\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|\Gamma^1\Gamma^2 g_{k_1}(t)\|_{L^2}\big(\|e^{-it\Lambda}g_{k_2}\|_{L^\infty} + 2^m\|e^{-it\Lambda}\partial_t g_{k_2}(t)\|_{L^\infty}\big)$$
$$\leq C 2^{-m/2+\beta m}\epsilon_0^2. \tag{6.66}$$

Recall the estimate (7.7) in Lemma 7.2. It motivates us to do the decomposition as follows,
$\widehat{Q}^{1;2}_{k,k_1,k_2} = \widehat{Q}^{1;2,1}_{k,k_1,k_2} + \widehat{Q}^{1;2,2}_{k,k_1,k_2}$,
where

$$\begin{aligned}
\widehat{Q}^{1;2,1}_{k,k_1,k_2} &:= \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\frac{a_{k,k_1}(\xi)}{2}\widehat{g^{\nu}_{k_2}}(t,\eta)\\
&\times\Big[\overline{\big(\partial_t\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi) - \sum_{(k_1',k_2')\in\chi_k^2}\sum_{\nu'\in\{+,-\}}\widetilde{B}^{+,\nu'}_{k,k_1',k_2'}(t,\xi)\big)}\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\\
&+\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}\big(\partial_t\widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta) - \sum_{(k_1',k_2')\in\chi_{k_1}^2}\sum_{\nu'\in\{+,-\}}\widetilde{B}^{+,\nu'}_{k_1,k_1',k_2'}(t,\xi-\eta)\big)\Big]d\eta\, d\xi\, dt,
\end{aligned}$$

$$\begin{aligned}
\widehat{Q}^{1;2,2}_{k,k_1,k_2} &:= \sum_{k_2'\le k_1'-10,|k_1-k_1'|\le 10}\sum_{\nu'\in\{+,-\}}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\frac{a_{k,k_1}(\xi)}{2}\Big[\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\\
&\times\overline{e^{it\Phi^{+,\nu}(\xi,\kappa)}\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)} + \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}\widehat{g^{\nu}_{k_2}}(t,\eta)\\
&\times e^{it\Phi^{+,\nu'}(\xi-\eta,\kappa)}\tilde{q}_{+,\nu'}(\xi-\eta-\kappa,\kappa)\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\eta-\kappa)\widehat{g^{\nu'}_{k_2'}(t,\kappa)}\Big]d\eta\, d\kappa\, d\xi\, dt.
\end{aligned}$$

From estimate (7.7) in Lemma 7.2 and the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$|\widehat{Q}^{1;2,1}_{k,k_1,k_2}| \le \sup_{t\in[2^{m-1},2^m]} C2^{40\beta m}\epsilon_0\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} \le C2^{-m/2+60\beta m}\epsilon_0^2.$$

Now, we proceed to estimate $\widehat{Q}^{1;2,2}_{k,k_1,k_2}$. To utilize symmetry, we do change of variables for the second part of integration as follows $(\xi,\eta,\kappa)\longrightarrow(\xi-\kappa,\eta,-\kappa)$. As a result, we have

$$\begin{aligned}
\widehat{Q}^{1;2,2}_{k,k_1,k_2} &:= \sum_{k_2'\le k_1'-10,|k_1-k_1'|\le 10}\sum_{\nu'\in\{+,-\}}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}\\
&\times\Big[\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\frac{a_{k,k_1}(\xi)}{2}\overline{\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}(t,\kappa)}\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)}\\
&+\frac{a_{k,k_1}(\xi-\kappa)}{2}\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\kappa)}\widehat{g^{\nu}_{k_2}}(t,\eta)\tilde{q}_{+,\nu'}(\xi-\eta,-\kappa)\\
&\times\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\eta)\widehat{g^{\nu'}_{k_2'}(t,-\kappa)}\Big]d\eta\, d\kappa\, d\xi\, dt\\
&= \sum_{k_2'\le k_1'-10,|k_1-k_1'|\le 10}\sum_{\nu'\in\{+,-\}}\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}\\
&\times\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\kappa)}\widehat{g^{\nu'}_{k_2'}(t,-\kappa)}\tilde{r}^{k,k_1'}_{\nu,\nu'}(\xi,\eta,\kappa)\,d\eta\, d\kappa\, d\xi\, dt, \qquad (6.67)
\end{aligned}$$

where

$$\tilde{r}^{k,k_1'}_{\nu,\nu'}(\xi,\eta,\kappa) := a_{k,k_1}(\xi)\overline{\tilde{q}_{+,-\nu'}(\xi-\kappa,\kappa)}\psi_{k_1'}(\xi-\kappa) + a_{k,k_1}(\xi-\kappa)\tilde{q}_{+,\nu'}(\xi-\eta,-\kappa)\psi_{k_1'}(\xi-\eta),$$

$$\Phi^{+,\nu}(\xi,\eta) - \Phi^{+,\nu'}(\xi,\kappa) = -\Lambda(\xi-\eta) - \nu\Lambda(\eta) + \Lambda(\xi-\kappa) - \nu'\Lambda(\kappa).$$

Recall (4.14) and (4.15). From the Lemma 2.1, we know that the following estimate holds

$$\|\tilde{r}_{\nu,\nu'}(\xi,\eta,\kappa)\psi_{k_2'}(\kappa)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty} \leq C2^{\max\{k_2,k_2'\}+k_1}, \tag{6.68}$$

where C is some absolute constant. From (6.68), and the $L^2-L^\infty-L^\infty-L^\infty$ type multilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}|\widehat{Q}^{1;2,2}_{k,k_1,k_2}| \leq & \sup_{t\in[2^{m-1},2^m]}\sum_{k_2'\leq k_1-10} C2^{m+\max\{k_2,k_2'\}+k_1}\|\Gamma^1\Gamma^2 g_{k_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\\ &\times\|e^{-it\Lambda}g_{k_2'}(t)\|_{L^\infty}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\leq C2^{-m/2+30\beta m}\epsilon_0^2.\end{aligned}$$

Next, we estimate Q^2_{k,k_1,k_2}. Recall (6.60). From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.1, (4.14) and (4.15), the following estimate holds for some absolute constant C,

$$\begin{aligned}\Big|\sum_{k_2\leq k_1+2,|k-k_1|\leq 10} Q^2_{k,k_1,k_2}\Big| \leq & \sup_{t\in[2^{m-1},2^m]}\sum_{|k-k_1|\leq 10} C2^{m+2k_1}\|P_{\leq k_1+2}[\Gamma^1\Gamma^2 g](t)\|_{L^2}\\ &\times\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\leq C2^{2\tilde{\delta}m}\epsilon_0^2.\end{aligned} \tag{6.69}$$

Lastly, we estimate Q^3_{k,k_1,k_2}. Recall (6.61). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}|Q^{j_1,j_2,3}_{k,k_1,k_2}|\\ &\leq \sup_{t\in[2^{m-1},2^m]}\sum_{i=1,2}C2^{m+2k_1}\Big(\sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}}2^{k_1+j_1}\|g_{k_1,j_1}(t)\|_{L^2}\\ &\quad\times(\|e^{-it\Lambda}\Gamma^i g_{k_2,j_2}(t)\|_{L^\infty}+\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty})\\ &\quad+\sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}}C2^{k_2+j_2}\|g_{k_2,j_2}(t)\|_{L^2}\\ &\quad\times(\|e^{-it\Lambda}\Gamma^i g_{k_1,j_1}(t)\|_{L^\infty}+\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty})\Big)\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\\ &\leq C2^{-m+20\beta m-k_2}\epsilon_0^2\leq C2^{-\beta m}\epsilon_0^2.\end{aligned} \tag{6.70}$$

Hence finishing the proof. □

LEMMA 6.6. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant C,*

$$\sum_{k_1,k_2\in\mathbb{Z},|k-k_1|\leq 10,k_2\leq k_1-10,k_1+k_2\geq -19m/20,k_1\leq 5\beta m}|P^1_{k,k_1,k_2}|\leq C2^{2\tilde{\delta}m}\epsilon_0^2. \tag{6.71}$$

Proof. – Recall (6.7) and (6.12). Same as in the High × High type interaction, we know that the integral inside P^1_{k,k_1,k_2} vanishes if $\Gamma^l=\Omega$. Hence, we only have to consider the case when $\Gamma^l=L$. Recall (6.17). We know that similar decompositions as in (6.25) and (6.28) also hold. Recall (6.28) and (6.17). From the estimate (2.3) in Lemma 2.1, the following estimate holds for some absolute constant C,

$$\|\tilde{q}^2_{+,\nu}(\xi,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty}\leq C2^{2k_2},\quad \text{if } k_2\leq k_1-10. \tag{6.72}$$

After doing integration by parts in "η" once, the following decomposition holds,

$$|\Gamma^{1,2}_{k,k_1,k_2}| \leq |\Gamma^{1,2;1}_{k,k_1,k_2}| + |\Gamma^{1,2;2}_{k,k_1,k_2}|, \tag{6.73}$$

where

$$\begin{aligned}
\Gamma^{1,2;1}_{k,k_1,,k_2} &:= \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\nabla_\eta \\
&\quad \cdot \Big(\frac{\nabla_\eta\Phi^{+,\nu}(\xi,\eta)}{|\nabla_\eta\Phi^{+,\nu}(\xi,\eta)|^2}\tilde{q}^2_{+,\nu}(\xi-\eta,\eta)\big[\tilde{q}_{+,\nu}(\xi-\eta,\eta) \\
&\quad \times \big(\widehat{\Gamma g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta) + \widehat{g_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^\nu_{k_2}}(t,\eta)\big) \\
&\quad + (\Gamma_\xi + \Gamma_\eta + d_\Gamma)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\big]\Big) \\
&\quad - \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\nabla_\eta \\
&\quad \cdot \Big(\frac{\nabla_\eta\Phi^{+,\nu}(\xi,\eta)}{|\nabla_\eta\Phi^{+,\nu}(\xi,\eta)|^2}\tilde{q}^2_{+,\nu}(\xi-\eta,\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^\nu_{k_2}}(t,\eta)\Big)\widehat{\Gamma g_{k_1}}(t,\xi-\eta)\, d\eta\, d\xi\, dt, \\
\Gamma^{1,2;2}_{k,k_1,k_2} &:= \sum_{j_1\geq -k_{1,-},\, j_2\geq -k_{2,-}} \Gamma^{1,2;2}_{k,k_1,j_1,k_2,j_2}, \\
\Gamma^{1,2;2}_{k,k_1,j_1,k_2,j_2} &:= \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\widehat{\Gamma g_{k_1,j_1}}(t,\xi-\eta)\nabla_\eta \\
&\quad \cdot \Big(\frac{\nabla_\eta\Phi^{+,\nu}(\xi,\eta)}{|\nabla_\eta\Phi^{+,\nu}(\xi,\eta)|^2}\tilde{q}^2_{+,\nu}(\xi-\eta,\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)\Big)\, d\eta\, d\xi\, dt.
\end{aligned}$$

From the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{k_2\leq k_1-10, |k_1-k|\leq 10} |\Gamma^{1,2;1}_{k,k_1,,k_2}| \\
&\leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2\leq k_1-10, |k_1-k|\leq 10}\sum_{i=1,2} C 2^{m+2k_2}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\quad \times \big[\big(2^{2k_1}\|\nabla^2_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2}\big)\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} \\
&\quad + 2^{k_1}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\big(2^{k_2}\|\nabla^2_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} \\
&\quad + \|\nabla_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-k_2}\|g_{k_2}(t)\|_{L^2}\big) \\
&\quad + \sum_{j_1\geq j_2} 2^{-m+k_2+k_1}2^{2j_2}\|\varphi^{k_2}_{j_2}(x)g_{k_2}(t)\|_{L^2}2^{j_1}\|\varphi^{k_1}_{j_1}(x)g_{k_1}(t)\|_{L^2} \\
&\quad + \sum_{j_2\geq j_1} 2^{-m+k_2+k_1}2^{2j_1}\|\varphi^{k_2}_{j_2}(x)g_{k_2}(t)\|_{L^2}2^{j_2}\|\varphi^{k_1}_{j_1}(x)g_{k_1}(t)\|_{L^2}\big] \leq C2^{2\tilde{\delta}m}\epsilon_0^2.
\end{aligned} \tag{6.74}$$

Now, we proceed to estimate $\Gamma^{1,2;2}_{k,k_1,,k_2}$. By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_2\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} \big|\Gamma^{1,2;2}_{k,k_1,j_1,k_2,j_2}\big|$$

$$
\begin{aligned}
&\leq \sup_{t\in[2^{m-1},2^m]} C 2^{m+2k_2+k_1}\Big[\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} 2^{k_1+j_1}\|g_{k_1,j_1}(t)\|_{L^2}\\
&\quad\times\big(2^{-k_2}\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty}+\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi g_{k_2,j_2}(t)]\|_{L^\infty}\big)\\
&\quad+\sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} 2^{j_2}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}\Gamma g_{k_1,j_1}(t)\|_{L^\infty}\Big]\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}
\end{aligned}
$$

$$
\leq C2^{-m/2+20\beta m}\epsilon_0^2. \tag{6.75}
$$

Recall the estimates (6.25) and (6.73). From the estimates (6.74), (6.75) and the estimate (6.76) in Lemma 6.7, we know that our desired estimate (6.71) holds. □

LEMMA 6.7. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k_1 + k_2 \geq -19m/20$ *and* $k_1 \leq 5\beta m$, *the following estimate holds for some absolute constant* C,

$$
|\Gamma^{1,1}_{k,k_1,k_2}| \leq C2^{-\beta m}\epsilon_0^2. \tag{6.76}
$$

Proof. – Same as in the High × High interaction, we do integration by parts in time once. As a result, we have the same formulations as in (6.34), (6.35) and (6.36).

We first estimate $\widetilde{\Gamma}^{1,1}_{k,k_1,k_2}$. Recall (6.34). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m+k_{1,-}-\beta m$. If $\max\{j_1, j_2\} \geq m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} &|\widetilde{\Gamma}^{j_1,j_2,1,1}_{k,k_1,k_2}|\\
&\leq \sup_{t\in[2^{m-1},2^m]} C2^{m+2k_1}\|\Gamma^1\Gamma^2 g_k\|_{L^2}\\
&\quad\times\Big[\sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}} 2^{k_1+j_1}\|g_{k_1,j_1}(t)\|_{L^2}\big(\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty}\\
&\quad+2^{k_2}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi \widehat{g}_{k_2,j_2}(t)]\|_{L^\infty}\big)\\
&\quad+\sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}} 2^{k_2+j_2}\|g_{k_2,j_2}(t)\|_{L^2}(\|e^{-it\Lambda}g_{k_1,j_1}\|_{L^\infty}\\
&\quad+2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi \widehat{g}_{k_1,j_1}(t)]\|_{L^\infty})\Big]\\
&\leq C2^{-m-k_2+20\beta m}\epsilon_0^2 \leq C2^{-\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2}_{k,k_1,k_2}$. Recall (6.36). Since now k_1 and k_2 are not comparable, different from the decomposition we did in (6.37) in the High × High type interaction, we do decomposition as follows,

$$
\widetilde{\Gamma}^{1,2}_{k,k_1,k_2} = \sum_{i=1,\ldots,7}\widetilde{\Gamma}^{1,2;i}_{k,k_1,k_2} \tag{6.77}
$$

$$
\widetilde{\Gamma}^{1,2;2}_{k,k_1,k_2} = \sum_{k_2'\leq k_1'+10}\widehat{\Gamma}^{k_1',k_2',1}_{k,k_1,k_2},\quad \widehat{\Gamma}^{k_1',k_2',1}_{k,k_1,k_2} = \sum_{j_2\geq -k_{2,-}, j_1'\geq -k_{1,-}', j_2'\geq -k_{2,-}'}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',1}_{k,k_1,k_2,j_2}, \tag{6.78}
$$

$$
\widetilde{\Gamma}^{1,2;3}_{k,k_1,k_2} = \sum_{k_2'\leq k_1'+10}\widehat{\Gamma}^{k_1',k_2',2}_{k,k_1,k_2},\quad \widehat{\Gamma}^{k_1',k_2',2}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-}, j_1'\geq -k_{1,-}', j_2'\geq -k_{2,-}'}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',2}_{k,k_1,j_1,k_2}, \tag{6.79}
$$

$$\widetilde{\Gamma}^{1,2;i}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} \widetilde{\Gamma}^{1,2;i}_{k,k_1,j_1,k_2,j_2}, \quad i\in\{4,5\},$$

$$\widetilde{\Gamma}^{1,2;1}_{k,k_1,k_2} := -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)t\partial_t\widehat{g^{\nu}_{k_2}}(t,\eta)$$
$$\times\big(\tilde{q}_{+,\nu}(\xi,\eta)\widehat{\Gamma g_{k_1}}(t,\xi-\eta) + (\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{+,\nu}(\xi,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\big)d\eta d\xi dt,$$

which results from the case when ∂_t hits the input "$\widehat{g_{k_2}}(t,\xi-\eta)$" in (6.36).

$$\widehat{\Gamma}^{k_1',j_1',k_2',j_2',1}_{k,k_1,k_2,j_2} := \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}t\tilde{c}(\xi-\eta)e^{it\Phi^{\mu',\nu'}(\xi-\eta,\sigma)}$$
$$\times\tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\psi_{k_1}(\xi-\eta)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)$$
$$\times\big(\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t,\eta) + (\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\big)d\eta d\xi dt, \tag{6.80}$$

which is resulted from the quartic terms when ∂_t hits the input "$\widehat{g_{k_1}}(t,\xi-\eta)$" in (6.36).

$$\widehat{\Gamma}^{k_1',j_1',k_2',j_2',2}_{k,k_1,j_1,k_2} := \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)t\tilde{q}_{+,\nu}(\xi-\eta,\eta)$$
$$\times\widehat{g_{k_1,j_1}}(t,\xi-\eta)e^{it\Phi^{\mu',\nu'}(\eta,\sigma)}\big[\Gamma_\eta\big(\tilde{q}_{\mu',\nu'}(\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\eta-\sigma)\big)$$
$$+it\Gamma_\eta\Phi^{\mu',\nu'}(\eta,\sigma)\tilde{q}_{\mu',\nu'}(\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\eta-\sigma)\big]$$
$$\times\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)d\sigma d\eta d\xi dt, \tag{6.81}$$

which is resulted from the quartic terms when ∂_t hits the input "$\widehat{\Gamma g_{k_2}}(t,\eta)$" in (6.36).

$$\widetilde{\Gamma}^{1,2;4}_{k,k_1,j_1,k_2,j_2} := -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}t\tilde{c}(\xi-\eta) \tag{6.82}$$
$$\times\big[\big(\Gamma\widehat{\Lambda_{\geq 3}[\partial_t g]}_{k_1,j_1}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta) + \widehat{g_{k_1,j_1}}(t,\xi-\eta)\Gamma\widehat{\Lambda_{\geq 3}[\partial_t g^{\nu}]}_{k_2,j_2}(t,\eta)$$
$$+\widehat{\Lambda_{\geq 3}[\partial_t g]}_{k_1,j_1}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t,\eta)\big)\tilde{q}_{+,\nu}(\xi-\eta,\eta)$$
$$+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Lambda_{\geq 3}[\partial_t g]}_{k_1,j_1}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\big)d\eta d\xi dt, \tag{6.83}$$

which is resulted from the quintic and higher order terms when ∂_t hits the inputs "$g_{k_1}(t)$," "$\Gamma g_{k_1}(t)$," and "$\Gamma g_{k_2}(t)$" in (6.36).

$$\widetilde{\Gamma}^{1,2;5}_{k,k_1,j_1,k_2,j_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}t\tilde{c}(\xi-\eta)$$
$$\times\overline{\big(\partial_t\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi) - \sum_{\nu\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2}\widetilde{B}^{+,\nu}_{k,k_1',k_2'}(t,\xi)\big)}$$
$$\times\big(\tilde{q}_{+,\nu}(\xi-\eta,\eta)(\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t,\eta) + \widehat{\Gamma g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta))$$
$$+(\Gamma_\xi+\Gamma_\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\big)d\eta d\xi dt, \tag{6.84}$$

which is resulted from the good error terms when ∂_t hits "$\Gamma^1\Gamma^2 g_k(t)$" in (6.36).

$$\widetilde{\Gamma}^{1,2;6}_{k,k_1,k_2} = \sum_{|k_1'-k_2'|\le 10} \widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;1} + \sum_{k_2'\le k_1'-10} \widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;2}, \tag{6.85}$$

$$\widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;i} = \sum_{j_1'\ge -k'_{1,-}, j_2'\ge -k'_{2,-}, j_2\ge -k_{2,-}} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;i}, \quad i\in\{1,2\}, \tag{6.86}$$

$$\begin{aligned}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;1} &:= \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta) \\ &\quad \times t\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\psi_{k_1}(\xi-\eta)e^{it\Phi^{\mu',\nu'}(\xi-\eta,\sigma)}\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma) \\ &\quad \times \Big[it\Gamma_{\xi-\eta}\Phi^{\mu',\nu'}(\xi-\eta,\sigma)\tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma) \\ &\quad + \Gamma_{\xi-\eta}\big(\tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\big)\Big] d\sigma d\eta d\xi dt,\end{aligned} \tag{6.87}$$

which is resulted from the quartic terms when ∂_t hits the input "$\widehat{\Gamma g_{k_1}}(t,\xi-\eta)$" in (6.36) and moreover two inputs inside $\Lambda_2[\partial_t\widehat{\Gamma g_{k_1}}(t,\xi-\eta)]$ have comparable sizes of frequencies, see (6.85).

$$\begin{aligned}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;2} &:= \sum_{\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)t\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta) \\ &\times\psi_{k_1}(\xi-\eta)e^{it\Phi^{+,\nu'}(\xi-\eta,\sigma)}\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)\Big[it\Gamma_{\xi-\eta}\Phi^{+,\nu'}(\xi-\eta,\sigma)\tilde{q}_{+,\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g_{k_1',j_1'}}(t,\xi-\eta-\sigma) \\ &+\Gamma_{\xi-\eta}\big(\tilde{q}_{+,\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g_{k_1',j_1'}}(t,\xi-\eta-\sigma)\big)-\widehat{\Gamma g_{k_1',j_1'}}(t,\xi-\eta-\sigma)\tilde{q}_{+,\nu'}(\xi-\eta-\sigma,\sigma)\Big]d\sigma d\eta d\xi dt,\end{aligned} \tag{6.88}$$

which is resulted from the quartic terms when ∂_t hits the input "$\widehat{\Gamma g_{k_1}}(t,\xi-\eta)$" in (6.36) and moreover two inputs inside $\Lambda_2[\partial_t\widehat{\Gamma g_{k_1}}(t,\xi-\eta)]$ have different size of frequencies (see (6.85)) and the bulk term of this scenario is removed.

$$\begin{aligned}\widetilde{\Gamma}^{1,2;7}_{k,k_1,k_2} &= \sum_{k_2'\le k_1'-10, |k_1-k_1'|\le 10}\ \sum_{\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)} t r^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa) \\ &\quad \times \widehat{\Gamma g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\kappa)}\widehat{g^{\nu'}_{k_2'}}(t,-\kappa)d\kappa d\eta d\xi dt,\end{aligned} \tag{6.89}$$

which is resulted from putting the bulk term inside "$\Lambda_2[\partial_t\widehat{\Gamma g_{k_1}}(t,\xi-\eta)]$" and the bulk term inside "$\Lambda_2[\partial_t\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)]$" together, and the symbol $r^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)$ is given as follows,

$$\begin{aligned} r^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa) &= \tilde{c}(\xi-\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\overline{\tilde{q}_{+,-\nu'}(\xi-\kappa,\kappa)}\psi_{k_1'}(\xi-\kappa)\psi_{k_1}(\xi-\eta)\psi_k(\xi) \\ &\quad + \tilde{c}(\xi-\eta-\kappa)\tilde{q}_{+,\nu}(\xi-\kappa-\eta,\eta)\tilde{q}_{+,\nu'}(\xi-\eta,-\kappa)\psi_{k_1'}(\xi-\eta)\psi_k(\xi-\kappa)\psi_{k_1}(\xi-\eta-\kappa).\end{aligned}$$

Recall (4.14) and (4.15). From the estimate (2.3) in Lemma 2.1, the following estimate holds for some absolute constant C,

$$\|r^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\psi_{k_2}(\eta)\psi_{k_2'}(\kappa)\|_{\mathcal{S}^\infty} \le C2^{\max\{k_2,k_2'\}+3k_1}. \tag{6.90}$$

With the above preparation of classifying all terms inside $\widetilde{\Gamma}^{1,2}_{k,k_1,k_2}$, see the decomposition (6.77). Now, we are ready to estimate them one by one. From estimate (7.2) in Lemma 7.1 and the L^2-L^∞ type bilinear estimate, we have

$$\begin{aligned}|\widetilde{\Gamma}^{1,2;1}_{k,k_1,k_2}| &\leq \sup_{t\in[2^{m-1},2^m]}\sum_{i=1,2} C2^{2m+2k_1}\|\partial_t\widehat{g_{k_2}}(t,\xi)\|_{L^2}\\ &\quad\times\big(\|e^{-it\Lambda}\Gamma^i g_{k_1}(t)\|_{L^\infty}+\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\big)\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\\ &\leq C2^{m+2\tilde{\delta}m}(2^{-21m/20}+2^{-2m-k_2+2\tilde{\delta}m})\epsilon_0^2\leq C2^{-\beta m}\epsilon_0^2,\end{aligned}$$

where C is some absolute constant. Now, we proceed to estimate $\widetilde{\Gamma}^{1,2;2}_{k,k_1,k_2}$. Recall (6.78) and (6.80). We split into two cases as follows based on the size of difference between k_1' and k_2'.

If $|k_1'-k_2'|\leq 5$. – Note that $k_1'\geq k_1-5\geq k_2+5$. By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1',j_2\}\leq m+k'_{1,-}-\beta m$. Hence, it would be sufficient to consider the case when $\max\{j_1',j_2\}\geq m+k'_{1,-}-\beta m$. From the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}&\sum_{|k_1'-k_2'|\leq 5}\Big|\sum_{\max\{j_1',j_2\}\geq m+k'_{1,-}-\beta m}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',1}_{k,k_1,k_2,j_2}\Big|\\ &\quad\leq \sup_{t\in[2^{m-1},2^m]}\sum_{|k_1'-k_2'|\leq 5} C2^{2m+2k_1+2k_1'}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\\ &\quad\times\Big[\sum_{j_1'\geq\max\{j_2,m+k'_{1,-}-\beta m\}}\|g_{k_1',j_1'}\|_{L^2}\|e^{-it\Lambda}g_{k_2'}\|_{L^\infty}\\ &\quad\times(\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty}+\|e^{-it\Lambda}\Gamma^n g_{k_2,j_2}\|_{L^\infty})\\ &\quad+\sum_{j_2\geq\max\{j_1',m+k'_{1,-}-\beta m\}}\|e^{-it\Lambda}g_{k_1',j_1'}\|_{L^\infty}2^{k_2+j_2}\|g_{k_2,j_2}\|_{L^2}\|e^{-it\Lambda}g_{k_2'}(t)\|_{L^\infty}\Big]\\ &\quad\leq C2^{-m-k_2+20\beta m}\epsilon_0^2\leq C2^{-2\beta m}\epsilon_0^2.\end{aligned}$$

If $k_2'\leq k_1'-5$. – For this case we have $|k_1-k_1'|\leq 2$ and $k_1'\geq k_2+5$. If moreover $k_1+k_2'\leq -9m/10$, then from estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\begin{aligned}&\sum_{k_2'\leq\min\{-9m/10-k_1,k_1-10\}}|\widehat{\Gamma}^{k_1',k_2',1}_{k,k_1,k_2}|\\ &\quad\leq\sup_{t\in[2^{m-1},2^m]}\sum_{k_2'\leq\min\{-9m/10-k_1,k_1-10\}} C2^{2m+3k_1}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\\ &\quad\times\|e^{-it\Lambda}g_{k_1'}\|_{L^\infty}\big(\|g_{k_2}\|_{L^2}+\|\Gamma^n g_{k_2}\|_{L^2}\big)\\ &\quad\times\big(2^{3k_2'}\|\widehat{g}_{k_2'}(t,\xi)\|_{L^\infty_\xi}+2^{k_1+2k_2'}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2'}(\xi)\|_{L^\infty_\xi}\big)\\ &\quad\leq C2^{-2\beta m}\epsilon_0^2.\end{aligned}$$

Lastly, if $k_1+k_2'\geq -9m/10$, we can do integration by part in "σ" many times to rule out the case when $\max\{j_1',j_2'\}\leq m+k_{1,-}-\beta m$. Also, by doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1', j_2\} \leq m + k_{1,-} - \beta m$. Hence, it would be sufficient to consider the case when $\max\{j_1', j_2\} \geq m + k_{1,-} - \beta m$ and $\max\{j_1', j_2'\} \geq m + k_{1,-} - \beta m$, which implies that one of the following two cases must holds: (i) $j_1' \geq m + k_{1,-} - \beta m$; (ii) $j_1' \leq m + k_{1,-} - \beta m$ and $j_2, j_2' \geq m + k_{1,-} - \beta m$. From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\Big|\sum_{\max\{j_1',j_2'\},\max\{j_1',j_2\}\geq m+k_{1,-}-\beta m} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',1}_{k,k_1,k_2,j_2}\Big| \\
&\leq \sup_{t\in[2^{m-1},2^m]} C2^{2m+4k_1}\Big[\sum_{j_1'\geq m+k_{1,-}-\beta m} \|g_{k_1',j_1'}\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}\|_{L^\infty} \\
&\quad\times\big(\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty} + \|e^{-it\Lambda}\Gamma g_{k_2,j_2}\|_{L^\infty}\big) \\
&\quad+\sum_{j_2',j_2\geq m+k_{1,-}-\beta m} 2^{2k_2+j_2}\|g_{k_2,j_2}\|_{L^2}\|g_{k_2',j_2'}\|_{L^2}\|e^{-it\Lambda}g_{k_1',j_1'}\|_{L^\infty}\Big]\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\leq C2^{-m-k_2'+20\beta m}\epsilon_0^2 \leq C2^{-2\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2;3}_{k,k_1,k_2}$. Recall (6.79) and (6.81). We separate into two cases as follows based on the size of difference between k_1' and k_2'.

If $|k_1'-k_2'| \leq 10$. – Note that $k_1' \geq k_2 - 5$. By doing integration by parts in "σ," we can rule out the case when $\max\{j_1', j_2'\} \leq m + k_{2,-} - k_{1,+}' - \beta m$. If $\max\{j_1', j_2'\} \geq m + k_{2,-} - k_{1,+}' - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{|k_1'-k_2'|\leq 10}\Big|\sum_{\max\{j_1',j_2'\}\geq m+k_{2,-}-k_{1,+}'-\beta m} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',2}_{k,k_1,j_1,k_2}\Big| \\
&\quad\leq \sup_{t\in[2^{m-1},2^m]}\sum_{|k_1'-k_2'|\leq 10} C2^{2m+2k+2k_1'}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\quad\times\Big(\sum_{j_1'\geq\{j_2',m+k_{2,-}-k_{1,+}'-\beta m\}} 2^{k_1'}(2^{j_1'}+2^{m+k_2})\|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}(t)\|_{L^\infty} \\
&\quad\quad+\sum_{j_2'\geq\{j_1',m+k_{2,-}-k_{1,+}'-\beta m\}} 2^{k_1'}(2^{j_1'}+2^{m+k_2})\|g_{k_2',j_2'}(t)\|_{L^2}2^{-m}\|g_{k_1',j_1'}(t)\|_{L^1}\Big) \\
&\quad\times\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty} \leq C2^{-m-k_2+20\beta m}\epsilon_0^2 \leq C2^{-2\beta m}\epsilon_0^2.
\end{aligned}
$$

If $k_2' \leq k_1' - 10$. – For this case, we have $k_2 - 2 \leq k_1' \leq k_2 + 2 \leq k_1 - 5$. By doing integration by parts in "η," we can rule out the case when $\max\{j_1, j_1'\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_1'\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\sum_{k_2'\leq k_1'-10}\Big|\sum_{\max\{j_1,j_1'\}\geq m+k_{1,-}-\beta m} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',2}_{k,k_1,j_1,k_2}\Big|
$$

$$
\begin{aligned}
&\le \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\le k_1'-10} C2^{2m+2k+2k_1'}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2'}(t)\|_{L^\infty}\\
&\quad\times\Big(\sum_{j_1\ge\max\{j_1',m+k_{1,-}-\beta m\}}(2^{k_1'+j_1'}+2^{m+k_2+k_1'})\|g_{k_1,j_1}(t)\|_{L^2}2^{-m}\|g_{k_1',j_1'}(t)\|_{L^1}\\
&\quad+\sum_{j_1'\ge\max\{j_1,m+k_{1,-}-\beta m\}}(2^{k_1'+j_1'}+2^{m+k_2+k_1'})\|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\Big)\\
&\le C2^{-m/2+20\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2;4}_{k,k_1,k_2}$ and $\widetilde{\Gamma}^{1,2;5}_{k,k_1,k_2}$. Recall (6.77), (6.83), and (6.84). By doing integration by parts in "η," we can rule out the case when $\max\{j_1,j_2\}\le m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\ge m+k_{1,-}-\beta m$, from estimate (7.7) in Lemma 7.2, (6.137) in Lemma 6.14, and the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=4,5}\sum_{\max\{j_1,j_2\}\ge m+k_{1,-}-\beta m}|\widetilde{\Gamma}^{1,2;i}_{k,k_1,j_1,k_2,j_2}|\\
&\quad\le \sup_{t\in[2^{m-1},2^m]} C2^{m+2k_1+\beta m}\Big[\sum_{j_1\ge\max\{j_2,m+k_{1,-}-\beta m\}}2^{k_1+j_1}\\
&\qquad\times(\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}+\|e^{-it\Lambda}\Gamma g_{k_2,j_2}(t)\|_{L^\infty})\\
&\qquad\times\big(2^{6k_+}\|g_{k_1,j_1}(t)\|_{L^2}+2^m\|\Lambda_{\ge3}[\partial_t g(t)]_{k_1,j_1}\|_{L^2}\big)\\
&\quad+2^{m+k_2}\|g_{k_1,j_1}\|_{L^2}2^{k_2+j_2}\|\Lambda_{\ge3}[\partial_t g(t)]_{k_2,j_2}\|_{L^2}\\
&\quad+\sum_{j_2\ge\max\{j_1,m+k_{1,-}-\beta m\}}2^{k_1+j_1}\big(2^{6k_+}\|g_{k_1,j_1}\|_{L^2}+2^m\|\Lambda_{\ge3}[\partial_t g(t)]_{k_1,j_1}\|_{L^2}\big)2^{k_2}\|g_{k_2,j_2}\|_{L^2}\\
&\quad+2^{k_2+j_2}\big(2^{6k_+}\|g_{k_2,j_2}\|_{L^2}+2^m\|\Lambda_{\ge3}[\partial_t g(t)]_{k_2,j_2}\|_{L^2}\big)\\
&\quad\times\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\Big]\le C2^{-m+40\beta m-k_2}\epsilon_0^2\le C2^{-2\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2;6}_{k,k_1,k_2}$. Recall (6.85) and (6.86). We split into three cases based on the difference between k_1' and k_2' and the size of $k_1'+k_2'$.

If $|k_1'-k_2'|\le 10$, i.e., we are estimating $\widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;1}$. – Note that we have $k_1'\ge k_1-5$. Recall (6.87). By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1',j_2'\}\le m+k_{1,-}-k_{1,+}'-\beta m$. If $\max\{j_1',j_2'\}\ge m+k_{1,-}-k_{1,+}'-\beta m$, from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\Big|\sum_{\max\{j_1',j_2'\}\ge m+k_{1,-}-k_{1,+}'-\beta m}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;1}\Big|\\
&\quad\le \sup_{t\in[2^{m-1},2^m]} C2^{2m+2k+2k_1'}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\\
&\qquad\times\Big(\sum_{j_1'\ge\max\{j_2',m+k_{1,-}-k_{1,+}'-\beta m\}}(2^{m+k_1+k_2'}+2^{k_2'+j_1'})\|g_{k_1',j_1'}\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}\|_{L^\infty}
\end{aligned}
$$

$$+ \sum_{j_2'\geq \max\{j_1', m+k_{1,-}-k_{1,+}'-\beta m\}} (2^{m+k_1+k_2'}+2^{k_2'+j_1'})\|g_{k_2',j_2'}\|_{L^2}2^{-m}\|g_{k_1',j_1'}\|_{L^1}\Big) \leq C2^{-m/2+20\beta m}\epsilon_0^2.$$

$\oplus$ If $k_2'\leq k_1'-10$ and $k_1'+k_2'\leq -19m/20$. For this case, we have $|k_1'-k_1|\leq 5$. Recall (6.85). From estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\begin{aligned}|\widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;2}| \leq &\sup_{t\in[2^{m-1},2^m]} C2^{2m+3k_1}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\\ &\times\Big((2^{m+k_2'+k_1}+1)\|g_{k_1'}\|_{L^2}+\sum_{i=1,2}2^{k_2'}\|\nabla_\xi\widehat{g}_{k_1'}(t,\xi)\|_{L^2}\Big)\\ &\times\Big(2^{3k_2'}\|\widehat{g}_{k_2'}(t)\|_{L^\infty_\xi}+2^{k_1+2k_2'}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2'}(\xi)\|_{L^\infty_\xi}\Big)\\ \leq &C2^{-2\beta m}\epsilon_0^2.\end{aligned}$$

If $k_2'\leq k_1'-10$ *and* $k_1'+k_2'\geq -19m/20$. – Recall (6.88). By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1',j_2'\}\leq m+k_{1,-}-\beta m$. By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1',j_2\}\geq m+k_{1,-}-\beta m$. Therefore, we only need to consider the case when $\max\{j_1',j_2'\}\geq m+k_{1,-}-\beta m$ and $\max\{j_1',j_2\}\geq m+k_{1,-}-\beta m$. In other words, either $j_1'\geq m+k_{1,-}-\beta m$ or $j_2',j_2\geq m+k_{1,-}-\beta m$. From the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds,

$$\begin{aligned}&\Big|\sum_{\max\{j_1',j_2'\},\max\{j_1',j_2\}\geq m+k_{1,-}-\beta m}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;2}\Big|\\ &\leq \sup_{t\in[2^{m-1},2^m]} C2^{2m+4k}\Big(\sum_{j_1'\geq m+k_{1,-}-\beta m}(2^{m+k_1+k_2'}+2^{k_2'+j_1'})\\ &\quad\times\|g_{k_1',j_1'}\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}\|_{L^\infty}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\\ &\quad+\sum_{j_2',j_2\geq m+k_{1,-}-\beta m}(2^{m+k_1+k_2'}\|e^{-it\Lambda}g_{k_1',j_1'}\|_{L^\infty}+2^{k_2'}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g_{k_1',j_1'}}]\|_{L^\infty})\\ &\quad\times\|g_{k_2',j_2'}\|_{L^2}2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2}\Big)\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\\ &\leq C2^{-2\beta m}\epsilon_0^2.\end{aligned}$$

Lastly, we estimate $\widetilde{\Gamma}^{1,2;7}_{k,k_1,k_2}$. Recall (6.89). After doing spatial localizations for inputs "Γg_{k_1}" and "g_{k_2}" inside $\widetilde{\Gamma}^{1,2;7}_{k,k_1,k_2}$, we have

$$\widetilde{\Gamma}^{1,2;7}_{k,k_1,k_2}=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}}\widetilde{\Gamma}^{1,2;7}_{k,k_1,j_1,k_2,j_2}, \tag{6.91}$$

$$\widetilde{\Gamma}^{1,2;7}_{k,k_1,j_1,k_2,j_2}:=\sum_{k_2'\leq k_1'-10,|k_1-k_1'|\leq 10}\sum_{\nu'\in\{+,-\}}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}tr^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)$$

$$\times\widehat{\Gamma g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)}\widehat{g^{\nu'}_{k_2'}}(t,-\kappa)\,d\eta\, d\kappa\, d\xi\, dt. \tag{6.92}$$

By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m+k_{1,-}-\beta m$. If $\max\{j_1, j_2\} \geq m+k_{1,-}-\beta m$, from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and (6.90), the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} |\widetilde{\Gamma}^{1,2;7}_{k,k_1,j_1,k_2,j_2}| \\
&\leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\leq k_1-10} C2^{2m+\max\{k_2,k_2'\}+3k_1} \|e^{-it\Lambda} g_{k_2'}(t)\|_{L^\infty} \\
&\quad\times \Big(\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} \|\Gamma g_{k_1,j_1}(t)\|_{L^2} \|e^{-it\Lambda} g_{k_2,j_2}(t)\|_{L^\infty} \\
&\quad+ \sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} \|e^{-it\Lambda}\Gamma g_{k_1,j_1}(t)\|_{L^\infty} \|g_{k_2,j_2}(t)\|_{L^2}\Big) \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\leq C2^{-m/4+20\beta m}\epsilon_1^3 + C2^{-m-k_2+20\beta m}\epsilon_0^2 \leq C2^{-2\beta m}\epsilon_0^2.
\end{aligned}$$

Hence finishing the proof. □

LEMMA 6.8. – *Under the bootstrap assumption* (4.49), *the following estimate holds for some absolute constant C,*

$$\sum_{k_1,k_2\in\mathbb{Z}, |k-k_1|\leq 10, k_2\leq k_1-10, k_1+k_2\geq -19m/20, k_1\leq 5\beta m} |P^2_{k,k_1,k_2}| \leq C2^{2\tilde{\delta}m}\epsilon_0^2. \tag{6.93}$$

Proof. – Recall (6.8) and (6.12). Note that P^2_{k,k_1,k_2} vanishes except when $\Gamma^1 = \Gamma^2 = L$. Hence, we only have to consider the case when $\Gamma^1 = \Gamma^2 = L$. We decompose it into two parts as follows,

$$\begin{aligned}
P^2_{k,k_1,k_2} = \sum_{i=1,2} P^{2,i}_{k,k_1,k_2}, \quad P^{2,i}_{k,k_1,k_2} &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{LLg_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} t^2 \widehat{q}^{\,i}_{+,\nu}(\xi,\eta) \\
&\quad\times \widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt, \quad i\in\{1,2\},
\end{aligned}$$

where $\widehat{q}^{\,i}_{+,\nu}(\xi-\eta,\eta)$, $i\in\{1,2\}$, are defined in (6.29) and (6.30). After doing integration by parts in "η" twice, from the estimate of the symbol $\widehat{q}^{\,2}_{+,\nu}(\xi-\eta,\eta)$ in (6.31), the following estimate holds,

$$\begin{aligned}
&\sum_{k_2\leq k_1-10, |k_1-k|\leq 10} \big|P^{2,2}_{k,k_1,,k_2}\big| \leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2\leq k_1-10, i=1,2} 2^{m+k_1+3k_2+k_{1,+}} \\
&\quad\big[\|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty} \big(\|\nabla^2_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{-k_2}\|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2}\big) \\
&\quad+ \|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty} \big(\|\nabla^2_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-k_2}\|\nabla_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-2k_2}\|g_{k_2}(t)\|_{L^2}\big) \\
&\quad+ \sum_{j_1\geq j_2} 2^{-m+2j_2+j_1} \|\varphi^{k_2}_{j_2}(x) g_{k_2}(t)\|_{L^2} \|\varphi^{k_1}_{j_1}(x) g_{k_1}(t)\|_{L^2} \\
&\quad+ \sum_{j_2\geq j_1} 2^{-m+2j_1+j_2} \|\varphi^{k_1}_{j_1}(x) g_{k_1}(t)\|_{L^2} \|\varphi^{k_2}_{j_2}(x) g_{k_2}(t)\|_{L^2}\big] \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\leq C2^{2\tilde{\delta}m}\epsilon_0^2.
\end{aligned}$$

For "$P^{2,1}_{k,k_1,k_2}$," we do integration by parts in time once. As a result, we have

(6.94)
$$P^{2,1}_{k,k_1,k_2} = \sum_{i=1,2,3,4,5} \widetilde{P}^{i}_{k,k_1,k_2}, \quad \widetilde{P}^{1}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} \widetilde{P}^{j_1,j_2,1}_{k,k_1,k_2},$$

$$\begin{aligned}\widetilde{P}^{j_1,j_2,1}_{k,k_1,k_2} = {} & \sum_{i=1,2}(-1)^i \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t_i,\xi)} e^{it_i\Phi^{+,\nu}(\xi,\eta)} i t_i^2 \widehat{p}^{1}_{+,\nu}(\xi,\eta) \\ & \times \widehat{g_{k_1,j_1}}(t_i,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t_i,\eta)\, d\eta\, d\xi \\ & - \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} i 2t\, \widehat{p}^{1}_{+,\nu}(\xi,\eta) \\ & \times \widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta) - e^{it\Phi^{+,\nu}(\xi,\eta)} i t^2 \widehat{p}^{1}_{+,\nu}(\xi,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta) \\ & \times \widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\overline{\big(\partial_t \widehat{\Gamma^1\Gamma^2 g_k}(t,\xi) - \sum_{\nu\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2} \widetilde{B}^{+,\nu}_{k,k_1',k_2'}(t,\xi)\big)}\, d\eta\, d\xi\, dt,\end{aligned} \tag{6.95}$$

(6.96)
$$\widetilde{P}^{2}_{k,k_1,k_2} = \sum_{k_2'\leq k_1'+10} \widehat{P}^{2,k_1',k_2'}_{k,k_1,k_2}, \quad \widehat{P}^{2,k_1',k_2'}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-}, j_1'\geq -k_{1,-}', j_2'\geq -k_{2,-}'} \widehat{P}^{2,k_1',j_1',k_2',j_2'}_{k,k_1,j_1,k_2},$$

$$\begin{aligned}\widehat{P}^{2,k_1',j_1',k_2',j_2'}_{k,k_1,j_1,k_2} := {} & \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} i t^2 \widehat{p}^{1}_{+,\nu}(\xi,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta) \\ & \times P_{\nu}\big[e^{it\Phi^{\mu',\nu'}(\eta,\sigma)}\tilde{q}_{\mu',\nu'}(\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)\big]\, d\eta\, d\xi\, dt,\end{aligned} \tag{6.97}$$

(6.98)
$$\widetilde{P}^{3}_{k,k_1,k_2} = \sum_{|k_1'-k_2'|\leq 10} \widehat{P}^{3,k_1',k_2'}_{k,k_1,k_2}, \quad \widehat{P}^{3,k_1',k_2'}_{k,k_1,k_2} = \sum_{j_1'\geq -k_{1,-}', j_2'\geq -k_{2,-}'} \widehat{P}^{3,k_1',j_1',k_2',j_2'}_{k,k_1,k_2},$$

$$\begin{aligned}\widehat{P}^{3,k_1',j_1',k_2',j_2'}_{k,k_1,k_2} = {} & \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} i t^2 \widehat{p}^{1}_{+,\nu}(\xi,\eta) e^{it\Phi^{\mu',\nu'}(\xi-\eta,\sigma)} \\ & \times \tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)\widehat{g^{\nu}_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt.\end{aligned} \tag{6.99}$$

$$\widetilde{P}^{4}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} \widetilde{P}^{4}_{k,k_1,j_1,k_2,j_2},$$

(6.100)
$$\begin{aligned}\widetilde{P}^{4}_{k,k_1,j_1,k_2,j_2} := {} & -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} i t^2 \widehat{p}^{1}_{+,\nu}(\xi,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} \\ & \times \big(\Lambda_{\geq 3}\widehat{[\partial_t g]}_{k_1,j_1}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta) + \widehat{g_{k_1,j_1}}(t,\xi-\eta)\Lambda_{\geq 3}\widehat{[\partial_t g^{\nu}]}_{k_2,j_2}(t,\eta)\big)\, d\eta\, d\xi\, dt,\end{aligned}$$

(6.101)
$$\begin{aligned}\widetilde{P}^{5}_{k,k_1,k_2} = {} & \sum_{k_2'\leq k_1'-10, |k_1-k_1'|\leq 10}\ \sum_{\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} i t^2 \widehat{p}^{1}_{+,\nu}(\xi,\eta) \\ & \Big[\widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\overline{e^{it\Phi^{+,\nu'}(\xi,\kappa)}\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)}\end{aligned}$$

$$
\begin{aligned}
&+ \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\widehat{g^{\nu}_{k_2}}(t,\eta)e^{it\Phi^{+,\nu'}(\xi-\eta,\kappa)}\tilde{q}_{+,\nu'}(\xi-\eta-\kappa,\kappa)\\
&\times \widehat{g_{k_1'}}(t,\xi-\eta-\kappa)\widehat{g^{\nu'}_{k_2'}(t,\kappa)}\Big]d\eta d\kappa d\xi dt. \qquad (6.102)\\
&= \sum_{k_2'\leq k_1'-10, |k_1-k_1'|\leq 10}\sum_{\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}\\
&\times it^2\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\widehat{g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\kappa)}\widehat{g^{\nu'}_{k_2'}(t,-\kappa)}d\eta d\kappa d\xi dt,
\end{aligned}
$$

where the symbol "$\widehat{p}^1_{\mu,\nu}(\xi,\eta)$" is defined in (6.29) and the symbol $\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)$ is defined as follows,

$$
\begin{aligned}
\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa) &= \widehat{p}^1_{+,\nu}(\xi,\eta)\overline{\tilde{q}_{+,-\nu'}(\xi-\kappa,\kappa)}\psi_{k_1'}(\xi-\kappa)\psi_{k_1}(\xi-\eta)\psi_k(\xi)\\
&\quad + \widehat{p}^1_{+,\nu}(\xi-\kappa,\eta)\tilde{q}_{+,\nu'}(\xi-\eta,-\kappa)\psi_{k_1'}(\xi-\eta)\psi_{k_1}(\xi-\eta-\kappa)\psi_k(\xi-\kappa).
\end{aligned}
$$

Recall (6.29), (4.14) and (4.15). From the estimate (2.3) in Lemma 2.1, the following estimate holds,

$$
\|\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\psi_{k_2}(\eta)\psi_{k_2'}(\kappa)\|_{\mathcal{S}^\infty} \leq C2^{\max\{k_2,k_2'\}+k_2+4k_1}, \qquad (6.103)
$$

where C is some absolute constant. After doing spatial localizations for inputs $\widehat{g}_{k_1}(t)$ and $\widehat{g}_{k_2}(t)$ in $\widetilde{P}^5_{k,k_1,k_2}$, the following decomposition holds,

(6.104)

$$
\widetilde{P}^5_{k,k_1,k_2} = \sum_{k_2'\leq k_1'-10,|k_1-k_1'|\leq 10}\widehat{P}^{5,k_1',k_2'}_{k,k_1,k_2}, \quad \widehat{P}^{5,k_1',k_2'}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}}\widehat{P}^{5,k_1',k_2'}_{k,k_1,j_1,k_2,j_2},
$$

$$
\begin{aligned}
\widehat{P}^{5,k_1',k_2'}_{k,k_1,j_1,k_2,j_2} &= \sum_{\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}it^2\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\\
&\times \widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)}\widehat{g^{\nu'}_{k_2'}(t,-\kappa)}d\eta d\kappa d\xi dt. \qquad (6.105)
\end{aligned}
$$

With the above preparation, now we are ready to estimate $\tilde{P}^i_{k,k_1,k_2}$, $i\in\{1,\ldots,5\}$, one by one.

We first estimate $\tilde{P}^1_{k,k_1,k_2}$. Recall (6.94) and (6.95). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, (7.7) in Lemma 7.2,, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}|\widetilde{P}^{j_1,j_2,1}_{k,k_1,k_2}|\\
&\leq \sup_{t\in[2^{m-1},2^m]} C2^{2m+2\tilde{\delta}m+k_2+3k_1+6k_+}\Big(\sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}}\|g_{k_1,j_1}(t)\|_{L^2}\\
&\quad\times\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty} + \sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\Big)\\
&\leq C2^{-2\beta m}\epsilon_0^2.
\end{aligned}
$$

Now we proceed to estimate $\tilde{P}^2_{k,k_1,k_2}$. Recall (6.96) and (6.97). Based on the size of the difference between k_1' and k_1, we split into two cases as follows,

If $k_1' \geq k_1 - 5$. – For this case, we have $k_1' \geq k_2 + 5$ and $|k_1' - k_2'| \leq 5$. By doing integration by parts in "σ," we can rule out the case when $\max\{j_1', j_2'\} \leq m + k_{2,-} - k'_{1,+} - \beta m$. If $\max\{j_1', j_2'\} \geq m + k_{2,-} - k'_{1,+} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some constant C,

$$\begin{aligned}
&\sum_{k_1' \geq k_1 - 5} \Big| \sum_{\max\{j_1', j_2'\} \geq m + k_{2,-} - k'_{1,+} - \beta m} \widehat{P}^{2,k_1',j_1',k_2',j_2'}_{k,k_1,j_1,k_2} \Big| \\
&\quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k_1'-k_2'|\leq 5} C 2^{3m+k_2+3k_1+2k_1'} \\
&\quad \times \Big(\sum_{j_1' \geq \max\{j_2', m+k_{2,-}-k'_{1,+}-\beta m\}} \|g_{k_1',j_1'}(t)\|_{L^2} \|e^{-it\Lambda} g_{k_2',j_2'}\|_{L^\infty} \\
&\quad + \sum_{j_2' \geq \max\{j_1', m+k_{2,-}-k'_{1,+}-\beta m\}} \|g_{k_2',j_2'}(t)\|_{L^2} \|e^{-it\Lambda} g_{k_1',j_1'}\|_{L^\infty} \Big) \\
&\quad \times \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty} \leq C 2^{-m-k_2+30\beta m} \epsilon_0^2 \leq C 2^{-2\beta m} \epsilon_0^2.
\end{aligned}$$

If $k_1' \leq k_1 - 5$. – For this case, we do integration by parts in "η" many times to rule out the case when $\max\{j_1', j_1\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1', j_1\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{k_2' \leq k_1' \leq k_1 - 5} \Big| \sum_{\max\{j_1', j_1\} \geq m + k_{1,-} - \beta m} \widehat{P}^{2,k_1',j_1',k_2',j_2'}_{k,k_1,j_1,k_2} \Big| \\
&\quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2' \leq k_1' \leq k_1 - 5} C 2^{3m+k_2+3k_1+2k_1'} \\
&\quad \times \Big(\sum_{j_1' \geq \max\{j_1, m+k_{1,-}-\beta m\}} \|g_{k_1',j_1'}(t)\|_{L^2} \|e^{-it\Lambda} g_{k_1,j_1}\|_{L^\infty} \\
&\quad + \sum_{j_1 \geq \max\{j_1', m+k_{1,-}-\beta m\}} \|g_{k_1,j_1}(t)\|_{L^2} \\
&\quad \times \|e^{-it\Lambda} g_{k_1',j_1'}\|_{L^\infty} \Big) \|e^{-it\Lambda} g_{k_2'}(t)\|_{L^\infty} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\quad \leq C 2^{-m/2+30\beta m} \epsilon_0^2.
\end{aligned}$$

Now, we proceed to estimate $\tilde{P}^3_{k,k_1,k_2}$. Recall (6.98) and (6.99). Note that $|k_1' - k_2'| \leq 10$ and "$\nabla_\sigma \Phi^{\mu',\nu'}(\xi - \eta, \sigma)$" always has a lower bound, which is $2^{k_1 - k'_{1,+}}$. By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1', j_2'\} \leq m + k_{1,-} - k'_{1,+} - \beta m$. If $\max\{j_1', j_2'\} \geq m + k_{1,-} - k'_{1,+} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some constant C,

$$\sum_{|k_1'-k_2'|\leq 10} \sum_{\max\{j_1', j_2'\} \geq m + k_{1,-} - k'_{1,+} - \beta m} |\widehat{P}^{3,k_1',j_1',k_2',j_2'}_{k,k_1,k_2}|$$

$$
\begin{aligned}
&\leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k_1'-k_2'|\leq 10} C2^{3m+k_2+3k_1+2k_1'} \\
&\quad \times \Big(\sum_{j_1'\geq \max\{j_2', m+k_{1,-}-k_{1,+}'-\beta m\}} \|g_{k_1',j_1'}(t)\|_{L^2} \|e^{-it\Lambda} g_{k_2',j_2'}\|_{L^\infty} \\
&\qquad + \sum_{j_2'\geq \max\{j_1', m+k_{1,-}-k_{1,+}'-\beta m\}} \|g_{k_2',j_2'}(t)\|_{L^2} \\
&\quad \times \|e^{-it\Lambda} g_{k_1',j_1'}\|_{L^\infty}\Big) \|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\leq C2^{-m/2+30\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\tilde{P}^4_{k,k_1,k_2}$. Recall (6.100). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_2\} \geq m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, estimate (6.137) in Lemma 6.14, and estimate (7.3) in Lemma 7.1, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} \|\widetilde{P}^4_{k,k_1,j_1,k_2,j_2}\|_{L^2} \\
&\qquad \leq \sup_{t\in[2^{m-1},2^m]} C2^{3m+k_2+3k_1} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\qquad\quad \times \Big[\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} \|\Lambda_{\geq 3}[\partial_t g^\mu]_{k_1,j_1}\|_{L^2} \|e^{-it\Lambda} g_{k_2,j_2}\|_{L^\infty} \\
&\qquad\quad + \|g_{k_1,j_1}\|_{L^2} 2^{k_2} \|\Lambda_{\geq 3}[\partial_t g_{k_2}]\|_{L^2} \\
&\qquad\quad + \sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} \|\Lambda_{\geq 3}[\partial_t g^\mu]_{k_2,j_2}\|_{L^2} \|e^{-it\Lambda} g_{k_1,j_1}\|_{L^\infty} \\
&\qquad\quad + 2^{k_2} \|g_{k_2,j_2}(t)\|_{L^2} \|\Lambda_{\geq 3}[\partial_t g^\mu_{k_1}]\|_{L^2}\Big] \\
&\qquad \leq C2^{-m-k_2+40\beta m}\epsilon_0^2 + C2^{-m/2+40\beta m}\epsilon_0^2 \leq C2^{-2\beta m}\epsilon_0^2.
\end{aligned}
$$

Lastly, we estimate $\tilde{P}^5_{k,k_1,k_2}$. Recall (6.104) and (6.105). For the case we are considering, we have $k_2' \leq k_1' - 10$ and $|k_1' - k_1| \leq 10$. By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_2\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and estimate (6.103), the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{k_2'\leq k_1-10} \sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} |\widehat{P}^{5,k_1',k_2'}_{k,k_1,j_1,k_2,j_2}| \\
&\quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\leq k_1-10} C2^{3m+k_2+\max\{k_2,k_2'\}+4k_1} \\
&\quad \times \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \Big(\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} \|g_{k_1,j_1}\|_{L^2} \|e^{-it\Lambda} g_{k_2,j_2}\|_{L^\infty} \\
&\qquad + \sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} \|g_{k_2,j_2}\|_{L^2} \|e^{-it\Lambda} g_{k_1,j_1}\|_{L^\infty}\Big) \|e^{-it\Lambda} g_{k_2'}(t)\|_{L^\infty}
\end{aligned}
$$

$$\leq C2^{-m/2+30\beta m}\epsilon_0^2 + C2^{-m-k_2+30\beta m}\epsilon_0^2 \leq C2^{-2\beta m}\epsilon_0^2.$$

Hence finishing the proof. □

6.4. The Z_2 norm estimate of cubic terms

Recall (4.35) and (4.37). Note that we have $k_3 \leq k_2 \leq k_1$ for the case we are considering. For any $\Gamma_\xi^1, \Gamma_\xi^2 \in \{\hat{L}_\xi, \hat{\Omega}_\xi\}$, we have

$$\Gamma_\xi^1\Gamma_\xi^2\Lambda_3[\partial_t \widehat{g}(t,\xi)]\psi_k(\xi) = \sum_{\tau,\kappa,\iota\in\{+,-\}}\sum_{k_3\leq k_2+1\leq k_1+2}\sum_{i=1,2,3,4} T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi),$$

$$T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}, j_3\geq -k_{3,-}} T^{\tau,\kappa,\iota,i}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi), \quad i\in\{3,4\},$$

where

$$(6.106)\qquad T^{\tau,\kappa,\iota,1}_{k,k_1,k_2,k_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\Gamma_\xi^1\Gamma_\xi^2\widehat{g^\tau_{k_1}}(t,\xi-\eta) \times \widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma,$$

$$(6.107)\qquad T^{\tau,\kappa,\iota,2}_{k,k_1,k_2,k_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\big[\Gamma_\xi^1\Gamma_\xi^2\big(\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\big)\widehat{g^\tau_{k_1}}(t,\xi-\eta) + \sum_{\{l,n\}=\{1,2\}}\Gamma_\xi^l\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\Gamma_\xi^n\widehat{g^\tau_{k_1}}(t,\xi-\eta)\big] \times \widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma,$$

$$(6.108)\qquad T^{\tau,\kappa,\iota,3}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}it\big(\Gamma_\xi^l\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)\big) \times \Gamma_\xi^n\big(\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^\tau_{k_1,j_1}}(t,\xi-\eta)\big)\widehat{g^\kappa_{k_2,j_2}}(t,\eta-\sigma) \times \widehat{g^\iota_{k_3,j_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma,$$

$$(6.109)\qquad T^{\tau,\kappa,\iota,4}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi) = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}t^2\big(\Gamma_\xi^1\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)\Gamma_\xi^2\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)\big) \times \tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^\tau_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\kappa_{k_2,j_2}}(t,\eta-\sigma) \times \widehat{g^\iota_{k_3,j_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma.$$

Therefore, we have

$$(6.110)\qquad \mathrm{Re}\Big[\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)}\Gamma_\xi^1\Gamma_\xi^2\Lambda_3[\partial_t\widehat{g}(t,\xi)]\psi_k(\xi)d\xi dt\Big] = \sum_{\tau,\kappa,\iota\in\{+,-\}}\sum_{k_3\leq k_2\leq k_1}\sum_{i=1,2,3,4}\mathrm{Re}\big[T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}\big],$$

$$(6.111)\qquad T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)}T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)d\xi dt.$$

The main goal of this subsection is to prove the following proposition,

PROPOSITION 6.9. – *Under the bootstrap assumption* (4.49)*, the following estimates hold for some absolute constant C,*

(6.112)
$$\sup_{t_1,t_2\in[2^{m-1},2^m]}\Big|\sum_{k\in\mathbb{Z}}\mathrm{Re}\Big[\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)}\Gamma_\xi^1\Gamma_\xi^2\Lambda_3[\partial_t\widehat{g}(t,\xi)]\psi_k(\xi)d\xi dt\Big]\Big|\leq C2^{2\tilde{\delta}m}\epsilon_0^2,$$

(6.113)
$$\sup_{t\in[2^{m-1},2^m]}\sum_{i=1,2,3,4}\sum_{k\in\mathbb{Z}}\sum_{k_3\leq k_2\leq k_1}\|T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{-m+\tilde{\delta}m}\big(1+2^{2\tilde{\delta}m+k+5k_+}\big)\epsilon_0.$$

Proof. – To simplify the problem, we first rule out the very high frequency case and the very low frequency case. Very similar to what we did in the estimate of quadratic terms (see (6.55)), we do integration by parts in η to move the derivatives ∇_η of $\nabla_\xi\widehat{g}_{k_1}(t,\xi-\eta)=-\nabla_\eta\widehat{g}_{k_1}(t,\xi-\eta)$ around such that there is no derivatives in front of $\widehat{g}_{k_1}(t,\xi-\eta)$. As a result, from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and the $L^\infty\to L^2$ type Sobolev embedding, the following estimate holds for some absolute constant C,

$$\sum_{i=1,2,3,4}\|T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{2m+2k_1+6k_{1,+}}\|g_{k_1}(t)\|_{L^2}2^{k_2+k_3}\|g_{k_3}(t)\|_{L^2}\big(2^{-2k_2}\|g_{k_2}(t)\|_{L^2}$$
$$+2^{-k_2}\|\nabla_\xi\widehat{g}_{k_2}(t)\|_{L^2}+\|\nabla_\xi^2\widehat{g}_{k_2}(t)\|_{L^2}\big)\leq C2^{2m+\beta m-(N_0-20)k_{1,+}}\epsilon_0. \tag{6.114}$$

From the above estimate, we can rule out the case when $k_1\geq 4\beta m$. It remains to consider the case when $k_1\leq 4\beta m$. Next, we proceed to rule out the very low frequency case. If either $k\leq -2m$ or $k_3\leq -3m-30\beta m$, then from the $L^2-L^\infty-L^\infty$ type trilinear estimate, the following estimate holds for some absolute constant C,

$$\sum_{i=1,2,3,4}\|T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C(1+2^{2m+2k})2^{k+k_3+4k_{1,+}}\big(2^{2k_1}\|\nabla_\xi^2\widehat{g}_{k_1}(t,\xi)\|_{L^2}$$
$$+2^{k_1}\|\nabla_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}+\|g_{k_1}(t)\|_{L^2}\big)\|g_{k_2}(t)\|_{L^2}\|g_{k_3}(t)\|_{L^2}\leq C2^{-m-\beta m}\epsilon_0. \tag{6.115}$$

Therefore, from now on, we restrict ourself to the case when k,k_1,k_2, and k_3 are in the range listed as follows,

$$-3m-30\beta m\leq k_3\leq k_2\leq k_1\leq 4\beta m,\quad -2m\leq k\leq 4\beta m. \tag{6.116}$$

Recall (6.107). From the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\|T^{\tau,\kappa,\iota,2}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{2k_1+4k_{1,+}}\big(\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}+\sum_{i=1,2}\|e^{-it\Lambda}\Gamma^i g_{k_1}\|_{L^\infty}\big)\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}$$
$$\times\|g_{k_3}\|_{L^2}\leq C2^{-3m/2+50\beta m}\epsilon_0^2\Longrightarrow|T^{\tau,\kappa,\iota,2}_{k,k_1,k_2,k_3}|\leq C2^{-m/2+50\beta m}\epsilon_0^2.$$

Since there are only at most "m^4 "cases in the range (6.116), to prove (6.112) and (6.113), it would be sufficient to prove the following estimate for any $i=1,3,4$, any $\tau,\kappa,\iota\in\{+,-\}$, and any fixed k,k_1,k_2,k_3 in the range (6.116),

(6.117)
$$|\mathrm{Re}\big[T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}\big]|\leq C2^{3\tilde{\delta}m/2}\epsilon_0^2,\quad\|T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{-m+\tilde{\delta}m/2}\big(1+2^{2\tilde{\delta}m+k+5k_+}\big)\epsilon_0,$$

where C is some absolute constant.

From the results in the next three lemmas, i.e., Lemma 6.10, Lemma 6.11, and Lemma 6.12, we know that our desired estimates in (6.117) indeed holds for fixed k, k_1, k_2, k_3 in the range listed in (6.116). Hence finishing the proof. □

LEMMA 6.10. – *For $i = 1, 3, 4$ and fixed k, k_1, k_2, k_3 in the range* (6.116) *, our desired estimates listed in* (6.117) *hold if moreover $k_2 \leq k_1 - 10$.*

Proof. – Recall the normal form transformation that we did in Subsection 4.1, see (4.30) and (4.40). For the case we are considering, which is $k_2 \leq k_1 - 10$, we have "$\tau = +$" and the fact that the estimate $|\nabla_\xi \Phi^{+,\kappa,\iota}(\xi, \eta, \sigma)| \leq 2^{k_2}$ holds for some absolute constant C.

We first estimate $T^{+,\kappa,\iota,1}_{k_1,k_2,k_3}$ and $T^{+,\kappa,\iota,1}_{k_1,k_2,k_3}(t, \xi)$. Recall (6.106) and (6.111). From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\|T^{+,\kappa,\iota,1}_{k_1,k_2,k_3}(t,\xi)\|_{L^2} \leq C 2^{2k_1+4k_{1,+}}\big(2^{2k_1}\|\nabla_\xi^2 \widehat{g_{k_1}}(t,\xi)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g_{k_1}}(t,\xi)\|_{L^2} + \|g_{k_1}(t)\|_{L^2}\big)$$
$$\times \|e^{-it\Lambda} g_{k_2}\|_{L^\infty}\|e^{-it\Lambda} g_{k_3}\|_{L^\infty} \leq C 2^{-m+7\tilde{\delta}m/3+k+5k_+}\epsilon_0. \tag{6.118}$$

Since the L^∞_x decay rate of the nonlinear solution itself is slightly slower than $t^{-1/2}$, a rough $L^2 - L^\infty - L^\infty$ is not sufficient to close the estimate of $T^{\tau,\kappa,\iota,1}_{k_1,k_2,k_3}$. An essential ingredient is to utilize symmetry such that one of the inputs putted in L^∞ associates with a spatial derivative. To see the symmetric structure, we decompose $T^{+,\kappa,\iota,1}_{k_1,k_2,k_3}$ into three parts as follows,

$$T^{+,\kappa,\iota,1}_{k,k_1,k_2,k_3} = \sum_{i=1,2,3} T^{+,\kappa,\iota,1;i}_{k,k_1,k_2,k_3}, \quad T^{+,\kappa,\iota,1;1}_{k,k_1,k_2,k_3} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)} e(\xi)$$
$$\times \widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\,d\eta\, d\sigma\, d\xi\, dt,$$
$$T^{+,\kappa,\iota,1;2}_{k,k_1,k_2,k_3} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)} \big(\tilde{d}_{+,\kappa,\iota}(\xi,\eta,\sigma) - e(\xi)\big)$$
$$\times \psi_k(\xi)\widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta)\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\,d\eta\, d\sigma\, d\xi\, dt,$$
$$T^{+,\kappa,\iota,1;3}_{k,k_1,k_2,k_3} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)} \tilde{d}_{+,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)$$
$$\times \psi_k(\xi)\widehat{g^\iota_{k_3}}(t,\sigma)\big(\Gamma^1_\xi\Gamma^2_\xi\widehat{g_{k_1}}(t,\xi-\eta) - \widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta)\big)\,d\eta\, d\sigma\, d\xi\, dt,$$

where $e(\xi)$ is defined in (4.47). After switching the role of ξ and $\xi - \eta$ inside $T^{+,\kappa,\iota,1;1}_{k_1,k_2,k_3}$, we have

$$\sum_{\kappa,\iota\in\{+,-\}} \mathrm{Re}[T^{+,\kappa,\iota,1;1}_{k_1,k_2,k_3}] = \sum_{\kappa,\iota\in\{+,-\}} \mathrm{Re}[\tilde{T}^{+,\kappa,\iota}_{k_1,k_2,k_3}], \quad \tilde{T}^{+,\kappa,\iota}_{k_1,k_2,k_3} := \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}$$
$$\times e^{it\Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{k,k_1}(\xi,\eta,\sigma)\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\,d\eta\, d\sigma\, d\xi\, dt,$$

where

$$\tilde{d}_{k,k_1}(\xi,\eta,\sigma) := e(\xi)\psi_{k_1}(\xi-\eta)\psi_k(\xi) + \overline{e(\xi-\eta)}\psi_{k_1}(\xi)\psi_k(\xi-\eta).$$

Recall (4.47). From the estimate (2.3) in Lemma 2.1, the following estimate holds for some absolute constant C,

$$\|\tilde{d}_{k,k_1}(\xi,\eta,\sigma)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty} \leq C 2^{\max\{k_2,k_3\}+k_1+4k_{1,+}}. \tag{6.119}$$

From (6.119), (4.46) and the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.5) in Lemma 2.2, we have

$$\begin{aligned}
\sum_{i=1,2,3}\sum_{\kappa,\iota\in\{+,-\}} &|\mathrm{Re}\big[T^{+,\kappa,\iota,1;i}_{k,k_1,k_2,k_3}\big]| \\
&\leq \sup_{t\in[2^{m-1},2^m]} C2^{m+k_1+4k_{1,+}+\max\{k_2,k_3\}}\big(2^{2k_1}\|\nabla_\xi^2\widehat{g}_{k_1}(t,\xi)\|_{L^2} \\
&\quad + 2^{k_1}\|\nabla_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|g_{k_1}(t)\|_{L^2}\big)\|\Gamma^1\Gamma^2 g_{k_1}\|_{L^2}\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty} \\
&\leq C2^{-m/2+50\beta m}\epsilon_0^2,
\end{aligned}$$

where C is some absolute constant.

Therefore, now it would be sufficient to estimate $T^{+,\kappa,\iota,i}_{k_1,k_2,k_3}$ and $T^{+,\kappa,\iota,i}_{k_1,k_2,k_3}(t,\xi)$, $i\in\{3,4\}$. Recall (6.108), (6.109), and (6.111). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
\sum_{i=3,4}&\Big\|\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} T^{+,\kappa,\iota,i}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi)\Big\|_{L^2} \\
&\leq C2^{m+3k_1+k_2+4k_{1,+}}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty} \\
&\quad\times\Big[\sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}}(2^{k_1+j_1}+(1+2^{m+k_1+k_2}))\|g_{k_1,j_1}\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty} \\
&\quad+\sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}}\big(2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty} \\
&\quad+(1+2^{m+k_1+k_2})\|e^{-it\Lambda}g_{k_1,j_1}\|_{L^\infty}\big)\|g_{k_2,j_2}\|_{L^2}\Big] \\
&\leq C2^{-3m/2+50\beta m}\epsilon_0.
\end{aligned}$$

Note that the above estimate is sufficient to imply our second desired estimate in (6.117). Hence finishing the proof. $\square$

LEMMA 6.11. – *For $i = 1,3,4$ and fixed k,k_1,k_2,k_3 in the range* (6.116) *, our desired estimate* (6.117) *holds if either $|k_1-k_2|\leq 10$ and $k_3\leq k_2-10$ or $|k_1-k_2|\leq 10$, $|k_3-k_2|\leq 10, k\leq k_1-10$.*

Proof. – The estimate of $T^{\tau,\kappa,\iota,1}_{k,k_1,k_2,k_3}(t,\xi)$ is straightforward. As $|k_1-k_2|\leq 10$, the size of symbol compensates the decay rate of $e^{-it\Lambda}g_{k_2}(t)$. From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
\|T^{\tau,\kappa,\iota,1}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2} &\leq C2^{2k_1+4k_{1,+}}\big(2^{2k_1}\|\nabla_\xi^2\widehat{g}_{k_1}(t,\xi)\|_{L^2}+2^{k_1}\|\nabla_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}+\|g_{k_1}(t)\|_{L^2}\big) \\
&\quad\times\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}\leq C2^{-3m/2+50\beta m}\epsilon_0.
\end{aligned}\tag{6.120}$$

Now, we proceed to estimate $T^{\tau,\kappa,\iota,3}_{k,k_1,k_2,k_3}(t,\xi)$ and $T^{\tau,\kappa,\iota,4}_{k,k_1,k_2,k_3}(t,\xi)$. Recall (6.108) and (6.109). Note that, if either $|k_1-k_2|\leq 10$ and $k_3\leq k_2-10$ or $|k_1-k_2|\leq 10$, $|k_3-k_2|\leq 10, k\leq k_1-10$, we know that $\nabla_\eta\Phi^{\tau,\kappa,\iota}(\xi,\eta,\kappa)$ has a lower bound, which is $2^{k-4\beta m}$. To take advantage of this fact, we do integration by parts in "η" many times to rule out the case when $\max\{j_1, j_2\} \le m + k_- - 5\beta m$. If $\max\{j_1, j_2\} \ge m + k_- - 5\beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=3,4}\Big\|\sum_{\max\{j_1,j_2\}\ge m+k_--5\beta m} T^{+,\kappa,\iota,i}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi)\Big\|_{L^2}\\
&\quad\le C2^{m+2k+k_1+4k_{1,+}}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}\\
&\quad\quad\times\Big(\sum_{j_2\ge\max\{j_1,m+k_--5\beta m\}}\big((1+2^{m+2k_1})\|e^{-it\Lambda}g_{k_1,j_1}\|_{L^\infty}+2^{k_1}\|e^{-it\Lambda}\\
&\quad\quad\times\mathscr{F}^{-1}[\nabla_\xi\widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty}\big)\|g_{k_2,j_2}\|_{L^2}\\
&\quad\quad+\sum_{j_1\ge\max\{j_2,m+k_--5\beta m\}}\big(2^{m+2k_1}+2^{j_1+k_1}\big)\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty}\|g_{k_1,j_1}\|_{L^2}\Big)\\
&\quad\le C2^{-3m/2+50\beta m}\epsilon_0.
\end{aligned}\tag{6.121}
$$

From (6.120) and (6.121), it is easy to see our desired estimates in (6.117) hold. □

LEMMA 6.12. – *For $i = 1, 3, 4$ and fixed k, k_1, k_2, k_3 in the range* (6.116) *, our desired estimate* (6.117) *holds if $|k_1 - k_2| \le 10$, $|k_3 - k_2| \le 10$, and $|k - k_1| \le 10$.*

Proof. – Since we still have $|k_1 - k_2| \le 10$, the estimate of $T^{\tau,\kappa,\iota,1}_{k,k_1,k_2,k_3}(t,\xi)$ in (6.120) still holds. It would be sufficient to estimate $T^{\tau,\kappa,\iota,3}_{k,k_1,k_2,k_3}(t,\xi)$ and $T^{\tau,\kappa,\iota,4}_{k,k_1,k_2,k_3}(t,\xi)$, which is more delicate. For those cases, we need to study the space resonance in "η" set and the space resonance in "σ" set carefully as we did in the Z_1-norm estimate of cubic terms in the proof of Lemma 5.7.

Recall that we already canceled out the case when $(\tau,\kappa,\iota) \in \mathscr{S}_4$ (see (5.32)) and $(\xi-\eta, \eta-\sigma, \sigma)$ is very close to $(\xi/3, \xi/3, \xi/3)$ in the normal form transformation. Therefore, for the case when $(\tau,\kappa,\iota) \in \mathscr{S}_4$, we only have to consider the case when $(\xi-\eta, \eta-\sigma, \sigma)$ is not close to $(\xi/3, \xi/3, \xi/3)$, in which case either $\nabla_\eta\Phi^{\tau,\kappa,\iota}(\xi,\eta,\kappa)$ or $\nabla_\sigma\Phi^{\tau,\kappa,\iota}(\xi,\eta,\kappa)$ has a good lower bound, which allows us to do integration by parts either in η or in σ. The estimate of this case is similar and also easier than the estimate of (6.121) in the proof of Lemma 6.11. We omit details here.

Now, we focus on the case when $(\tau,\kappa,\iota) \in \mathscr{S}_i$, $i \in \{1,2,3\}$. By the symmetries between inputs, it would be sufficient to consider the case when $(\tau,\kappa,\iota) \in \mathscr{S}_1$, i.e., $(\tau,\kappa,\iota) \in \{(+,-,-),(-,+,+)\}$. After changing the variables as follows $(\xi,\eta,\sigma) \longrightarrow (\xi, 2\xi+\eta+\sigma, \xi+\sigma)$, we have the following decomposition for $i \in \{3,4\}$,

$$
\begin{aligned}
&T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi) := \sum_{l_1,l_2\ge\bar{l}_\tau} H^{l_1,l_2,\tau,i-2}(t,\xi),\\
&H^{l_1,l_2,\tau,i-2}(t,\xi) = \sum_{j_1\ge -k_{1,-},\, j_2\ge -k_{2,-}} H^{l_1,l_2,\tau,i-2}_{j_1,j_2}(t,\xi),\\
&H^{l_1,l_2,\tau,1}_{j_1,j_2}(t,\xi) := \sum_{\{l,n\}=\{1,2\}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}it\big(\Gamma^l_\xi\Phi^{\tau,\kappa,\iota}(\xi,2\xi+\eta+\sigma,\xi+\sigma)\big)\varphi_{l_1;\bar{l}_\tau}(\eta)\\
&\qquad\times\varphi_{l_2;\bar{l}_\tau}(\sigma)\psi_k(\xi)\Gamma^n_\xi\big(\tilde{d}_{\tau,\kappa,\iota}(-\xi-\eta-\sigma,\xi+\eta,\xi+\sigma)\widehat{g^\tau_{k_1,j_1}}(t,-\xi-\eta-\sigma)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \widehat{g^{\kappa}_{k_2,j_2}}(t,\xi+\eta)\widehat{g^{\iota}_{k_3,j_3}}(t,\xi+\sigma)d\eta d\sigma,\\
H^{l_1,l_2,\tau,2}_{j_1,j_2} &:= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}t^2\big(\Gamma^1_\xi\Phi^{\tau,\kappa,\iota}(\xi,2\xi+\eta+\sigma,\xi+\sigma)\\
&\times \Gamma^2_\xi\Phi^{\tau,\kappa,\iota}(\xi,2\xi+\eta+\sigma,\xi+\sigma)\big)\tilde{d}_{\tau,\kappa,\iota}(-\xi-\eta-\sigma,\xi+\eta,\xi+\sigma)\\
&\times \widehat{g^{\tau}_{k_1,j_1}}(t,-\xi-\eta-\sigma)\widehat{g^{\kappa}_{k_2,j_2}}(t,\xi+\eta)\widehat{g^{\iota}_{k_3}}(t,\xi+\sigma)\psi_k(\xi)\\
&\times \varphi_{l_1;\bar{l}_\tau}(\eta)\varphi_{l_2;\bar{l}_\tau}(\sigma)d\eta d\sigma,
\end{aligned}
$$

where $\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is defined in (5.35), the cutoff function $\varphi_{l;\bar{l}}(\cdot)$ is defined in (5.36) and the thresholds are chosen as follows, $\bar{l}_+ := k_1 - 10$ and $\bar{l}_- := -m/2 + 10\delta m + k_{1,+}/2$.

If $\tau = +$, i.e., $(\tau,\kappa,\iota) = (+,-,-)$. – Recall the normal form transformation that we did in (4.1), see (4.20) and (4.30). For the case we are considering, $(\tau,\kappa,\iota)\in\widetilde{S}$, we have already removed the case when $\max\{l_1,l_2\} = \bar{l}_+$. Hence it would be sufficient to consider the case when $\max\{l_1,l_2\} > \bar{l}_+$. Due to the symmetry between inputs, we assume that $l_2 = \max\{l_1,l_2\}$. As $l_2 > \bar{l}_+$, we can take the advantage of the fact that "$\nabla_\eta\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$" is big by doing integration by parts in "η". From (5.37), we can rule out the case when $\max\{j_1,j_2\}\le m+k_- - \beta m$ by doing integration by parts in "η" many times.

If $\max\{j_1,j_2\}\ge m+k_- -\beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{\max\{j_1,j_2\}\ge m+k_- -\beta m}\sum_{i=1,2}\|H^{l_1,l_2,\tau,i}_{j_1,j_2}(t,\xi)\|_{L^2}\\
&\le C2^{m+4k_1+4k_{1,+}}\Big(\sum_{j_1\ge\max\{j_2,m+k_- -\beta m\}}\big(2^{m+2k_1}+2^{k_1+j_1}\big)\\
&\quad\times\|g_{k_1,j_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\\
&\quad+\sum_{j_2\ge\max\{j_1,m+k_- -\beta m\}}\big((2^{m+2k_1}+1)\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\\
&\quad+2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi\widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty}\big)\|g_{k_2,j_2}(t)\|_{L^2}\Big)\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty} \qquad (6.122)\\
&\le C2^{-2m+50\beta m}\epsilon_0.
\end{aligned}
$$

If $\tau = -$, i.e., $(\tau,\kappa,\iota) = (-,+,+)$. – Note that the estimates (5.38) and (5.39) hold for the case we are considering. Same as before, due to the symmetry between inputs, without loss of generality, we assume that $l_2 = \max\{l_1,l_2\}$.

We first consider the case when $l_2 > \bar{l}_-$. Recall (5.37), by doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\le m+l_2-4\beta m$. If $\max\{j_1,j_2\}\ge m+l_2-4\beta m$, from the $L^2-L^\infty-L^\infty$ type trilinear estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{\max\{j_1,j_2\}\ge m+l_2-4\beta m}\sum_{i=1,2}\|H^{l_1,l_2,-,i}_{j_1,j_2}(t,\xi)\|_{L^2}\le C2^{m+3k_1+l_2+4k_{1,+}}\\
&\quad\times\Big(\sum_{j_1\ge\max\{j_2,m+l_2-4\beta m\}}\big(2^{m+k_1+l_2}+2^{j_1+k_1}\big)\|g_{k_1,j_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j_2 \geq \max\{j_1, m+l_2-4\beta m\}} \Big((1+2^{m+k_1+l_2})\|e^{-it\Lambda} g_{k_1,j_1}(t)\|_{L^\infty} \\
&+ 2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi \widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty}\Big)\|g_{k_2,j_2}(t)\|_{L^2}\Big)\|e^{-it\Lambda} g_{k_3}(t)\|_{L^\infty} \\
&\leq C 2^{-2m+50\beta m}\epsilon_0^2. \qquad (6.123)
\end{aligned}
$$

Lastly, we consider the case when $l_2 = \bar{l}_- = -m/2 + 10\delta m + k_{1,+}/2$. Recall the estimate (5.38). For this case, we use the volume of support in "η" and "σ". As a result, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=1,2} \|H^{\bar{l}_-,\bar{l}_-,-,i}(t,\xi)\|_{L^2} \\
&\quad \leq C 2^{4k_{1,+}}\big(2^{2m+6\bar{l}+4k_1} + 2^{m+5\bar{l}+4k_1}\big)\big(2^{-k_1}\|g_{k_1}(t)\|_{L^2} + \|\nabla_\xi \widehat{g_{k_1}}(t,\xi)(t)\|_{L^2}\big) \\
&\quad \times \|g_{k_2}(t)\|_{L^1}\|g_{k_3}(t)\|_{L^1} \leq C 2^{-m+100\delta m}\epsilon_0^2, \qquad (6.124)
\end{aligned}
$$

hence finishing the proof. □

6.5. The Z_2 norm estimate of the quartic terms

Recall (4.38). For any $\Gamma_\xi^1, \Gamma_\xi^2 \in \{\hat{L}_\xi, \hat{\Omega}_\xi\}$, we have

$$
\Gamma_\xi^1\Gamma_\xi^2\Lambda_4[\partial_t \widehat{g}(t,\xi)]\psi_k(\xi) = \sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}} \sum_{k_4\leq k_3\leq k_2\leq k_1} \sum_{i=1,2,3,4} K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi),
$$

$$
K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,j_1,k_2,j_2,k_3,k_4}(t,\xi), \quad i\in\{3,4\}, \qquad (6.125)
$$

where

$$
\begin{aligned}
K^{\mu_1,\mu_2,\nu_1,\nu_2,1}_{k,k_1,k_2,k_3,k_4}(t,\xi) &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)} \\
&\times \tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\Gamma_\xi^1\Gamma_\xi^2\widehat{g_{k_1}^{\mu_1}}(t,\xi-\eta) \\
&\times \widehat{g_{k_2}^{\mu_2}}(t,\eta-\sigma)\widehat{g_{k_3}^{\nu_1}}(t,\sigma-\kappa)\widehat{g_{k_4}^{\nu_2}}(t,\kappa)\psi_k(\xi)\,d\kappa\, d\sigma\, d\eta, \qquad (6.126)
\end{aligned}
$$

$$
\begin{aligned}
K^{\mu_1,\mu_2,\nu_1,\nu_2,2}_{k,k_1,k_2,k_3,k_4}(t,\xi) &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\psi_k(\xi) \\
&\times \Big[\Gamma_\xi^1\Gamma_\xi^2\big(\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\big)\widehat{g_{k_1}^{\mu_1}}(t,\xi-\eta) \\
&+ \sum_{\{l,n\}=\{1,2\}} \Gamma_\xi^l\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\Gamma_\xi^n\widehat{g_{k_1}^{\mu_1}}(t,\xi-\eta)\Big] \\
&\times \widehat{g_{k_2}^{\mu_2}}(t,\eta-\sigma)\widehat{g_{k_3}^{\nu_1}}(t,\sigma-\kappa)\widehat{g_{k_4}^{\nu_2}}(t,\kappa)\,d\kappa\, d\sigma\, d\eta, \qquad (6.127)
\end{aligned}
$$

$$
\begin{aligned}
K^{\mu_1,\mu_2,\nu_1,\nu_2,3}_{k,k_1,j_1,k_2,j_2,k_3,k_4}(t,\xi) &:= \sum_{\{l,n\}=\{1,2\}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi_k(\xi)e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}it \\
&\times \big(\Gamma_\xi^l\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)\big)\Gamma_\xi^n\big(\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa) \\
&\times \widehat{g_{k_1,j_1}^{\mu_1}}(t,\xi-\eta)\big)\widehat{g_{k_2,j_2}^{\mu_2}}(t,\eta-\sigma)\widehat{g_{k_3}^{\nu_1}}(t,\sigma-\kappa)\widehat{g_{k_4}^{\nu_2}}(t,\kappa)\,d\kappa\, d\sigma\, d\eta,
\end{aligned}
$$

$$K^{\mu_1,\mu_2,\nu_1,\nu_2,4}_{k,k_1,j_1,k_2,j_2,k_3,k_4}(t,\xi) := -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi_k(\xi) e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)} t^2 \Gamma^1_\xi \Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$$
$$\times \Gamma^2_\xi \Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)$$
$$\times \widehat{g^{\mu_1}_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\mu_2}_{k_2,j_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)d\kappa d\eta d\sigma. \tag{6.128}$$

The main goal of this subsection is to prove the following proposition.

PROPOSITION 6.13. – *Under the bootstrap assumption (*4.49*), the following estimates hold for some absolute constant C and any $t \in [2^{m-1}, 2^m]$,*

(6.129)
$$\sup_{t_1,t_2\in[2^{m-1},2^m]} \Big| \sum_k \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \overline{\Gamma^1_\xi\Gamma^2_\xi \widehat{g}(t,\xi)} \Gamma^1_\xi\Gamma^2_\xi \Lambda_4[\partial_t \widehat{g}(t,\xi)]\psi_k(\xi) d\xi dt \Big| \le C 2^{2\tilde{\delta} m}\epsilon_0^2.$$

$$\sup_{t\in[2^{m-1},2^m]} \|\Gamma^1_\xi\Gamma^2_\xi \Lambda_4[\partial_t \widehat{g}(t,\xi)]\|_{L^2} \le C 2^{-m+\tilde{\delta} m}\epsilon_0^2. \tag{6.130}$$

Proof. – As usual, we first rule out the very high frequency case and the very low frequency case. Same as what we did in the estimate of cubic terms, we move the derivative $\nabla_\xi = -\nabla_\eta$ in front of $\widehat{g}_{k_1}(t,\xi-\eta)$ around by doing integration by parts in η such that there is no derivative in front of $\widehat{g}_{k_1}(t,\xi-\eta)$. As a result, the following estimate holds,

$$\sum_{i=1,2,3,4} \|K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi)\|_{L^2} \le C(1+2^{2m+2k})2^{6k_{1,+}}\|g_{k_1}(t)\|_{L^2}$$
$$\times \big(\|\nabla^2_\xi \widehat{g_{k_2}}(t,\xi)\|_{L^2} + 2^{-k_2}\|\nabla_\xi \widehat{g_{k_2}}(t,\xi)\|_{L^2} + 2^{-2k_2}\|g_{k_2}(t)\|_{L^2}\big) \tag{6.132}$$
$$\times 2^{k_3+k_4}\|g_{k_3}(t)\|_{L^2}\|g_{k_4}(t)\|_{L^2} \le C 2^{2m+\beta m-(N_0-10)k_{1,+}}\epsilon_0^2,$$

where C is some absolute constant. Hence, we can rule out the case when $k_1 \ge 4\beta m$. It remains to consider the case when $k_1 \le 4\beta m$. We can also rule out the very low frequencies case. If either $k_4 \le -3m - 30\beta m$ or $k \le -2m$, then the following estimate holds for some absolute constant C,

$$\sum_{i=1,2,3,4} \|K^{\mu_1,\mu_2,\nu_1,\nu_2,1}_{k,k_1,k_2,k_3,k_4}(t,\xi)\|_{L^2} \le C(1+2^{2m+2k})2^{k+k_4+4k_{1,+}}$$
$$\times \big(2^{2k_1}\|\nabla^2_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|g_{k_1}(t)\|_{L^2}\big)$$
$$\times \|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\|g_{k_3}(t)\|_{L^2}\|g_{k_4}(t)\|_{L^2} \le C 2^{-m-\beta m}\epsilon_0^2.$$

Now it would be sufficient to consider fixed k, k_1, k_2, k_3, and k_4 in the following range,

$$-3m - 30\beta m \le k_4 \le k_3 \le k_2 \le k_1 \le 4\beta m, \quad -2m \le k \le 3\beta m. \tag{6.133}$$

From the $L^2 - L^\infty - L^\infty - L^\infty$ type multilinear estimate, the following estimate holds

$$\sum_{i=1,2} \|K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi)\|_{L^2} \le C 2^{2k_1+4k_{1,+}}$$
$$\times \big(2^{2k_1}\|\nabla^2_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|g_{k_1}(t,\xi)\|_{L^2}\big)$$
$$\times \|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_4}(t)\|_{L^\infty} \le C 2^{-3m/2+50\beta m}\epsilon_0^2, \tag{6.134}$$

where C is some absolute constant.

It remains to estimate the case when $i = 3, 4$. We first consider the case when $k_1 - 10 \leq k_3$. For this case, the following estimate holds from the $L^2 - L^\infty - L^\infty - L^\infty$ type estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{i=3,4} \| K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi)\|_{L^2} \leq C 2^{m+4k_1+4k_{1,+}} \\
&\quad \times \big[\big(\|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty} + 2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi \widehat{g}_{k_1}(t,\xi)]\|_{L^\infty}\big) + 2^{m+2k_1}\|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty}\big] \\
&\quad \times \|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty}\|e^{-it\Lambda} g_{k_3}(t)\|_{L^\infty}\|g_{k_4}(t)\|_{L^2} \\
&\quad \leq C 2^{-m+\tilde{\delta}m/2}\epsilon_0^2.
\end{aligned} \tag{6.135}$$

Lastly, we consider the case when $k_3 \leq k_1 - 10$. Recall (4.32) and (4.41). Because of the construction of the normal form transformation we did in Subsection 4.1, we know that the case when η is very close to $\xi/2$ and $|\sigma|, |\kappa| \leq 2^{-10}|\xi|$ is removed, which means that "$\nabla_\eta \Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$" has a lower bound, which is $2^{k-k_{1,+}}$. To take advantage of this fact, we do integration by parts in "η" many times to rule out the case when $\max\{j_1, j_2\} \leq m + k_- - 5\beta m$. If $\max\{j_1, j_2\} \geq m + k_- - 5\beta m$, from the $L^2 - L^\infty - L^\infty - L^\infty$ type estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{i=3,4}\sum_{\max\{j_1,j_2\}\geq m+k_- - 5\beta m} \| K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,j_1,k_2,j_2,k_3,k_4}(t,\xi)\|_{L^2} \\
&\quad \leq \sum_{j_1 \geq \max\{j_2, m+k_- - 5\beta m\}} C 2^{m+k+k_2+2k_1+4k_{1,+}}\big(2^{m+k+k_1} + 2^{k_1+j_1}\big)\|g_{k_1,j_1}(t)\|_{L^2} \\
&\quad \times \|e^{-it\Lambda} g_{k_2,j_2}(t)\|_{L^\infty}\|e^{-it\Lambda} g_{k_3}(t)\|_{L^\infty}\|e^{-it\Lambda} g_{k_4}(t)\|_{L^\infty} \\
&\quad + \sum_{j_2 \geq \max\{j_1, m+k_- - 5\beta m\}} C 2^{m+k+k_2+2k_1+4k_{1,+}}\big(2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi \widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty} \\
&\quad + 2^{m+k+k_1}\|e^{-it\Lambda} g_{k_1,j_1}(t)\|_{L^\infty}\big)2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda} g_{k_3}(t)\|_{L^\infty}\|g_{k_4}(t)\|_{L^2} \\
&\quad \leq C 2^{-3m/2+50\beta m}\epsilon_0^2.
\end{aligned} \tag{6.136}$$

To sum up, from the estimates (6.134), (6.135) and (6.136), we know that our desired estimates (6.129) and (6.130) hold. □

LEMMA 6.14. – *Under the bootstrap assumption* (4.49)*, the following estimates hold for any* $t \in [2^{m-1}, 2^m]$ *and any* $\Gamma^1_\xi, \Gamma^2_\xi \in \{\hat{L}_\xi, \hat{\Omega}_\xi\}$,

$$\|\Gamma^1_\xi \Gamma^2_\xi \Lambda_{\geq 3}[\partial_t \widehat{g}_k(t,\xi)]\|_{L^2} \leq C 2^{-m+\tilde{\delta}m}\big(1 + 2^{2\tilde{\delta}m+k+5k_+}\big)\epsilon_0, \tag{6.137}$$

Proof. – The desired estimate (6.137) follows straightforwardly from the estimate (6.113) in Proposition (6.9), the estimate (6.130) in Proposition (6.13), and the estimate (7.13) in Lemma 7.4. □

7. Fixed time weighted norm estimates

There are mainly two tasks to complete in this section. (i) Firstly, we prove some fixed time weighted norm estimates, which are stated in Lemma 7.1 and Lemma 7.2 and have been used in previous two sections. (ii) Lastly, we estimate both the low order weighted norm (Z_1-norm) and the high order weighted norm (Z_2-norm) of the profile of the quintic and higher order remainder term $\mathcal{R}_1$, see the equation satisfied by the good substitution variable v in (4.21). Therefore, finishing the bootstrap argument of the weighted norms of the profile $g(t) = e^{it\Lambda}v(t)$ over time.

LEMMA 7.1. – *Under the bootstrap assumption* (4.49)*, the following estimates hold,*

$$\sup_{t\in[2^{m-1},2^m]} \|\partial_t \widehat{g}_k(t,\xi) - \sum_{\mu,\nu\in\{+,-\}} \sum_{(k_1,k_1)\in\chi_k^1} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2} \leq C 2^{-21m/20}\epsilon_0, \tag{7.1}$$

$$\sup_{t\in[2^{m-1},2^m]} \|\partial_t \widehat{g}_k(t,\xi)\|_{L^2} \leq C \min\{2^{-2m-k+2\tilde{\delta}m}, 2^{-m+\delta m}\}\epsilon_0 + C2^{-21m/20}\epsilon_0, \tag{7.2}$$

$$\sup_{t\in[2^{m-1},2^m]} \|\Lambda_{\geq 3}[\partial_t \widehat{g}_k(t,\xi)]\|_{L^2} \leq C 2^{-3m/2+\beta m}\epsilon_0, \tag{7.3}$$

where C is some absolute constant, χ_k^1 is defined in (6.2) *and $B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)$ is defined in* (4.36).

Proof. – For the cubic and higher order terms, after putting the input with the smallest frequency in L^2 and all other inputs in L^∞, the decay rate of L^2 norm is at least $2^{-3m/2+\beta m}$, which gives us our desired estimate (7.3). Hence to prove (7.1) and (7.2), we only have to consider the quadratic terms "$B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)$". Recall (4.36), after doing spatial localizations for two inputs, we have

$$B^{\mu,\nu}_{k,k_1,k_2}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} B^{\mu,\nu,j_1,j_2}_{k,k_1,k_2}(t,\xi),$$

$$B^{\mu,\nu,j_1,j_2}_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)} \tilde{q}_{\mu,\nu}(\xi,\eta) \widehat{g^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\psi_k(\xi)d\eta.$$

We first consider the case when $|k_1-k_2|\leq 10$. From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\sum_{|k_1-k_2|\leq 10} \|B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2} \leq \sum_{|k_1-k_2|\leq 10} C2^{2k_1}\|g_{k_1}\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} \leq C2^{-m+\delta m}\epsilon_0. \tag{7.4}$$

Meanwhile, after doing integration by parts in "η" once, the following estimate also holds,

$$\sum_{|k_1-k_2|\leq 10} \|B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2} \leq \sum_{|k_1-k_2|\leq 10} C2^{2k_1}2^{-m-k+k_{1,+}}\big(\|e^{-it\Lambda}g_{k_1}\|_{L^\infty} + \|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\big)$$

$$\times \big(\|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|\nabla_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-k_1}\|g_{k_1}(t)\|_{L^2}\big) \leq C2^{-2m-k+2\tilde{\delta}m}\epsilon_0. \tag{7.5}$$

Now, we consider the case when $k_2 \leq k_1 - 10$ and $k_{1,-} + k_2 \leq -18m/19$. Similar to the proof of the estimate (5.10) in Lemma 5.2, from the estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\sum_{k_{1,-}+k_2\leq -18m/19} \Big\| \sum_{\nu\in\{+,-\}} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\Big\|$$

$$\leq \sum_{k_{1,-}+k_2\leq -18m/19} C\|g_{k_1}(t)\|_{L^2}\min\{2^{2k_1+k_2}\|g_{k_2}(t)\|_{L^2},$$
$$\times 2^{k_1+3k_2}\|\widehat{g}_{k_2}(t,\xi)\|_{L^\infty_\xi}+2^{2k_1+2k_2}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2}(\xi)\|_{L^\infty_\xi}\}$$
$$\leq \sum_{k_{1,-}+k_2\leq -18m/19} C2^{3\tilde{\delta}m}\min\{2^{2k_{1,-}+k_2},2^{2k_2}\big(2^{k_{1,-}+k_2+m}+2^{2k_{1,-}+2k_2+2m}\big)\}$$
$$\leq C2^{-21m/20}\epsilon_0.$$

Lastly, we consider the case when $k_2 \leq k_1 - 10$ and $k_{1,-} + k_2 \geq -18m/19$. After doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_2\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}&\|B^{\mu,\nu,j_1,j_2}_{k,k_1,k_2}(t,\xi)\|_{L^2}\\ &\leq \sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}} C2^{2k_1}\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty}\|g_{k_1,j_1}\|_{L^2}\\ &+\sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}} C2^{2k_1}\|e^{-it\Lambda}g_{k_1,j_1}\|_{L^\infty}\|g_{k_2,j_2}\|_{L^2}\\ &\leq C2^{-3m-2k_2-k_{1,-}+3\beta m}\epsilon_0\leq C2^{-21m/20}\epsilon_0.\end{aligned}\tag{7.6}$$

Combining the estimates (7.4), (7.5), and (7.6), it is easy to see that our desired estimate (7.2) holds. □

LEMMA 7.2. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for any* $t\in[2^{m-1},2^m]$*,*

$$\|\partial_t\widehat{\Gamma_1\Gamma_2 g_k}(t,\xi)-\sum_{\nu\in\{+,-\}}\sum_{(k_1,k_2)\in\chi_k^2}\widetilde{B}^{+,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\leq C2^{-m+\tilde{\delta}m+\delta m}\big(1+2^{2\tilde{\delta}m+k+5k_+}\big)\epsilon_0,\tag{7.7}$$

where C is some absolute constant, $\Gamma_1,\Gamma_2\in\{L,\Omega\}$ *and* $\widetilde{B}^{+,\nu}_{k,k_1,k_2}(t,\xi)$ *is defined as follows,*

$$\widetilde{B}^{+,\nu}_{k,k_1,k_2}(t,\xi):=\int_{\mathbb{R}^2}e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma_1\Gamma_2 g_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\psi_k(\xi)d\eta.\tag{7.8}$$

Proof. – From (6.113) in Proposition 6.9, (6.130) in Proposition 6.13, and (7.13) in Lemma 7.4, we know that all terms except quadratic terms inside $\partial_t\widehat{\Gamma_1\Gamma_2 g_k}(t,\xi)$ already satisfy the desired estimate 7.7. Hence, we only need to estimate the quadratic terms. Based on the possible size of k_1 and k_2, we separate into two cases as follows.

If $(k_1,k_2)\in\chi_k^1$*, i.e.,* $|k_1-k_2|\leq 10$*.* – Note that the following equality holds,

$$\Gamma^1_\xi\Gamma^2_\xi B^{\mu,\nu}_{k_1,k_2}(t,\xi)=\sum_{i=1,2,3}K^{\mu,\nu,1;i}_{k_1,k_2},$$
$$K^{\mu,\nu,1;1}_{k_1,k_2}:=\int_{\mathbb{R}^2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}\Gamma^1_\xi\Gamma^2_\xi\big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\big)\widehat{g^{\nu}_{k_2}}(t,\eta)d\eta,$$
$$K^{\mu,\nu,1;2}_{k_1,k_2}:=\sum_{l,m=\{1,2\}}\int_{\mathbb{R}^2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}it\big(\Gamma^l_\xi\Phi^{\mu,\nu}(\xi,\eta)\big)\Gamma^n_\xi\big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$

$$
\begin{aligned}
&\times \widehat{g_{k_1}^{\mu}}(t,\xi-\eta)\big)\widehat{g_{k_2}^{\nu}}(t,\eta)d\eta,\\
K_{k_1,k_2}^{\mu,\nu,1;3} &:= -\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}t^2\big(\Gamma_\xi^1\Phi^{\mu,\nu}(\xi,\eta)\Gamma_\xi^2\Phi^{\mu,\nu}(\xi,\eta)\big)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\\
&\times \widehat{g_{k_1}^{\mu}}(t,\xi-\eta)\big)\widehat{g_{k_2}^{\nu}}(t,\eta)d\eta.
\end{aligned}
$$

From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{|k_1-k_2|\le 10}\|K_{k_1,k_2}^{\mu,\nu,1;1}\|_{L^2} &\le C2^{2k_1}\big(2^{2k}\|\nabla_\xi^2\widehat{g}_{k_1}(t,\xi)\|_{L^2}+2^k\|\nabla_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}+\|\widehat{g}_{k_1}(t,\xi)\|_{L^2}\big)\\
&\times\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\le C2^{-m+\tilde{\delta}m}\epsilon_0.
\end{aligned}
$$

We do integration by parts in "η" once for $K_{k_1,k_2}^{\mu,\nu,1;2}$ and do integration by parts in "η" twice for $K_{k_1,k_2}^{\mu,\nu,1;3}$. As a result, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{|k_1-k_2|\le 10}\sum_{i=2,3}\|K_{k_1,k_2}^{\mu,\nu,1;i}\|_{L^2}\\
&\le\sum_{|k_1-k_2|\le 10}C2^{2k_1}\big(\sum_{i=0,1,2}2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_1}(t,\xi)\|_{L^2}+2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_2}(t,\xi)\|_{L^2}\big)\\
&\quad\times\big(\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}+\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\big)\\
&\quad+\sum_{|k_1-k_2|\le 10}\sum_{j_1\ge\max\{-k_{1,-},j_2\}}C2^{4k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g}_{k_2,j_2}(t,\xi)]\|_{L^\infty}\|\nabla_\xi\widehat{g}_{k_1,j_1}(t,\xi)\|_{L^2}\\
&\quad+\sum_{|k_1-k_2|\le 10}\sum_{j_2\ge\max\{-k_{2,-},j_1\}}C2^{4k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g}_{k_1,j_1}(t,\xi)]\|_{L^\infty}\|\nabla_\xi\widehat{g}_{k_2,j_2}(t,\xi)\|_{L^2}\\
&\le C2^{-m+\tilde{\delta}m}\epsilon_0+\sum_{j_1\ge -k_{1,-}}C2^{-m+4k_1+2j_1}\|\varphi_{j_1}^{k_1}(x)g_{k_1}(t)\|_{L^2}\big(\sum_{j_2\ge j_1}2^{j_2}\|\varphi_{j_2}^{k_2}(x)g_{k_2}(t)\|_{L^2}\big)\\
&\quad+\sum_{j_2\ge -k_{2,-}}C2^{-m+4k_1+2j_2}\|\varphi_{j_2}^{k_2}(x)g_{k_2}(t)\|_{L^2}\big(\sum_{j_1\ge j_2}2^{j_1}\|\varphi_{j_1}^{k_1}(x)g_{k_1}(t)\|_{L^2}\big)\\
&\le C2^{-m+\tilde{\delta}m}\epsilon_0.
\end{aligned}
$$

If $(k_1,k_2)\in\chi_k^2$, *i.e.,* $k_2\le k_1-10$. – For this case we have $\mu=+$. We separate it into two cases based on the size of k_1+k_2. If $k_1+k_2\le -18m/19$, the following estimate holds from estimates (5.15) in Lemma 5.3,

$$
\begin{aligned}
&\sum_{i=1,2,3}\Big\|\sum_{\nu\in\{+,-\}}K_{k_1,k_2}^{+,\nu,1;i}\Big\|_{L^2}+\Big\|\sum_{\nu\in\{+,-\}}\widetilde{B}_{k,k_1,k_2}^{+,\nu}(t,\xi)\Big\|_{L^2}\\
&\le\Big(\big(\sum_{i=0,1,2}2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_1}(t,\xi)\|_{L^2}\big)\\
&\quad+2^{m+k_1+k_2}\big(\sum_{i=0,1}2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_1}(t,\xi)\|_{L^2}\big)+2^{2m+2k_1+2k_2}\|g_{k_1}(t)\|_{L^2}\Big)\\
&\quad\times C\min\{2^{k_1+3k_2}\|\widehat{g}_{k_2}(t)\|_{L_\xi^\infty}+2^{2k_1+2k_2}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2}(\xi)\|_{L_\xi^\infty},2^{2k_1+k_2}\|g_{k_2}\|_{L^2}\}
\end{aligned}
$$

$$\le C(2^{k_1+\tilde{\delta}m} + 2^{2m+3k_1+2k_2+2\tilde{\delta}m})\min\{2^{k_1+k_2}, 2^{3k_2+m} + 2^{k_1+4k_2+2m}\}\epsilon_0$$
$$\le C2^{-m-\beta m}\epsilon_0,$$

where C is some absolute constant.

Now, we will rule out the case when k_1 is relatively large. Same as before, we move the derivative $\nabla_\xi = -\nabla_\eta$ in front of $\widehat{g}_{k_1}(t,\xi-\eta)$ around by doing integration by parts in η such that there is no derivative in front of $\widehat{g}_{k_1}(t,\xi-\eta)$. As a result, if $k_1+k_2 \ge -18m/19$ and $k_1 \ge 5\beta m$, the following estimate holds for some absolute constant C,

$$\begin{aligned}
\sum_{k_1+k_2\ge -18m/19, k_1\ge 5\beta m}\ \sum_{i=1,2,3} & \|K^{+,\nu,1;i}_{k_1,k_2}\|_{L^2} + \|\widetilde{B}^{+,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2} \\
&\le C2^{2m+2k_1+k_2+4k_{1,+}}\|g_{k_1}(t)\|_{L^2} \\
&\quad\times\big(\|\nabla^2_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-k_2}\|\nabla_\xi\widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-2k_2}\|g_{k_2}(t)\|_{L^2}\big) \\
&\le \sum_{k_1+k_2\ge -18m/19, k_1\ge 5\beta m} C2^{2m+\beta m+2k_1-k_2-(N_0-10)k_{1,+}}\epsilon_1^2 \le C2^{-m-\beta m}\epsilon_0.
\end{aligned}$$

Lastly, we consider the case when $k_1+k_2\ge -18m/19$ and $k_1 \le 5\beta m$. Note that

$$\Gamma^1_\xi\Gamma^2_\xi B^{+,\nu}_{k_1,k_2}(t,\xi) - \int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)d\eta = \sum_{i=1}^{4} K^{+,\nu,2;i}_{k_1,k_2},$$

where

$$\begin{aligned}
K^{+,\nu,2;1}_{k_1,k_2} &= \int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{\Gamma^1\Gamma^2 g^\nu_{k_2}}(t,\eta)d\eta,\\
K^{+,\nu,2;2}_{k_1,k_2} &= \sum_{j_1\ge k_{1,-}, j_2\ge -k_{2,-}} K^{+,\nu,2;2}_{k_1,j_1,k_2,j_2},\\
K^{+,\nu,2;2}_{k_1,j_1,k_2,j_2} &:= \sum_{(l,n)\in\{(1,2),(2,1)\}}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\Big[\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma^l g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2,j_2}}(t,\eta)\\
&\quad + (\Gamma^l_\xi + \Gamma^l_\eta + d_{\Gamma^l})\tilde{q}_{+,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma^n g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)\\
&\quad + \widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2,j_2}}(t,\eta)\big)\\
&\quad + it(\Gamma^l_\xi+\Gamma^l_\eta)\Phi^{+,\nu}(\xi,\eta)(\Gamma^n_\xi + \Gamma^n_\eta + d_{\Gamma^n})\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)\\
&\quad + (\Gamma^1_\xi+\Gamma^1_\eta+d_{\Gamma^1})(\Gamma^2_\xi+\Gamma^2_\eta+d_{\Gamma^2})\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)d\eta.\\
K^{+\nu,2;3}_{k_1,k_2} &= \sum_{(l,n)\in\{(1,2),(2,1)\}}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} it(\Gamma^l_\xi+\Gamma^l_\eta)\Phi^{+,\nu}(\xi,\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\\
&\quad\times\big(\widehat{g^\nu_{k_2}}(t,\eta)\widehat{\Gamma^n g_{k_1}}(t,\xi-\eta) + \widehat{g_{k_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2}}(t,\eta)\big)d\eta\\
K^{+,\nu,2;4}_{k_1,k_2} &= -\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}t^2(\Gamma^1_\xi+\Gamma^1_\eta)\Phi^{\mu,\nu}(\xi,\eta)(\Gamma^2_\xi+\Gamma^2_\eta)\Phi^{+,\nu}(\xi,\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\\
&\quad\times\widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)d\eta.
\end{aligned}$$

From the L^2-L^∞ type estimate (2.5) in Lemma 2.2, the following estimate holds,

$$\|K^{+,\nu,2;1}_{k_1,k_2}\|_{L^2} \le C2^{2k_1}\|\Gamma^1\Gamma^2 g_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty} \le C2^{-m+\tilde{\delta}m}\epsilon_0,$$

where C is some absolute constant. Now, we proceed to estimate $K^{+,\nu,2;2}_{k_1,k_2}$. By doing integration by parts in η many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. From the $L^2 - L^\infty$ type estimate (2.5) in Lemma 2.2, the following estimate holds when $\max\{j_1, j_2\} \geq m + k_{1,-} - \beta m$,

$$
\begin{aligned}
\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} & \|K^{+,\nu,2;2}_{k_1,j_1,k_2,j_2}\|_{L^2} \\
& \leq \sum_{j_1\geq \max\{m+k_{1,-}-\beta m, j_2\}} C 2^{2k_1}\big(2^{j_1+k_1+k_2+j_2} + 2^{m+k_1+k_2}\big) \\
& \quad \times \|g_{k_1,j_1}(t)\|_{L^2} 2^{-m} \|g_{k_2,j_2}(t)\|_{L^1} \\
& \quad + \sum_{j_2\geq \max\{m+k_{1,-}-\beta m, j_1\}} C 2^{2k_1}\big(2^{j_1+k_1+k_2+j_2} + 2^{m+k_1+k_2}\big) \\
& \quad \times \|g_{k_2,j_2}(t)\|_{L^2} 2^{-m} \|g_{k_1,j_1}(t)\|_{L^1} \\
& \leq C 2^{-2m-k_2+20\beta m}\epsilon_0 \leq C 2^{-m-\beta m}\epsilon_0,
\end{aligned}
$$

where C is some absolute constant.

Lastly, it remains to consider $K^{+,\nu,2;i}_{k_1,k_2}$, $i \in \{3,4\}$. We do integration by parts in "η" once for $K^{+,\nu,2;3}_{k_1,k_2}$ and do integration by parts in "η" twice for $K^{+,\nu,2;4}_{k_1,k_2}$. As a result, the following estimate holds,

$$
\begin{aligned}
& \|K^{+,\nu,2;3}_{k_1,k_2}\|_{L^2} + \|K^{+,\nu,2;4}_{k_1,k_2}\|_{L^2} \\
& \quad \leq C\Big(\sum_{i=0,1,2} 2^{ik_2}\|\nabla_\xi^i \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{ik_1}\|\nabla_\xi^i \widehat{g}_{k_1}(t,\xi)\|_{L^2}\Big) \\
& \qquad \times \big(2^{2k_1}\|e^{-it\Lambda} g_{k_1}\|_{L^\infty} + 2^{k_1+k_2}\|e^{-it\Lambda} g_{k_2}\|_{L^\infty}\big) \\
& \qquad + \sum_{j_1\geq \max\{-k_{1,-}, j_2\}, j_2\geq -k_{2,-}} C 2^{-m+3k_1+k_2+j_1+2j_2}\|\varphi^{k_1}_{j_1}(x) g_{k_1}(t)\|_{L^2}\|\varphi^{k_2}_{j_2}(x) g_{k_2}(t)\|_{L^2} \\
& \qquad + \sum_{j_2\geq \max\{-k_{2,-}, j_1\}, j_1\geq -k_{1,-}} C 2^{-m+3k_1+k_2+j_2+2j_1}\|\varphi^{k_1}_{j_1}(x) g_{k_1}(t)\|_{L^2}\|\varphi^{k_2}_{j_2}(x) g_{k_2}(t)\|_{L^2} \\
& \quad \leq C 2^{-m+2\tilde{\delta} m+\delta m/2+k}\epsilon_0,
\end{aligned}
$$

where C is some absolute constant, hence finishing the proof. □

The rest of this section is devoted to prove the weighted norm estimates for the remainder term $\mathcal{R}_1$ in (4.35), which will be done by using the fixed point type formulation (3.8). Before that, we first prove the weighted norm estimates for a very general multilinear form, which will be used as black boxes.

For $g_i \in H^{N_0-10} \cap Z_1 \cap Z_2$, $i \in \{1, \ldots, 5\}$, we define a multilinear form as follows,

$$
\begin{aligned}
& Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_1(t), g_2(t), g_3(t), g_4(t), g_5(t))(\xi) \\
& \quad := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')} q^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma') \\
& \qquad \times \widehat{g_1^\tau}(t,\xi-\eta)\widehat{g_2^\kappa}(t,\eta-\sigma)\widehat{g_3^\iota}(t,\sigma-\eta')\widehat{g_4^\mu}(t,\eta'-\sigma')\widehat{g_5^\nu}(t,\sigma')\psi_k(\xi)\, d\sigma' d\eta' d\eta d\sigma,
\end{aligned}
$$

where the phase $\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$ is defined as follows,

$$\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma') = \Lambda(|\xi|)-\tau\Lambda(|\xi-\eta|)-\kappa\Lambda(|\eta-\sigma|)-\iota\Lambda(|\sigma-\eta'|)-\mu\Lambda(|\eta'-\sigma'|)-\nu\Lambda(|\sigma'|),$$

and the symbol $q^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$ satisfies the following estimate for some absolute constant C,

$$\|q^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma-\eta')\psi_{k_4}(\eta'-\sigma')\psi_{k_5}(\sigma')\|_{\mathcal{S}^\infty} \leq C2^{2k_1+6\max\{k_1,\dots,k_5\}_+}.$$

For $i\in\{0,1,2\}$, we define auxiliary function spaces as follows,

$$\|f\|_{\widetilde{Z}_i} := \sup_{k\in\mathbb{Z}}\sup_{j\geq -k_-}\|f\|_{\widetilde{B}^i_{k,j}}, \quad \|f\|_{\widetilde{B}^i_{k,j}} := 2^{(1-\delta)k+k_++(20-5i)k_++ij+\delta j}\|\varphi^k_j(x)P_kf\|_{L^2}. \tag{7.9}$$

From the above definition and the definition of Z_i-norm, $i\in\{1,2\}$, in (1.22) and (1.23), we know that the following estimates hold for some absolute constant C,

$$\sum_{k\in\mathbb{Z}}2^{k+(20-5i)k_+}\|\nabla^i_\xi\widehat{f_k}(t,\xi)\|_{L^2}\leq C\|f\|_{\widetilde{Z}_i}, \qquad \|f\|_{Z_l}\leq C\|f\|_{\widetilde{Z}_l},$$

where $i\in\{0,1,2\}$, $l\in\{1,2\}$.

LEMMA 7.3. – *Let $g_i(t)\in H^{N_0-10}\cap Z_1\cap Z_2$, $i\in\{1,\dots,5\}$. Assume that the following estimate holds for any $t\in[2^{m-1},2^m]$, $m\in\mathbb{Z}_+$,*

$$2^{-\delta m}\|g_i(t)\|_{H^{N_0-10}}+\|g_i(t)\|_{Z_1}+2^{-\tilde{\delta}m}\|g_i(t)\|_{Z_2}\leq\epsilon_1:=\epsilon_0^{5/6}, \quad i\in\{1,\dots,5\},$$

then the following estimates hold for any $t\in[2^{m-1},2^m]$ and any $\mu,\nu,\kappa,\iota,\tau\in\{+,-\}$,

$$\sum_{i=0,1,2}2^{(3-i)m}\|\mathcal{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_1(t),g_2(t),g_3(t),g_4(t),g_5(t))(\xi)\big]\|_{\widetilde{Z}_i}\leq C2^{-m/2+190\beta m}\epsilon_0^2, \tag{7.10}$$

where C is some absolute constant.

Proof. – As usual, we rule out the very high frequency case and the very low frequency case first. Without loss of generality, we assume that $k_5\leq k_4\leq k_3\leq k_2\leq k_1$. From the $L^2-L^\infty-L^\infty-L^\infty-L^\infty$ type multilinear estimate and the $L^\infty\to L^2$ type Sobolev estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}\sum_{i=0,1,2}&2^{(3-i)m}\|\mathcal{F}^{-1}[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1}(t),g_{2,k_2}(t),g_{3,k_3}(t),g_{4,k_4}(t),g_{5,k_5}(t))(\xi)]\|_{\widetilde{B}^i_{k,j}}\\ &\leq C2^{3m+(2+\delta)j}2^{30k_{1,+}+(1-\delta)k+k_5}\|g_{k_1}\|_{L^2}\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\\ &\quad\times\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}\|e^{-it\Lambda}g_{k_4}\|_{L^\infty}\|g_{k_5}\|_{L^2}.\end{aligned} \tag{7.11}$$

From estimate (7.11), we can rule out the case when $k_{1,+}\geq(3m+2j)/(N_0-45)$ or $k_5\leq -3m-2(1+2\delta)j$, or $k\leq -3m-2(1+2\delta)j$. Hence it would be sufficient to consider fixed k,k_1,k_2,k_3,k_4, and k_5 in the following range,

$$-3m-2(1+2\delta)j\leq k_5,k\leq k_1+2\leq(3m+2j)/(N_0-45). \tag{7.12}$$

From now on, $k, k_i, i \in \{1,\dots,5\}$, are restricted inside the range (7.12). We first consider the case when $j \geq (1+\delta)\big(m+k_{1,+}\big)+\beta m$. For this case, we do spatial localization for inputs "g_{k_1}" and "g_{k_2}". Note that the following estimate holds for the case we are considering,

$$2^{j-10} \leq \big|\nabla_\xi\big[x\cdot\xi + t\,\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')\big]\big|\varphi^k_j(x) \leq 2^{j+10}.$$

Therefore, by doing integration by parts in "ξ" many times, we can rule out the case when $\min\{j_1,j_2\} \leq j-\delta j-\delta m$, where j_1 and j_2 are the spatial concentrations of g_{k_1} and g_{k_2} respectively. For the case when $\min\{j_1,j_2\}\geq j-\delta j-\delta m$, the following estimate holds from the $L^2-L^\infty-L^\infty-L^\infty-L^\infty$ type multilinear estimate,

$$\begin{aligned}
&\sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m}\ \sum_{i=0,1,2} 2^{(3-i)m}\big\|\mathscr{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1,j_1}(t), g_{2,k_2,j_2}(t),\\
&\qquad\qquad g_{3,k_3}(t), g_{4,k_4}(t), g_{5,k_5}(t))(\xi)\big]\big\|_{\widetilde{B}^i_{k,j}}\\
&\leq \sum_{i=0,1,2}\ \sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m} C2^{(3-i)m+ij+\delta j+3\beta m+(3-\delta)k_1+30k_{1,+}}\\
&\quad\times \|g_{1,k_1,j_1}\|_{L^2}2^{k_2}\|g_{2,k_2,j_2}\|_{L^2}\|e^{-it\Lambda}g_{3,k_3}\|_{L^\infty}\|e^{-it\Lambda}g_{4,k_4}\|_{L^\infty}\|e^{-it\Lambda}g_{5,k_5}\|_{L^\infty}\\
&\leq C2^{-m/2+50\beta m}\epsilon_0^2,
\end{aligned}$$

where C is some absolute constant.

It remains to consider the case when $j \leq (1+\delta)\big(m+k_{1,+}\big)+\beta m$. Recall (7.12). Note that j now is bounded, we have $-6m\leq k_5\leq k_1\leq 5\beta m$. We split into three cases based on sizes of the difference between k_1 and k_2 and the difference between k_2 and k_3 as follows.

If $k_2 \leq k_1-10$. – For this case, we have a good lower bound for $\nabla_\eta\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$. Hence, we can do integration by parts in "η" many times to rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. From the $L^2-L^\infty-L^\infty-L^\infty-L^\infty$ type multilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}\ \sum_{i=0,1,2} 2^{(3-i)m}\big\|\mathscr{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1,j_1}(t), g_{2,k_2,j_2}(t),\\
&\qquad\qquad g_{3,k_3}(t), g_{4,k_4}(t), g_{5,k_5}(t))(\xi)\big]\big\|_{\widetilde{B}^i_{k,j}}\\
&\leq \sum_{j_1\geq\max\{-k_{1,-},j_2,m+k_{1,-}-\beta m\}} C2^{3m+4\beta m+3k_1+30k_{1,+}}\|g_{1,k_1,j_1}\|_{L^2}\|e^{-it\Lambda}g_{2,k_2,j_2}\|_{L^\infty}\\
&\quad\times\|e^{-it\Lambda}g_{3,k_3}\|_{L^\infty}\|e^{-it\Lambda}g_{4,k_4}\|_{L^\infty}\|e^{-it\Lambda}g_{5,k_5}\|_{L^\infty}\\
&\quad+\sum_{j_2\geq\max\{-k_{2,-},j_1,m+k_{1,-}-\beta m\}} C2^{3m+4\beta m+3k_1+30k_{1,+}+k_4+k_5}\|g_{2,k_2,j_2}\|_{L^2}\\
&\quad\times\|e^{-it\Lambda}g_{1,k_1,j_1}\|_{L^\infty}\|e^{-it\Lambda}g_{3,k_3}\|_{L^\infty}\|g_{4,k_4}\|_{L^2}\|g_{5,k_5}\|_{L^2}\\
&\leq C2^{-m/2+180\beta m}\epsilon_0^2.
\end{aligned}$$

If $|k_1-k_2|\leq 10$ and $k_3\leq k_1-20$. – Note that, $\nabla_\sigma\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$ has a good lower bound for the case we are considering. Hence, by doing integration by parts in σ, we can rule out the case when $\max\{j_2,j_3\}\leq m+k_{2,-}-\beta m$, where j_2 and j_3 are the spatial concentrations of inputs g_{k_2} and g_{k_3} respectively. From the $L^2-L^\infty-L^\infty-L^\infty-L^\infty$

type multilinear estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{\max\{j_2,j_3\}\ge m+k_{2,-}-\beta m}\ \sum_{i=0,1,2} 2^{(3-i)m}\|\mathscr{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1}(t), g_{2,k_2,j_2}(t), g_{3,k_3,j_3}(t),\\
&\qquad\qquad g_{4,k_4}(t), g_{5,k_5}(t))(\xi)\big]\|_{\widetilde{B}^i_{k,j}}\\
&\le \sum_{j_2\ge\max\{-k_{2,-},j_3,m+k_{1,-}-\beta m\}} C2^{3m+4\beta m+3k_1+30k_{1,+}}\|g_{3,k_3,j_3}\|_{L^2}\\
&\quad\times\|e^{-it\Lambda}g_{2,k_2,j_2}\|_{L^\infty}\|e^{-it\Lambda}g_{1,k_1}\|_{L^\infty}\|e^{-it\Lambda}g_{4,k_4}\|_{L^\infty}\|e^{-it\Lambda}g_{k_5}\|_{L^\infty}\\
&\quad+\sum_{j_3\ge\max\{-k_{3,-},j_2,m+k_{1,-}-\beta m\}} C2^{3m+4\beta m+3k_1+30k_{1,+}+k_4+k_5}\|g_{3,k_3,j_3}\|_{L^2}\\
&\quad\times\|e^{-it\Lambda}g_{2,k_2,j_2}\|_{L^\infty}\|e^{-it\Lambda}g_{1,k_1}\|_{L^\infty}\|g_{4,k_4}\|_{L^2}\|g_{5,k_5}\|_{L^2}\\
&\le C2^{-m/2+180\beta m}\epsilon_0^2.
\end{aligned}
$$

If $|k_1-k_2|\le 10$ *and* $|k_2-k_3|\le 10$. – This case is straightforward. By the $L^2-L^\infty-L^\infty-L^\infty-L^\infty$ type multilinear estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=0,1,2} 2^{(3-i)m}\|\mathscr{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1}(t), g_{2,k_2}(t), g_{3,k_3}(t), g_{4,k_4}(t), g_{5,k_5}(t))(\xi)\big]\|_{\widetilde{B}^i_{k,j}}\\
&\quad\le C2^{3m+4\beta m+3k_1+30k_{1,+}}\|g_{5,k_5}(t)\|_{L^2}\|e^{-it\Lambda}g_{1,k_1}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{2,k_2}(t)\|_{L^\infty}\\
&\quad\times\|e^{-it\Lambda}g_{3,k_3}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{4,k_4}(t)\|_{L^\infty}\\
&\quad\le C2^{-m/2+180\beta m}\epsilon_0^2.
\end{aligned}
$$

To sum up, our desired estimate (7.10) holds, hence finishing the proof. □

With the multilinear estimate (7.10) in the above lemma, now we are ready to estimate the Z_i-norm of the quintic and higher order remainder term $\mathscr{R}_1$, $i\in\{1,2\}$.

LEMMA 7.4. – *Under the bootstrap assumption* (4.49)*, there exists some absolute constant C such that the following estimate holds for the profile of the remainder term* $\mathscr{R}_1$*,*

$$
\sup_{t\in[2^{m-1},2^m]}\sum_{i=1,2}\|e^{it\Lambda}\mathscr{R}_1\|_{Z_i}\le C2^{-3m/2+200\beta m}\epsilon_0. \tag{7.13}
$$

Proof. – Recall the definition of $u=\tilde{\Lambda}h+i\tilde{\psi}$ and the definition of v in (4.20). To estimate the weighted norms of the reminder term $e^{it\Lambda}\mathscr{R}_1$, from estimate (7.10) in Lemma 7.3, we know that it would be sufficient to estimate the weighted norms of $e^{it\Lambda}\Lambda_{\ge5}[B(h)\psi]=e^{it\Lambda}\Lambda_{\ge5}[\partial_z\varphi](t)|_{z=0}$.

Recall the fixed point type formulation for $\nabla_{x,z}\varphi$ in (3.8). We decompose $\Lambda_{\ge5}[g_i(z)]$ into two parts: one of them doesn't depend on $\Lambda_{\ge5}[\nabla_{x,z}\varphi]$ while the other part does depend (linearly depend) on $\Lambda_{\ge5}[\nabla_{x,z}\varphi]$. For the first part, estimating (7.10) in Lemma 7.3 is very sufficient. Hence, it remains to estimate the second part. As usual, by doing integration by parts in ξ many times, we can rule out the case when $j\ge(1+\delta)\big(\max\{m+k_{1,+},-k_-\}\big)+\beta m$.

If $j \le (1+\delta)\big(\max\{m+k_{1,+}, -k_-\}\big) + \beta m$, from the estimate (7.10) in Lemma 7.3 and the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=1,2} \|e^{it\Lambda}\Lambda_{\ge 5}[\nabla_{x,z}\varphi](t)\|_{L^\infty_z Z_i} \le C2^{-3m/2+200\beta m}\epsilon_0 \\
&\quad + C2^{2m+3\beta m}\big(\|\Lambda_{\ge 6}[\nabla_{x,z}\varphi](t,\xi)\|_{L^\infty_z H^{15}}\|e^{-it\Lambda}g\|_{W^{20,0}} + \|g\|_{H^{20}}\||\nabla|\Lambda_5[\nabla_{x,z}\varphi]\|_{L^\infty_z H^{15}}\big) \\
&\quad + C2^{3\beta m}\|g\|_{H^{20}}\|\Lambda_{\ge 5}[\nabla_{x,z}\varphi](t,\xi)\|_{L^\infty_z H^{20}}.
\end{aligned} \tag{7.14}
$$

Similar to the proof of (3.17) in Lemma 3.2, the following estimate holds for $i \in \{5,6\}$,

$$
\begin{aligned}
&\|\Lambda_{\ge i}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^{20}} \\
&\quad \le C\big[\|(h,\psi)\|_{W^{30,1}}\|(h,\psi)\|^{i-2}_{W^{30,0}}\|(h,\psi)\|_{H^{30}} + \|(h,\psi)\|_{W^{30}}\|\Lambda_{\ge i}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^{20}}\big],
\end{aligned}
$$

where C is some absolute constant. Under the bootstrap assumption (4.49), the above estimate further implies the following estimate,

$$
\begin{aligned}
&\|\Lambda_{\ge i}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^{20}} \\
&\quad \le 2C\|(h,\psi)\|_{W^{30,1}}\|(h,\psi)\|^{i-2}_{W^{30,0}}\|(h,\psi)\|_{H^{30}} \le C2^{-im/2+\beta m}\epsilon_0^2, \quad i \in \{5,6\}.
\end{aligned} \tag{7.15}
$$

Therefore, from the estimates (7.14) and (7.15) and the estimate (7.10) in Lemma 7.3, we obtain the following estimate,

$$
\sum_{i=1,2} \|e^{it\Lambda}\Lambda_{\ge 5}[\nabla_{x,z}\varphi](t)\|_{L^\infty_z Z_i} \le C2^{-3m/2+200\beta m}\epsilon_0, \tag{7.16}
$$

where C is some absolute constant, hence finishing the proof of the desired estimate (7.13). □

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QUADRIC RANK LOCI ON MODULI OF CURVES AND $K3$ SURFACES

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ABSTRACT. – Assuming that $\phi : \mathrm{Sym}^2(\mathscr{E}) \to \mathscr{F}$ is a morphism of vector bundles on a variety X, we compute the class of the locus in X where $\mathrm{Ker}(\phi)$ contains a quadric of prescribed rank. Our formulas have many applications to moduli theory: (i) we find a simple proof of Borcherds' result that the Hodge class on the moduli space of polarized $K3$ surfaces of fixed genus is of Noether-Lefschetz type, (ii) we construct an explicit canonical divisor on the Hurwitz space parametrizing degree k covers of $\mathbf{P}^1$ from curves of genus $2k-1$, (iii) we provide a closed formula for the Petri divisor on $\overline{\mathcal{M}}_g$ of canonical curves which lie on a rank 3 quadric and (iv) we construct myriads of effective divisors of small slope on $\overline{\mathcal{M}}_g$.

RÉSUMÉ. – Étant donné deux fibrés vectoriels $\mathscr{E}$ et $\mathscr{F}$ sur une variété X et une application de $\mathrm{Sym}^2(\mathscr{E})$ dans $\mathscr{F}$, nous calculons la classe de cohomologie du lieu en X où le kernel de cette application contient une quadrique de rang donné. Nos formules ont plusieurs applications à la théorie d'espaces des modules: (i) nous trouvons une preuve simple du théorème de Bocherds qui établit que la classe de Hodge dans l'espace de modules de surfaces K3 polarisés avec genre fixé, est du type Noether-Lefschetz, (ii) nous construisons un diviseur canonique explicite dans l'espace d'Hurwitz paramétrisant les applications de degré k de courbes du genre $2k-1$ sur la droite projective, (iii) nous fournissons une formule fermée pour le diviseur de Petri dans l'espace de modules de courbes consistant de courbes canoniques contenues d'une quadrique de rang 3 et (iv) nous construisons une myriade de diviseurs de petite pente dans $\overline{\mathcal{M}}_g$.

1. Introduction

Let X be an algebraic variety and let $\mathscr{E}$ and $\mathscr{F}$ be two vector bundles on X having ranks e and f respectively. Assume we are given a morphism of vector bundles

$$\phi : \mathrm{Sym}^2(\mathscr{E}) \to \mathscr{F}.$$

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For a positive integer $r \leq e$, we define the subvariety of X consisting of points for which $\mathrm{Ker}(\phi)$ contains a quadric of corank at least r, that is,

$$\overline{\Sigma}^r_{e,f}(\phi) := \Big\{x \in X : \exists\, 0 \neq q \in \mathrm{Ker}(\phi(x)) \;\text{ with }\; \mathrm{rk}(q) \leq e - r\Big\}.$$

Since the codimension of the variety of symmetric $e \times e$-matrices of corank r is equal to $\binom{r+1}{2}$, it follows that the expected codimension of the locus $\overline{\Sigma}^r_{e,f}(\phi)$ is equal to $\binom{r+1}{2} - \binom{e+1}{2} + f + 1$. A main goal of this paper is to explicitly determine the cohomology class of this locus in terms of the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. This is achieved for every e, f and r in Theorem 4.4, using a localized Atiyah-Bott type formula. Of particular importance in moduli theory is the case when this locus is expected to be a divisor, in which case our general formula has a very simple form:

THEOREM 1.1. – *We fix integers* $0 \leq r \leq e$ *and set* $f := \binom{e+1}{2} - \binom{r+1}{2}$. *Suppose* $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ *is a morphism of vector bundles over* X. *The class of the virtual divisor* $\overline{\Sigma}^r_{e,f}(\phi)$ *is given by the formula*

$$[\overline{\Sigma}^r_{e,f}(\phi)] = A^r_e\Big(c_1(\mathcal{F}) - \frac{2f}{e}c_1(\mathcal{E})\Big) \in H^2(X, \mathbb{Q}),$$

where

$$A^r_e := \frac{\binom{e}{r}\binom{e+1}{r-1}\cdots\binom{e+r-1}{1}}{\binom{1}{0}\binom{3}{1}\binom{5}{2}\cdots\binom{2r-1}{r-1}}.$$

The quantity A^r_e is the degree of the variety of symmetric $e \times e$-matrices of corank at least r inside the projective space of all symmetric $e \times e$ matrices, see [31].

Before introducing a second type of degeneracy loci, we give a definition. If V is a vector space, a pencil of quadrics $\ell \subseteq \mathbf{P}(\mathrm{Sym}^2(V))$ is said to be *degenerate* if the intersection of ℓ with the discriminant divisor $D(V) \subseteq \mathbf{P}(\mathrm{Sym}^2(V))$ is non-reduced. We consider a morphism $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ such that all kernels are expected to be pencils of quadrics and impose the condition that the pencil be degenerate.

THEOREM 1.2. – *We fix integers* e *and* $f = \binom{e+1}{2} - 2$ *and let* $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ *be a morphism of vector bundles. The class of the virtual divisor* $\mathfrak{Dp} := \{x \in X : \mathrm{Ker}(\phi(x))$ *is a degenerate pencil*$\}$ *equals*

$$[\mathfrak{Dp}] = (e-1)\Big(ec_1(\mathcal{F}) - (e^2 + e - 4)c_1(\mathcal{E})\Big) \in H^2(X, \mathbb{Q}).$$

Theorems 1.1 and 1.2 are motivated by fundamental questions in moduli theory and in what follows we shall discuss some of these applications, which are treated at length in the paper.

Tautological classes on moduli of polarized $K3$ surfaces. – Let $\mathcal{F}_g$ be the moduli space of quasi-polarized $K3$ surfaces $[X, L]$ of genus g, that is, satisfying $L^2 = 2g - 2$. We denote by $\pi : \mathcal{X} \to \mathcal{F}_g$ the universal $K3$ surface and choose a polarization line bundle $\mathcal{L}$ on $\mathcal{X}$. We consider the Hodge class

$$\lambda := c_1\big(\pi_*(\omega_\pi)\big) \in CH^1(\mathcal{F}_g).$$

Note that $CH^1(\mathcal{F}_g) \cong H^2(\mathcal{F}_g, \mathbb{Q})$. Inspired by Mumford's definition of the κ classes on $\mathcal{M}_g$, for integers $a, b \geq 0$, Marian, Oprea and Pandharipande [39] introduced the classes $\kappa_{a,b} \in CH^{a+2b-2}(\mathcal{F}_g)$ whose definition we recall in Section 9. In codimension 1, there are two such classes, namely

$$\kappa_{3,0} := \pi_*\Big(c_1(\mathcal{L})^3\Big) \text{ and } \kappa_{1,1} := \pi_*\Big(c_1(\mathcal{L}) \cdot c_2(\mathcal{T}_\pi)\Big) \in CH^1(\mathcal{F}_g).$$

Both these classes depend on the choice of $\mathcal{L}$, but the following linear combination

$$\gamma := \kappa_{3,0} - \frac{g-1}{4}\kappa_{1,1} \in CH^1(\mathcal{F}_g)$$

is intrinsic and independent of the polarization line bundle.

For a general element $[X, L] \in \mathcal{F}_g$ one has $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$. Imposing the condition that $\mathrm{Pic}(X)$ be of rank at least 2, one is led to the notion of Noether-Lefschetz (NL) divisor on $\mathcal{F}_g$. For non-negative integers h and d, we denote by $D_{h,d}$ the locus of quasi-polarized $K3$ surfaces $[X, L] \in \mathcal{F}_g$ such that there exists a primitive embedding of a rank 2 lattice

$$\mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot D \subseteq \mathrm{Pic}(X),$$

where $D \in \mathrm{Pic}(X)$ is a class such that $D \cdot L = d$ and $D^2 = 2h - 2$. From the Hodge Index Theorem $D_{h,d}$ is empty unless $d^2 - 4(g-1)(h-1) > 0$. Whenever non-empty, $D_{h,d}$ is pure of codimension 1.

Maulik and Pandharipande [40] conjectured that $\mathrm{Pic}(\mathcal{F}_g)$ is spanned by the Noether-Lefschetz divisors $D_{h,d}$. This has been recently proved in [6] using deep automorphic techniques. Note that the rank of $\mathrm{Pic}(\mathcal{F}_g)$ can become arbitrarily large and understanding all the relations between NL divisors remains a daunting task. Borcherds [8] using automorphic forms on $O(2, n)$ has shown that the Hodge class λ is supported on NL divisors. A second proof of this fact, via Gromov-Witten theory, is due to Pandharipande and Yin, see [46] Section 7. Using Theorem 1.1, we find very simple and explicit Noether-Lefschetz representatives of both classes λ and γ. Our methods are within the realm of algebraic geometry and we use no automorphic forms.

We produce relations among tautological classes on $\mathcal{F}_g$ using the projective geometry of embedded $K3$ surfaces of genus g. We study geometric conditions that single out *only* NL special K3 surfaces. Let us first consider the divisor in $\mathcal{F}_g$ consisting of $K3$ surfaces which lie on a rank 4 quadric. We fix a $K3$ surface $[X, L] \in \mathcal{F}_g$ with $g \geq 4$ and let $\varphi_L : X \to \mathbf{P}^g$ be the morphism induced by the polarization L. One computes $h^0(X, L^{\otimes 2}) = 4g-2$. Assuming that the image $X \subseteq \mathbf{P}^g$ is projectively normal (which holds under very mild genericity assumptions, see again Section 9), we observe that the space $I_{X,L}(2)$ of quadrics containing X has the following dimension:

$$\dim I_{X,L}(2) = \dim \mathrm{Sym}^2 H^0(X, L) - h^0(X, L^{\otimes 2}) = \binom{g-2}{2}.$$

This equals the codimension of the space of symmetric $(g+1)\times(g+1)$ matrices of rank 4. Therefore the condition that $X \subseteq \mathbf{P}^g$ lie on a rank 4 quadric is expected to be divisorial on $\mathcal{F}_g$. This expectation is easily confirmed in Proposition 9.1, and we are led to the divisor:

$$D_g^{\mathrm{rk}4} := \Big\{[X,L] \in \mathcal{F}_g : \exists\, 0 \neq q \in I_{X,L}(2),\ \ \mathrm{rk}(q) \leq 4\Big\}.$$

THEOREM 1.3. – *Set $g \geq 4$. The divisor $D_g^{\mathrm{rk}4}$ is an effective combination of NL divisors and its class is*

$$[D_g^{\mathrm{rk}4}] = A_{g+1}^{g-3}\Big((2g-1)\lambda + \frac{2}{g+1}\gamma\Big) \in CH^1(\mathcal{F}_g).$$

In order to get a second relation between λ and γ, we distinguish depending on the parity of g. For odd genus g, we obtain a second relation between λ and γ by considering the locus of $K3$ surfaces $[X,L] \in \mathcal{F}_g$ for which the embedded surface $\varphi_L : X \to \mathbf{P}^g$ has a non-trivial middle linear syzygy. In terms of Koszul cohomology groups, we set

$$\mathfrak{Kosz}_g := \Big\{[X,L] \in \mathcal{F}_g : K_{\frac{g-1}{2},1}(X,L) \neq 0\Big\}.$$

For instance $\mathfrak{Kosz}_3$ consists of quartic $K3$ surfaces for which the map $\mathrm{Sym}^2 H^0(X,L) \to H^0(X, L^{\otimes 2})$ is not an isomorphism. Voisin's solution [52] of the generic Green Conjecture on syzygies of canonical curves ensures that $\mathfrak{Kosz}_g$ is a proper locus of NL type. She proved that for a $K3$ surface $[X,L] \in \mathcal{F}_g$ with $\mathrm{Pic}(X) = \mathbb{Z}\cdot L$, the vanishing

$$K_{\frac{g-1}{2},1}(X,L) = 0$$

holds, or equivalently, $[X,L] \notin \mathfrak{Kosz}_g$. We realize $\mathfrak{Kosz}_g$ as the degeneracy locus of a morphism of two vector bundles of the same rank over $\mathcal{F}_g$, whose Chern classes can be expressed in terms of $\kappa_{1,1}$, $\kappa_{3,0}$ and λ. We then obtain the following formula (see Theorem 9.5)

$$(1) \qquad [\mathfrak{Kosz}_g] = \frac{4}{g-1}\binom{g-4}{\frac{g-3}{2}}\Big(\frac{(g-1)(g+7)}{2}\lambda + \gamma\Big) + \alpha\cdot[D_{1,1}] \in CH^1(\mathcal{F}_g),$$

where recall that $D_{1,1}$ is the NL divisor of $K3$ surfaces $[X,L]$ for which the polarization L is not globally generated. Theorems 1.3 and 9.5 then quickly imply (in the case of odd g):

THEOREM 1.4. – *Both tautological classes λ and γ on $\mathcal{F}_g$ are of Noether-Lefschetz type.*

Theorem 1.4 is proved for even genus $g \geq 8$ in Section 10 using two further geometric relations between tautological classes (in the spirit of Theorem 1.3) involving the geometry of rank 2 Lazarsfeld-Mukai bundle E_L one associates canonically to each NL-general polarized $K3$ surface $[X,L] \in \mathcal{F}_g$. The vector bundle E_L satisfies $\det(E_L) = L$ and $h^0(X, E_L) = \frac{g}{2}+2$ and has already been put to great use in [37], [42], or [52]. A direct proof of Theorem 1.4 when $g \leq 10$ has already appeared in [27].

In Section 11 we discuss an application of Theorem 1.3 to the Geometric Invariant Theory of $K3$ surfaces. The second Hilbert point $[X,L]_2$ of a suitably general polarized $K3$ surface $[X,L]$ is defined as the quotient

$$[X,H]_2 := \Big[\mathrm{Sym}^2 H^0(X,L) \longrightarrow H^0(X,L^{\otimes 2}) \longrightarrow 0\Big] \in \mathrm{Gr}\Big(\mathrm{Sym}^2 H^0(X,H), 4g-2\Big).$$

We establish the following result:

THEOREM 1.5. – *The second Hilbert point of a polarized $K3$ surface $[X, L] \in \mathcal{F}_g \setminus D_g^{\mathrm{rk}4}$ is semistable.*

Note that a similar result at the level of canonical curves has been obtained in [19].

The Petri class on $\overline{\mathcal{M}}_g$. – A non-hyperelliptic canonical curve $C \subseteq \mathbf{P}^{g-1}$ of genus g is projectively normal and lies on precisely $\binom{g-2}{2}$ quadrics. This number equals the codimension of the locus of symmetric $g \times g$-matrices of rank 3. The condition that C lies on a rank 3 quadric in its canonical embedding is divisorial and leads to the Petri divisor $\mathcal{GP}_g$ of curves $[C] \in \mathcal{M}_g$, having a pencil A such that the Petri map

$$\mu(A) : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^{\vee}) \to H^0(C, \omega_C)$$

is not injective. Using Theorem 1.1, we establish the following result:

THEOREM 1.6. – *The class of the compactified Petri divisor $\widetilde{\mathcal{GP}}_g$ on $\overline{\mathcal{M}}_g$ is given by the formula*

$$[\widetilde{\mathcal{GP}}_g] = A_g^{g-3}\Big(\frac{7g+6}{g}\lambda - \delta\Big) \in CH^1(\overline{\mathcal{M}}_g).$$

Here λ is the Hodge class on $\overline{\mathcal{M}}_g$ and δ denotes the total boundary divisor. The Petri divisor splits into components $D_{g,k}$, where $\lfloor \frac{g+2}{2} \rfloor \leq k \leq g-1$, depending on the degree of the (base point free) pencil A for which the Petri map $\mu(A)$ is not injective. With a few notable exceptions when k is extremal, the individual classes $[\overline{D}_{g,k}] \in CH^1(\overline{\mathcal{M}}_g)$ are not known. However, we predict a simple formula for the multiplicities of $\overline{D}_{g,k}$ in the expression of $[\widetilde{\mathcal{GP}}_g]$, see Conjecture 6.3.

Effective divisors on Hurwitz spaces. – We fix an integer $k \geq 4$ and denote by $\mathcal{H}_k$ the Hurwitz space parametrizing degree k covers $[f : C \to \mathbf{P}^1]$ from a smooth curve of genus $2k-1$. The space $\mathcal{H}_k$ admits a compactification $\overline{\mathcal{H}}_k$ by means of admissible covers, which is defined to be the normalization of the space constructed by Harris and Mumford in [30]. We refer to [1] for details. We denote by $\sigma : \overline{\mathcal{H}}_k \to \overline{\mathcal{M}}_{2k-1}$ the morphism assigning to each admissible cover the stabilization of the source curve. The image $\sigma(\overline{\mathcal{H}}_k)$ is the divisor $\overline{\mathcal{M}}^1_{2k-1,k}$ consisting of k-gonal curves in $\overline{\mathcal{M}}_{2k-1}$, which was studied in great detail by Harris and Mumford [30] in the course of their proof that $\overline{\mathcal{M}}_g$ is general for large genus. The birational geometry of $\overline{\mathcal{H}}_k$ is largely unknown, see however [51] for some recent results.

Let us choose a general point $[f : C \to \mathbf{P}^1] \in \mathcal{H}_k$ and denote by $A := f^*(\mathcal{O}_{\mathbf{P}^1}(1)) \in W_k^1(C)$ the pencil inducing the cover. We consider the residual linear system $L := \omega_C \otimes A^{\vee} \in W_{3k-4}^{k-1}(C)$ and denote by $\varphi_L : C \to \mathbf{P}^{k-1}$ the induced map. Under these genericity assumptions L is very ample, $H^1(C, L^{\otimes 2}) = 0$ and the image curve $\varphi_L(C)$ is projectively normal. In particular,

$$\dim I_{C,L}(2) = \dim \mathrm{Sym}^2 H^0(C, L) - h^0(C, L^{\otimes 2}) = \binom{k-3}{2},$$

which equals the codimension of the space of symmetric $k \times k$ matrices of rank 4. Imposing the condition that $C \subseteq \mathbf{P}^{k-1}$ be contained in a rank 4 quadric, we obtain a (virtual) divisor

$$\mathfrak{H}_k^{\mathrm{rk}4} := \Big\{[C, A] \in \mathcal{H}_k : \exists\, 0 \neq q \in I_{C,\omega_C \otimes A^{\vee}}(2),\ \mathrm{rk}(q) \leq 4\Big\}.$$

The condition $[C,A] \in \mathfrak{H}_k^{\mathrm{rk}4}$ amounts to representing the canonical bundle ω_C as a sum

$$\omega_C = A \otimes A_1 \otimes A_2 \tag{2}$$

of *three* pencils, that is, $h^0(C,A_1) \geq 2$ and $h^0(C,A_2) \geq 2$. To show that $\mathfrak{H}_k^{\mathrm{rk}4}$ is indeed a divisor, it suffices to exhibit a point $[C,A] \in \mathcal{H}_k$ such that (2) cannot hold. To that end, we take a general polarized $K3$ surface $[X,L] \in \mathcal{F}_{2k-1}$ carrying an elliptic pencil E with $E \cdot L = k$ (that is, a general element of the NL divisor $D_{1,k} \subseteq \mathcal{F}_{2k-1}$). If $C \in |L|$ is a smooth curve in the polarization class and $A = \mathcal{O}_C(E) \in W_k^1(C)$, we check that one has an isomorphism $I_{C,\omega_C \otimes A^\vee}(2) \cong I_{X,L(-E)}(2)$ between the spaces of quadrics containing C and $X \subseteq \mathbf{P}^{k-1}$ respectively. Showing that this latter space contains no rank 4 quadric becomes a lattice-theoretic problem inside $\mathrm{Pic}(X)$, which we solve.

We summarize our results concerning $\mathfrak{H}_k^{\mathrm{rk}4}$. We denote by $\lambda := \sigma^*(\lambda)$ the Hodge class on $\overline{\mathcal{H}}_k$ and by D_0 the boundary divisor on $\overline{\mathcal{H}}_k$ whose general point corresponds to a 1-nodal singular curve C of genus $2k-1$ and a locally free sheaf A of degree k with $h^0(C,A) \geq 2$ (see Section 12 for details).

THEOREM 1.7. – *For each $k \geq 6$, the locus $\mathfrak{H}_k^{\mathrm{rk}4}$ is an effective divisor on $\mathcal{H}_k$. Away from the union of the boundary divisors $\sigma^{-1}(\Delta_i)$ where $i = 1,\ldots,k-1$, one has the relation*

$$K_{\overline{\mathcal{H}}_k} = \frac{k-12}{k-6}\Big(7\lambda - [D_0]\Big) + \frac{k}{(k-6)A_k^{k-4}}[\overline{\mathfrak{H}}_k^{\mathrm{rk}4}].$$

Theorem 1.7 follows from applying Theorem 1.1 in the context of Hurwitz spaces to compute the class $[\overline{\mathfrak{H}}_k^{\mathrm{rk}4}]$ in terms of certain tautological classes on $\overline{\mathcal{H}}_k$, see Theorem 12.6, then comparing with the formula we find for $K_{\overline{\mathcal{H}}_k}$ in terms of those same classes. Proving that $\mathfrak{H}_k^{\mathrm{rk}4}$ is indeed a genuine divisor on $\overline{\mathcal{H}}_k$ is achieved in Theorem 12.5.

We mention the following consequence to the birational geometry of $\overline{\mathcal{H}}_k$.

THEOREM 1.8. – *For $k > 12$, there exists an effective $\mathbb{Q}$-divisor class E on $\overline{\mathcal{H}}_k$ supported on the divisor $\sum_{i=1}^{k-1} \sigma^*(\Delta_i)$ of curves of compact type, such that the class $K_{\overline{\mathcal{H}}_k} + E$ is big.*

This result should be compared to the classical result [30] asserting that $\overline{\mathcal{M}}_{2k-1}$ is of general type for $k \geq 13$, whereas the Kodaira dimension of $\overline{\mathcal{M}}_{23}$ is at least 2, see [14]. Assuming that the singularities of $\overline{\mathcal{H}}_k$ impose no adjunction conditions (something one certainly expects), Theorem 1.8 should imply that for $k > 12$ the Hurwitz space $\overline{\mathcal{H}}_k$ is a variety of general type.

Effective divisors of small slope on $\overline{\mathcal{M}}_g$. – Theorem 1.1 has multiple applications to the birational geometry of the moduli space of curves. Recall that if $\lambda, \delta_0, \ldots, \delta_{\lfloor \frac{g}{2} \rfloor}$ denote the standard generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, then the *slope* of an effective divisor $D \subseteq \overline{\mathcal{M}}_g$ such that $\Delta_i \not\subseteq \mathrm{supp}(D)$ for all $i = 0,\ldots,\lfloor \frac{g}{2} \rfloor$, is defined as $s(D) := \frac{a}{\min_i b_i} \geq 0$, where $[D] = a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i \delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$. The slope of the moduli space, defined as the quantity

$$s(\overline{\mathcal{M}}_g) := \inf\Big\{s(D) : D \text{ is an effective divisor of } \overline{\mathcal{M}}_g\Big\}$$

is a fundamental invariant encoding for instance the Kodaira dimension of the moduli space. For a long time it was conjectured (see [29]) that $s(\overline{\mathcal{M}}_g) \geq 6 + \frac{12}{g+1}$, with equality if and only if $g+1$ is composite and D is a Brill-Noether divisor on $\overline{\mathcal{M}}_g$ consisting of curves $[C] \in \mathcal{M}_g$ having a linear series $L \in W^r_d(C)$ with Brill-Noether number $\rho(g,r,d) = -1$. This conjecture has been disproved in [15], [16] and [35], where for an infinite series of genera g effective divisors of slope less than $6 + \frac{12}{g+1}$ were constructed. At the moment there is no clear conjecture concerning even the asymptotic behavior of $s(\overline{\mathcal{M}}_g)$ as g is large, see also [45]. For instance, it is not clear that $\liminf_{g\to\infty} s(\overline{\mathcal{M}}_g) > 0$.

Imposing the condition that a curve C of genus g lies on a quadric of prescribed rank in one of the embeddings $\varphi_L : C \hookrightarrow \mathbf{P}^r$ given by a linear system $L \in W^r_d(C)$ with Brill-Noether number $\rho(g,r,d) := g-(r+1)(g-d+r) = 0$, we obtain an infinite sequence of effective divisors on $\overline{\mathcal{M}}_g$ of very small slope (see condition (20) for the numerical condition g has to satisfy). Theorems 7.1 and 7.2 exemplify two infinite subsequences of such divisors on $\overline{\mathcal{M}}_{(4\ell-1)(9\ell-1)}$ and $\overline{\mathcal{M}}_{4(3\ell+1)(2\ell+1)}$ respectively, where $\ell \geq 1$. We mention the following concrete example on $\overline{\mathcal{M}}_{24}$.

THEOREM 1.9. – *The following locus defined as*

$$D_{7,3} := \Big\{[C] \in \mathcal{M}_{24} : \exists\, L \in W^7_{28}(C),\ \exists\, 0 \neq q \in I_{C,L}(2),\ \mathrm{rk}(q) \leq 6\Big\}$$

is an effective divisor on $\mathcal{M}_{24}$. The slope of its closure $\overline{D}_{7,3}$ in $\overline{\mathcal{M}}_{24}$ is given by $s(\overline{D}_{7,3}) = \frac{34423}{5320} < 6 + \frac{12}{25}$.

Theorem 7.3 establishes that $D_{7,3}$ is a genuine divisor on $\mathcal{M}_{24}$. We show using *Macaulay* that there exists a smooth curve $C \subseteq \mathbf{P}^7$ of genus 24 and degree 28 which does not lie on a quadric of rank at most 6 in $\mathbf{P}^7$. Using the irreducibility of the space of pairs $[C, L]$, where C is a smooth curve of genus 24 and $L \in W^7_{28}(C)$, we conclude that $D_{7,3} \neq \mathcal{M}_{24}$, hence $D_{7,3}$ is indeed a divisor on $\mathcal{M}_{24}$.

Theorem 1.2 has applications to the slope of $\overline{\mathcal{M}}_{12}$. A general curve $[C] \in \mathcal{M}_{12}$ has a finite number of embeddings $C \subseteq \mathbf{P}^5$ of degree 15. They are all residual to pencils of minimal degree. The curve $C \subseteq \mathbf{P}^5$ lies on a pencil of quadrics and we impose the condition that one of these pencils be degenerate.

THEOREM 1.10. – *The locus of smooth curves of genus* 12 *having a degenerate pencil of quadrics*

$$\mathfrak{Dp}_{12} := \Big\{[C] \in \mathcal{M}_{12} : \exists\, L \in W^5_{15}(C) \textit{ with } \mathbf{P}\big(I_{C,L}(2)\big) \text{ degenerate}\Big\}$$

is an effective divisor. The slope of its closure $\overline{\mathfrak{Dp}}_{12}$ inside $\overline{\mathcal{M}}_{12}$ equals $s(\overline{\mathfrak{Dp}}_{12}) = \frac{373}{54} < 6+\frac{12}{13}$.

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2. Equivariant fundamental classes, degeneracy loci

2.1. Equivariant fundamental class

We consider a connected algebraic group G acting on a smooth variety V, and let Σ be an invariant subvariety. Then Σ represents a fundamental cohomology class—denoted by $[\Sigma]$ or $[\Sigma \subseteq V]$—in the G-equivariant cohomology of V, namely

$$[\Sigma] \in H_G^{2\mathrm{codim}(\Sigma \subseteq V)}(V).$$

Throughout the paper we use cohomology with complex coefficients. There are several equivalent ways to define this fundamental cohomology class, see for example [34], [12], [22], [41, 8.5], [24] for different flavors and different cohomology theories.

A particularly important case is when V is a vector space and Σ is an invariant cone. Then $[\Sigma]$ is an element of $H_G^*(\text{vector space}) = H_G^*(\text{point}) = H^*(BG)$, that is, $[\Sigma]$ is a G-characteristic class. This characteristic class has the following well known "degeneracy locus" interpretation. Let $E \to M$ be a bundle with fiber V and structure group G. Since Σ is invariant under the structure group, the notion of *belonging to* Σ makes sense in every fiber. Let $\Sigma(E)$ be the union of Σ's of all the fibers. Let s be a sufficiently generic section. Then the fundamental cohomology class $[s^{-1}(\Sigma(E)) \subseteq M]$ of the "degeneracy locus" $s^{-1}(\Sigma(E))$ in the *ordinary* cohomology $H^*(M)$ is equal to $[\Sigma]$ (as a G-characteristic class) of the bundle $E \to M$.

2.2. Examples

We recall two well known formulas for some equivariant cohomology classes. The second one will be used in Sections 4 and 5.

DEFINITION 2.1. – *For variables c_i and a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ let*

$$s_\lambda(c) = \det(c_{\lambda_i+j-i})_{i,j=1,\dots,r}$$

be the Schur polynomial. By convention $c_0 = 1$ and $c_{<0} = 0$.

EXAMPLE 2.2. – *The Giambelli-Thom-Porteous formula*. Fix $r \leq n$, $\ell \geq 0$ and let $\Omega^r \subseteq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+\ell})$ be the space of linear maps having an r-dimensional kernel. It is invariant under the group $GL_n(\mathbb{C}) \times GL_{n+\ell}(\mathbb{C})$ acting by $(A, B) \cdot \phi = B \circ \phi \circ A^{-1}$. One has [47]

$$[\overline{\Omega^r}] = s_\lambda(c),$$

where

$$\lambda = (\underbrace{r+\ell, \dots, r+\ell}_{r}), \qquad 1 + c_1 t + c_2 t^2 + \cdots = \frac{1 + b_1 t + b_2 t^2 + \cdots + b_{n+\ell} t^{n+\ell}}{1 + a_1 t + a_2 t^2 + \cdots + a_n t^n}.$$

Here a_i (respectively b_i) is the ith universal Chern class of $GL_n(\mathbb{C})$ (respectively $GL_{n+\ell}(\mathbb{C})$).

EXAMPLE 2.3. – *Symmetric 2-forms*. Let $r \leq n$ and let $\Sigma^r = \Sigma^r_n \subseteq \mathrm{Sym}^2(\mathbb{C}^n)$ be the collection of symmetric 2-forms having a kernel of dimension r. It is invariant under the group $GL_n(\mathbb{C})$ acting by $A \cdot M = AMA^T$. One has [33, 49, 31] that

$$[\overline{\Sigma^r_n}] = 2^r s_{(r,r-1,\dots,2,1)}(c),$$

where c_i is the ith universal Chern class of $GL_n(\mathbb{C})$.

3. Affine, projective, and restricted projective fundamental classes

In this section we recall the formalism of comparing equivariant fundamental classes in affine and projective spaces.

Consider the representation of the torus $T = (\mathbb{C}^*)^k$ acting by

$$(a_1,\dots,a_k)\cdot(x_1,\dots,x_n) = (\prod_{i=1}^k a_i^{s_{1,i}} x_1, \prod_{i=1}^k a_i^{s_{2,i}} x_2, \dots, \prod_{i=1}^k a_i^{s_{n,i}} x_n).$$

We will assume that the representation "contains the scalars", that is, there exist integers $r_1,\dots,r_k$ and r such that

$$\sum_{i=1}^k r_i s_{j,i} = r, \qquad \text{for all} \qquad j = 1,\dots,n.$$

In other words, the action of $(b^{r_1},\dots,b^{r_k}) \in T$ $(b \in \mathbb{C}^*)$ on $\mathbb{C}^n$ is multiplication by b^r.

Under this assumption we have that the non-zero orbits of the linear representation, and the orbits of the induced action on $\mathbf{P}^{n-1}$ are in bijection. We will compare the (T-equivariant) fundamental class of an invariant subvariety $\Sigma \subseteq \mathbb{C}^n$ with the (T-equivariant) fundamental class of the projectivization $\mathbf{P}(\Sigma) \subseteq \mathbf{P}^{n-1}$. For this we need some notation.

The fundamental class $[\Sigma]$ of Σ is an element of $H_T^*(\mathbb{C}^n) = H^*(BT) = \mathbb{C}[\alpha_1,\dots,\alpha_k]$, where α_i is the equivariant first Chern class of the $\mathbb{C}^*$-action corresponding to the ith factor. Hence we can consider $[\Sigma]$ as a polynomial in the α_i's.

Let $w_j = \sum_{i=1}^k s_{j,i}\alpha_i$, $j = 1,\dots,n$ be the weights of the representation above. Then we have

$$H_T^*(\mathbf{P}^{n-1}) = H^*(BT)[\xi]/\prod_{j=1}^n(\xi - w_j),$$

where ξ is the first Chern class of the tautological line bundle over $\mathbf{P}^{n-1}$.

PROPOSITION 3.1. – [20, Thm. 6.1] *Let Σ be a T-invariant subvariety of $\mathbb{C}^n$. For the T-equivariant fundamental class of $\boldsymbol{P}(\Sigma)$ we have*

$$[\boldsymbol{P}(\Sigma)] = [\Sigma]|_{\alpha_i \mapsto \alpha_i - \frac{r_i}{r}\xi} \qquad \in H_T^*(\boldsymbol{P}^{n-1}).$$

Here, and in the future, by $p(\alpha_i)|_{\alpha_i \mapsto \beta_i}$ we mean the substitution of β_i into the variables α_i of the polynomial $p(\alpha_i)$.

We shall need a further twist on this notion. Let F_j be the jth coordinate line of $\mathbb{C}^n$, which is a fixed point of the T-action on $\mathbf{P}^{n-1}$. We have the restriction map $H_T^*(\mathbf{P}^{n-1}) \to H_T^*(F_j) = H^*(BT)$, which we denote by $p \mapsto p|_{F_j}$.

COROLLARY 3.2. – *We have*

$$[\boldsymbol{P}(\Sigma)]|_{F_j} = [\Sigma]|_{\alpha_i \mapsto \alpha_i - \frac{r_i}{r} w_j} \qquad \in H^*(BT).$$

Proof. – The restriction homomorphism $H_T^*(\mathbf{P}^{n-1}) \to H_T^*(F_j)$ is given by substituting w_j for ξ. □

EXAMPLE 3.3. – Let $(\mathbb{C}^*)^3$ act on $\mathbb{C}^2$ by $(a_1,a_2,a_3)\cdot(x_1,x_2) = (a_1^3a_2^{-1}a_3\cdot x_1, a_1a_2^2a_3^2\cdot x_2)$. The numbers $r_1 = 2, r_2 = 1, r_3 = 1, r = 6$ prove that this action contains the scalars. Let Σ be the x_1-axis. Then $[\Sigma]$ is the normal Euler class, that is $[\Sigma] = \alpha_1 + 2\alpha_2 + 2\alpha_3$. According to Proposition 3.1 we have

$$[\mathbf{P}(\Sigma)] = \alpha_1 + 2\alpha_2 + 2\alpha_3|_{\alpha_1\mapsto\alpha_1-\frac{1}{3}\xi,\alpha_2\mapsto\alpha_2-\frac{1}{6}\xi,\alpha_3\mapsto\alpha_3-\frac{1}{6}\xi} = \alpha_1 + 2\alpha_2 + 2\alpha_3 - \xi.$$

According to Corollary 3.2 the two fixed point restrictions of this class are

$$\begin{aligned}[\mathbf{P}(\Sigma)]|_{(1:0)} &= \alpha_1 + 2\alpha_2 + 2\alpha_3|_{\alpha_1\mapsto\alpha_1-\frac{1}{3}(3\alpha_1-\alpha_2+\alpha_3),\alpha_2\mapsto\alpha_2-\frac{1}{6}(3\alpha_1-\alpha_2+\alpha_3),\alpha_3\mapsto\alpha_3-\frac{1}{6}(3\alpha_1-\alpha_2+\alpha_3)}\\ &= -2\alpha_1 + 3\alpha_2 + \alpha_3\end{aligned}$$

and

$$\begin{aligned}[\mathbf{P}(\Sigma)]|_{(0:1)} &= \alpha_1 + 2\alpha_2 + 2\alpha_3|_{\alpha_1\mapsto\alpha_1-\frac{1}{3}(\alpha_1+2\alpha_2+2\alpha_3),\alpha_2\mapsto\alpha_2-\frac{1}{6}(\alpha_1+2\alpha_2+2\alpha_3),\alpha_3\mapsto\alpha_3-\frac{1}{6}(\alpha_1+2\alpha_2+2\alpha_3)}\\ &= 0.\end{aligned}$$

The vanishing of the second one is expected since the x_2-axis is not in Σ, and the first one can be verified by seeing that the action on $\mathbf{P}^1$ in the coordinate $t = x_2/x_1$ is

$$(a_1,a_2,a_3).t = \frac{a_1a_2^2a_3^2}{a_1^3a_2^{-1}a_3}\cdot t = a_1^{-2}a_2^3a_3\cdot t.$$

The calculations of this example were deceivingly simple caused by the fact that Σ was smooth.

4. Loci characterized by singular vectors in the kernel

4.1. The $\Sigma^r_{e,f}$ locus

For positive integers e, f, let $E := \mathbb{C}^e$ and $F := \mathbb{C}^f$ be the standard representations of $GL_e(\mathbb{C})$ and $GL_f(\mathbb{C})$ respectively. Consider the induced action of $G = GL_e(\mathbb{C})\times GL_f(\mathbb{C})$ on $\mathrm{Hom}(\mathrm{Sym}^2E, F)$. Define the locus

$$\Sigma^r_{e,f} = \Big\{\phi \in \mathrm{Hom}(\mathrm{Sym}^2E,F) : \exists\, q \in \mathrm{Sym}^2E \text{ with } \dim(\mathrm{Ker}\, q) = r \text{ and } \phi(q) = 0\Big\},$$

which is invariant under the G-action. Using the notation of Example 2.3 we have

$$\Sigma^r_{e,f} = \Big\{\phi \in \mathrm{Hom}(\mathrm{Sym}^2E,F) : \exists\, 0 \neq q \in \Sigma^r_e \cap \mathrm{Ker}(\phi)\Big\}.$$

We will assume that $d := \binom{e+1}{2} - f$ is positive, that is, the condition above is not that ϕ *has* a kernel, but rather that this kernel intersects $\Sigma^r_e \subseteq \mathrm{Sym}^2E$. We shall also assume that this intersection is generically at most 0-dimensional, that is, $d \leq \mathrm{codim}(\Sigma^r_e \subseteq \mathrm{Sym}^2E) = \binom{r+1}{2}$.

In this section our goal is to find a formula for the G-equivariant fundamental class

$$[\overline{\Sigma^r_{e,f}}] \in H^*_G\Big(\mathrm{Hom}(\mathrm{Sym}^2E,F)\Big) = \mathbb{C}[\alpha_1,\dots,\alpha_e,\beta_1,\dots,\beta_f]^{S_e\times S_f}.$$

Here α_i are the Chern roots of $GL_e(\mathbb{C})$ (that is, their elementary symmetric polynomials are the Chern classes), and β_i are the Chern roots of $GL_f(\mathbb{C})$ respectively.

The calculation—which will complete the proof of Theorem 1.1—is done via torus-equivariant localization. To bypass complications caused by a complete resolution of $\overline{\Sigma^r_{e,f}}$

we will use a method of [5, 23] which requires only a partial desingularization exhibited as a vector bundle over a compact space.

4.2. A partial resolution

Let $\mathscr{F}$ be the partial flag manifold parametrizing chains of linear subspaces $C \subseteq D \subseteq \mathrm{Sym}^2 E$, where $\dim C = 1$ and $\dim D = d$. Recall that in Example 2.3 we defined the subset $\Sigma^r = \Sigma^r_e \subseteq \mathrm{Sym}^2 E$. Define

$$\begin{aligned} I &= \Big\{\big((C,D),\phi\big) \in \mathscr{F} \times \mathrm{Hom}(\mathrm{Sym}^2 E, F) : C \subseteq \overline{\Sigma^r} \text{ and } \phi|_D = 0\Big\} \text{ and} \\ Y &= \{(C,D) \in \mathscr{F} : C \subseteq \overline{\Sigma^r}\} \end{aligned}$$

and let $p : I \to Y$ be the map forgetting ϕ. We have the commutative diagram

$$\begin{tikzcd} I \arrow[r, hook, "i"'] \arrow[d, "p"'] \arrow[rr, bend left, "\rho"] & \mathscr{F} \times \mathrm{Hom}(\mathrm{Sym}^2 E, F) \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & \mathrm{Hom}(\mathrm{Sym}^2 E, F) \\ Y \arrow[r, hook, "j"'] & \mathscr{F}, & \end{tikzcd} \tag{3}$$

with i and j being natural inclusions and π_1, π_2 natural projections. The map $\rho = \pi_2 \circ i$ is birational to $\overline{\Sigma^r_{e,f}}$. We have

$$\dim Y = \binom{e+1}{2} - \binom{r+1}{2} - 1 + (d-1)f, \quad \dim I = \binom{e+1}{2} - \binom{r+1}{2} - 1 + (d-1)f + f^2.$$

Hence the codimension

$$\mathrm{codim}\Big(\overline{\Sigma^r_{e,f}} \subseteq \mathrm{Hom}(\mathrm{Sym}^2 E, F)\Big) = \binom{r+1}{2} - \binom{e+1}{2} + f + 1 = \binom{r+1}{2} - d + 1,$$

which is thus the degree of the fundamental class $[\overline{\Sigma^r_{e,f}}]$ we are looking for.

4.3. Localization and residue formulas

Let $W = \{\alpha_i + \alpha_j\}_{1 \le i \le j \le e}$ be the set of weights of $\mathrm{Sym}^2 E$. Let $h_r(\alpha_1, \dots, \alpha_e)$ be the polynomial $2^r s_{(r,r-1,\dots,1)}(c)$, where $1 + c_1 t + c_2 t^2 + \cdots = \prod_{i=1}^e (1 + \alpha_i t)$ (cf. Example 2.3).

THEOREM 4.1. – *Using the notations and assumption above we have*

$$[\overline{\Sigma^r_{e,f}}] = \sum_{H \subseteq W} |_{|H|=d} \sum_{\gamma \in H} \frac{h_r|_{\alpha_i \mapsto \alpha_i - \gamma/2} \cdot \prod_{j=1}^f \prod_{\delta \in H} (\beta_j - \delta)}{\prod_{\delta \in W - \{\gamma\}} (\delta - \gamma) \cdot \prod_{\delta \in H - \{\gamma\}} \prod_{\epsilon \in W - H} (\epsilon - \delta)}. \tag{4}$$

Proof. – To calculate the fundamental class $[\overline{\Sigma^r_{e,f}}]$ it would be optimal to find an equivariant resolution $\tilde{\Sigma} \to \mathrm{Hom}(\mathrm{Sym}^2 E, F)$ of $\overline{\Sigma^r_{e,f}} \subseteq \mathrm{Hom}(\mathrm{Sym}^2 E, F)$, with a well understood Gysin map formula. While the description of such a full resolution is difficult, in diagram (3) we constructed an equivariant partial resolution $\rho : I \to \mathrm{Hom}(\mathrm{Sym}^2 E, F)$ of the locus $\overline{\Sigma^r_{e,f}} \subseteq \mathrm{Hom}(\mathrm{Sym}^2 E, F)$. Although ρ is only a partial resolution (since I is not smooth), it is of special form: I is a *vector bundle* over a (possibly singular) subvariety of a *compact* space $\mathcal{F}$.

In [5, Section 3.2] and [23, Section 5] it is shown that such a partial resolution reduces the problem of calculating $[\overline{\Sigma^r_{e,f}}]$ to calculating the fundamental class $[Y \subseteq \mathcal{F}]$ near the fixed points of the maximal torus. Namely, [5, Proposition 3.2], or equivalently [23, Proposition 5.1], applied to diagram (3) gives

$$[\overline{\Sigma^r_{e,f}}] = \sum_q \frac{[Y \subseteq \mathcal{F}]|_q \cdot [I_q \subseteq \operatorname{Hom}(\operatorname{Sym}^2 E, F)]}{e(T_q \mathcal{F})}, \tag{5}$$

where q runs through the finitely many torus fixed points of $\mathcal{F}$ and $I_q = p^{-1}(q)$.

Let us start with the obvious ingredients of this formula. The fixed points of $\mathcal{F}$ are pairs (C, D) where $C \subseteq D$ are coordinate subspaces of $\operatorname{Sym}^2 E$ of dimension 1 and d respectively. The coordinate lines of $\operatorname{Sym}^2 E$ are in bijection with W, and hence the fixed points q are parameterized by choices $H \subset W$ ($|H| = d$) and $\gamma \in H$. Denoting the tautological rank 1 and rank d bundles over $\mathcal{F}$ by $\mathcal{L}$ and $\mathcal{D}$ we have

$$T\,\mathcal{F} = \operatorname{Hom}(\mathcal{L}, \mathcal{D}/\mathcal{L}) \oplus \operatorname{Hom}(\mathcal{L}, \operatorname{Sym}^2 E/\mathcal{D}) \oplus \operatorname{Hom}(\mathcal{D}/\mathcal{L}, \operatorname{Sym}^2 E/\mathcal{D}).$$

Hence, for a fixed point q corresponding to (H, γ) we have

— $[I_q \subseteq \operatorname{Hom}(\operatorname{Sym}^2 E, F)] = \prod_{j=1}^{f} \prod_{\delta \in H} (\beta_j - \delta)$,

— $e(T_q \mathcal{F}) = \prod_{\delta \in W - \{\gamma\}} (\delta - \gamma) \cdot \prod_{\delta \in H - \{\gamma\}} \prod_{\epsilon \in W - H} (\epsilon - \delta)$,

both following from the fact that for a G-representation K and invariant subspace $L \subseteq K$ the fundamental class $[L \subset K]$ is the product of the weights of K/L.

It remains to find the non-obvious ingredient of formula (5), the local fundamental class $[Y \subseteq \mathcal{F}]|_q$. However, this problem was essentially solved in Section 3. The space Y is the complete preimage of $\mathbf{P}(\overline{\Sigma^r})$ under the fibration $z : \mathcal{F} \to \mathbf{P}(\operatorname{Sym}^2 E)$. Hence $[Y \subseteq \mathcal{F}]|_q = [\mathbf{P}(\overline{\Sigma^r})]|_{z(q)}$. We have $[\overline{\Sigma^r}] = h_r(\alpha_1, \ldots, \alpha_e)$ (see Example 2.3), and hence Corollary 3.2 calculates $[\mathbf{P}(\overline{\Sigma^r})]|_{z(q)}$ to be $h_r|_{\alpha_i \mapsto \alpha_i - \gamma/2}$. This completes the proof. ☐

Example 4.2. – We have

$$[\overline{\Sigma^1_{2,2}}] = \frac{(\beta_1 - 2\alpha_1)(\beta_2 - 2\alpha_1)}{\alpha_2 - \alpha_1} + \frac{(\beta_1 - 2\alpha_2)(\beta_2 - 2\alpha_2)}{\alpha_1 - \alpha_2} = -4(\alpha_1 + \alpha_2) + 2(\beta_1 + \beta_2).$$

More structure of the localization formula (4) will be visible if we rewrite it as a residue formula, with the help of the following lemma, which we prepare by setting some notation.

Let $0 \le k_1 \le k_2 \le \cdots \le k_r$ be integers and let V be a vector bundle of rank k_r on X. Let $p : \mathcal{F}_{k_1,\ldots,k_r}(V) \to X$ be the bundle whose fiber over $x \in X$ is the variety of chains of linear subspaces $V_1^{k_1} \subseteq V_2^{k_2} \subseteq \cdots \subseteq V_r^{k_r} = V_x$, where upper indices indicate dimension and V_x is the fiber of V over x. The Chern roots of the tautological bundle of rank k_i over $\mathcal{F}_{k_1,\ldots,k_r}(V)$ will be denoted by $\sigma_{i,j}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, k_i$. The $\sigma_{r,j}$ classes are the pullbacks of the Chern roots of V. In notation we do not indicate the pullback, so $\sigma_{r,j}$ will also denote the Chern roots of V.

LEMMA 4.3. – *Consider the variables $z_{i,j}$ for $i = 1, \dots, r-1$, $j = 1, \dots, k_i$, and let $z_{r,j} = \sigma_{r,j}$. Let $g(z_{i,j})$ be a polynomial symmetric in the sets of variables z_{i*} for all i, and let $D = \sum_{i<j}(k_i - k_{i-1})(k_j - k_{j-1})$ be the dimension of the fiber of p. We have*

(6)
$$p_*(g(\sigma_{i,j})) = (-1)^D \left\{ \frac{g(z_{i,j}) \prod_{i=1}^{r-1} \prod_{1 \le u < v \le k_i} \left(1 - \frac{z_{i,u}}{z_{i,v}}\right)}{\prod_{i=1}^{r-1} \prod_{j=1}^{k_i} z_{i,j}^{k_{i+1}-k_i} \cdot \prod_{i=1}^{r-1} \prod_{u=1}^{k_{r+1}} \prod_{v=1}^{k_r} \left(1 - \frac{z_{i+1,u}}{z_{i,v}}\right)} \right\}_{z^0_{1*} \cdots z^0_{k-1,*}},$$

where, by $\{P\}_{z^0_{1*} \cdots z^0_{k-1,*}}$ we mean the constant term in the variables $z_{i,j}$ for $i = 1, \dots, k-1$ and $j = 1, \dots, k_i$, of the Laurent expansion of P in the region $|z_{1,j_1}| > |z_{2,j_1}| > \cdots > |z_{r,j_r}|$.

Proof. – First we prove the statement for $r = 2$. To that end, we temporarily rename $k_1 = k$, $k_2 = n$, $\sigma_{1,j} = \sigma_j$, $\sigma_{2,j} = \tau_j$, $z_{1,j} = z_j$, and we shall use the abbreviations $\sigma = (\sigma_1, \dots, \sigma_k)$, $\tau = (\tau_1, \dots, \tau_n)$, $z = (z_1, \dots, z_k)$. By [48, Lemma 2.5] we have

$$p_*(g(\sigma, \tau)) = \sum_I \frac{g(\tau_I, \tau)}{\prod_{j \notin I} \prod_{i \in I} (\tau_j - \tau_i)}, \tag{7}$$

where the summation is over k-element subsets $I = \{s_1, \dots, s_k\}$ of $\{1, \dots, n\}$ and $\tau_I = (\tau_{s_1}, \dots, \tau_{s_k})$. Define

$$H = (-1)^{k(n-k)} g(z, \tau) \prod_{1 \le i < j \le k} (z_j - z_i) \cdot \frac{z_1^{k-1} z_2^{k-2} \cdots z_{k-1}}{\prod_{j=1}^{n} \prod_{i=1}^{k} (z_i - \tau_j)}$$

and consider the differential form $\omega = H dz_1 \wedge \cdots \wedge dz_k$.

Let $R = \mathrm{Res}_{z_k=\infty} \mathrm{Res}_{z_{k-1}=\infty} \cdots \mathrm{Res}_{z_1=\infty}(\omega)$.

First we calculate R by applying the Residue Theorem (the sum of the residues of a meromorphic form on the Riemann sphere is 0) for $z_1, z_2, \dots, z_k$. We obtain

$$R = (-1)^k \sum_{s_k} \sum_{s_{k-1}} \cdots \sum_{s_1} \mathrm{Res}_{z_k=\tau_{s_k}} \mathrm{Res}_{z_{k-1}=\tau_{s_{k-1}}} \cdots \mathrm{Res}_{z_1=\tau_{s_1}}(\omega).$$

The terms corresponding to choices with non-distinct s_j's is 0, due to the factor $\prod(z_j - z_i)$ in the numerator of ω. Thus we have

$$R = (-1)^{k(n-k)+k} \sum_I \sum_{w \in S_k} \frac{g(\tau_I, \tau) \prod_{i<j} (\tau_{w(s_j)} - \tau_{w(s_i)}) \tau_{w(s_1)}^{k-1} \tau_{w(s_2)}^{k-2} \cdots \tau_{w(s_{k-1})}}{\prod_{i \ne j} (\tau_{w(s_j)} - \tau_{w(s_i)}) \prod_{j \notin I} \prod_{i=1}^{k} (\tau_{w(s_i)} - \tau_j)},$$

where the summation is over k-element subsets $I = \{s_1, \dots, s_k\} \subset \{1, \dots, n\}$. This further equals

$$R = (-1)^k \sum_I \left(\frac{g(\tau_I, \tau)}{\prod_{j \notin I} \prod_{i \in I} (\tau_j - \tau_i)} \underbrace{\sum_{w \in S_k} \frac{\tau_{w(s_1)}^{k-1} \tau_{w(s_2)}^{k-2} \cdots \tau_{w(s_{k-1})}}{\prod_{i > j} (\tau_{w(s_j)} - \tau_{w(s_i)})}}_{(*)} \right).$$

However, the sum marked by (*) is equal to 1—because of the well known product form of a Vandermonde determinant—, and using (7) we obtain that $p_*(g(\sigma, \tau)) = (-1)^k R$.

Calculating the residues at infinity as a coefficient of the Laurent expansion we get

$$p_*(g(\sigma,\tau)) = (-1)^k R = \left\{ H \cdot \prod_{i=1}^{k} z_i \right\}_{z_1^0 \cdots z_k^0},$$

where $\{\ \}_{z_1^0 \cdots z_k^0}$ means the constant term of the Laurent-expansion in the $|z_i| > |\tau_j|$ (for all i, j) region. This proves (6) for $r = 2$.

For $r > 2$ the push-forward map p_* can be factored as $p_{1*} \circ p_{2*} \circ \cdots \circ p_{r*}$ for the Grassmanian fibrations

$$p_i : \mathcal{F}_{k_i, k_{i+1}, \ldots, k_r}(V) \to \mathcal{F}_{k_{i+1}, \ldots, k_r}(V),$$

with the notation $\mathcal{F}_{\emptyset}(V) = X$. The map p_i is a special case of the construction in the theorem for $r = 2$ and the tautological rank k_{i+1} bundle over $\mathcal{F}_{k_{i+1}, \ldots, k_r}(V)$. Hence p_{i*} can be computed with the formula in the theorem (as it is proved for $r = 2$ above). The iterated application of (6) for $r = 2$ gives the general (6), which completes the proof of the theorem. □

Theorem 4.4. – *We have*

$$[\overline{\Sigma^r_{e,f}}] = (-1)^{d+1} \left\{ \frac{h_r|_{\alpha_i \mapsto \alpha_i - z/2} \cdot \prod_{1 \le i < j \le d} (1 - \frac{u_i}{u_j})}{z^{d-1} \prod_{j=1}^{d} (1 - \frac{u_j}{z})} \cdot \prod_{j=1}^{d} \sum_{i=0}^{\infty} \frac{c_i(F^\vee - \mathrm{Sym}^2 E^\vee)}{u_j^i} \right\}_{z^0 u^0},$$

where $\{P\}_{z^0 u^0}$ means the constant term in P with respect to z and $u_1, \ldots, u_d$.

Proof. – The formula (5) for $[\overline{\Sigma^r_{e,f}}]$ is the Atiyah-Bott localization formula for the equivariant push-forward $p_*([Y \subseteq \mathcal{F}] e(\mathrm{Hom}(\mathcal{D}, F))$, where $\mathcal{D}$ is the tautological rank d bundle over $\mathcal{F}$, and $p : \mathcal{F} \to$pt. Calculating the equivariant push-forward p_* with the formula in Lemma 4.3, we obtain

(8)
$$(-1)^{d\binom{e+1}{2} - d^2 + d - 1} \left\{ \frac{h_r|_{\alpha_i \mapsto \alpha_i - z/2} \prod_{i=1}^{f} \prod_{j=1}^{d} (\beta_i - u_j) \prod_{1 \le i < j \le d} (1 - \frac{u_i}{u_j})}{z^{d-1} (u_1 \cdots u_d)^{\binom{e+1}{2} - d} \prod_{j=1}^{d} (1 - \frac{u_j}{z}) \prod_{j=1}^{d} \prod_{\epsilon \in W} (1 - \frac{\epsilon}{u_j})} \right\}_{z^0 u^0}.$$

Observing that

$$\begin{aligned} \prod_{j=1}^{d} \frac{\prod_{i=1}^{f} (\beta_i - u_j)}{\prod_{\epsilon \in W} (1 - \epsilon/u_j)} &= (-1)^{df} \prod_{j=1}^{d} u_j^f \prod_{j=1}^{d} \frac{\prod_{i=1}^{f} (1 - \beta_i/u_j)}{\prod_{\epsilon \in W} (1 - \epsilon/u_j)} \\ &= (-1)^{df} \prod_{j=1}^{d} u_j^f \sum_{i=0}^{\infty} \frac{c_i(F^\vee - \mathrm{Sym}^2 E^\vee)}{u_j^i}, \end{aligned}$$

and that $f = \binom{e+1}{2} - d$, we have that (8) further equals the formula in the theorem. □

4.4. The divisorial case

The residue formula of Theorem 4.4 is more manageable in case the codimension of $\Sigma^r_{e,f}$ is 1—the case relevant for most applications given in this paper. After two technical lemmas we will provide a simple formula for the $[\overline{\Sigma^r_{e,f}}]$ in this case.

LEMMA 4.5. – *For the z-expansion of the polynomial $h_r|_{\alpha_i\mapsto\alpha_i-z/2}$ we have*

$$h_r|_{\alpha_i\mapsto\alpha_i-z/2} = (-1)^{\binom{r+1}{2}}\left(A^r_e z^{\binom{r+1}{2}} + B^r_e\cdot\sum_{i=1}^{e}\alpha_i\cdot z^{\binom{r+1}{2}-1} + \text{l.o.t.}\right) \tag{9}$$

where

$$A^r_e = 2^{-\binom{r}{2}}\det\left(\binom{e}{r+1-2i+j}\right)_{i,j=1,\dots,r} = \frac{\binom{e}{r}\binom{e+1}{r-1}\cdots\binom{e+r-1}{1}}{\binom{1}{0}\binom{3}{1}\binom{5}{2}\cdots\binom{2r-1}{r-1}},$$

$$B^r_e = -\frac{2}{e}\binom{r+1}{2}A^r_e.$$

Proof. – The polynomial h_r is a homogeneous degree $\binom{r+1}{2}$ symmetric polynomial in the $\alpha_1,\dots,\alpha_e$ variables. Hence the expansion (9) must hold for some numbers A^r_e, B^r_e. We will calculate them via the substitution $\alpha_1=\dots=\alpha_e$. Let $D=\det\left(\binom{e}{r+1-2i+j}\right)_{i,j=1,\dots,r}$. From the definition of h_r we see that $h_r(\underbrace{\alpha,\dots,\alpha}_{e}) = 2^r D\alpha^{\binom{r+1}{2}}$, and hence, for the z-expansion of $h_r(\alpha-\frac{z}{2},\dots,\alpha-\frac{z}{2})$ we get

$$2^r D\left(-\frac{1}{2}\right)^{\binom{r+1}{2}} z^{\binom{r+1}{2}} + 2^r D\binom{r+1}{2}\left(-\frac{1}{2}\right)^{\binom{r+1}{2}-1}\frac{1}{e}(e\alpha)z^{\binom{r+1}{2}-1} + \text{l.o.t.},$$

which proves the first expression for A^r_e and the expression for B^r_e. The equivalence of the two displayed expressions for A^r_e is proved in [31, Proposition 12]. □

LEMMA 4.6. – *We have*

$$\prod_{1\le i<j\le d}\left(1-\frac{u_i}{u_j}\right) = 1-\sum_{i=1}^{d-1}\frac{u_i}{u_{i+1}} + Q, \tag{10}$$

where Q is the sum of u-monomials in which the degree of the denominator is at least two. Also,

$$\left(\sum_{i=1}^{d}u_i\right)\cdot\prod_{1\le i<j\le d}\left(1-\frac{u_i}{u_j}\right) = u_d + \text{fractions},$$

where fractions *stands for terms of monomials with at least one u_i in the denominator.*

For example, if $d=3$ then we have

$$\left(1-\frac{u_1}{u_2}\right)\left(1-\frac{u_1}{u_3}\right)\left(1-\frac{u_2}{u_3}\right) = 1-\frac{u_1}{u_2}-\frac{u_2}{u_3}+\underbrace{\left(\frac{u_1u_2}{u_3^2}+\frac{u_1^2}{u_2u_3}-\frac{u_1^2}{u_3^2}\right)}_{Q},$$

and $(u_1+u_2+u_3)\prod_{i<j\le 3}(1-u_i/u_j) = u_3+$fractions.

Proof. – Arguing by induction on d we have that the left hand side of (10) is

$$\begin{aligned}\left(1-\sum_{i=1}^{d-2}\frac{u_i}{u_{i+1}}+Q'\right)\prod_{i=1}^{d-1}\left(1-\frac{u_i}{u_d}\right) &= \left(1-\sum_{i=1}^{d-2}\frac{u_i}{u_{i+1}}+Q_1\right)\left(1-\sum_{i=1}^{d-1}\frac{u_i}{u_d}+Q_2\right)\\ &= 1-\sum_{i=1}^{d-2}\frac{u_i}{u_{i+1}}-\sum_{i=1}^{d-1}\frac{u_i}{u_d}+\sum_{i=1}^{d-2}\frac{u_i}{u_d}+Q\\ &= 1-\sum_{i=1}^{d-1}\frac{u_i}{u_{i+1}}+Q,\end{aligned}$$

where Q_1 and Q_2 are sums of terms that multiplied with anything in the other factor will result in monomials with denominator degree at least 2.

The second statement of the lemma follows directly from the first one. □

We now determine the class of $\overline{\Sigma^r_{e,f}}$ when it is a divisor, which leads to a proof of Theorem 1.1.

THEOREM 4.7. – *Assume that* $\overline{\Sigma^r_{e,f}}$ *is a divisor, that is,*

$$\binom{r+1}{2}-d+1=\binom{r+1}{2}-\binom{e+1}{2}+f+1=1. \tag{11}$$

Then

$$[\overline{\Sigma^r_{e,f}}]=A^r_e\left(c_1(F)-\frac{2f}{e}c_1(E)\right). \tag{12}$$

Proof. – Under the assumption (11) Theorem 4.4 reads

$$[\overline{\Sigma^r_{e,f}}]=-\left\{\left(A^r_e z^1+B^r_e\cdot\sum_{i=1}^{e}\alpha_i\cdot z^0+\text{l.o.t.}\right)\cdot\frac{\prod_{1\le i<j\le d}\left(1-\frac{u_i}{u_j}\right)}{\prod_{j=1}^{d}\left(1-\frac{u_j}{z}\right)}\cdot\prod_{j=1}^{d}\sum_{i=0}^{\infty}\frac{c_i(F^\vee-\mathrm{Sym}^2E^\vee)}{u_j^i}\right\}_{z^0,u^0}.$$

Looking at the z-exponents, this is further equal to

$$-\left\{\left(A^r_e\sum_{j=1}^{d}u_j\prod_{1\le i<j\le d}\left(1-\frac{u_i}{u_j}\right)+B^r_e\sum_{i=1}^{e}\alpha_i\prod_{1\le i<j\le d}\left(1-\frac{u_i}{u_j}\right)\right)\cdot\prod_{j=1}^{d}\sum_{i=0}^{\infty}\frac{c_i(F^\vee-\mathrm{Sym}^2E^\vee)}{u_j^i}\right\}_{u^0}.$$

Looking at u-exponents, and using Lemma 4.6, this is further equal to

$$-\left\{\left(A^r_e\Big(u_d+\text{fractions}\Big)+B^r_e\sum_{i=1}^{e}\alpha_i(1+\text{fractions})\right)\cdot\prod_{j=1}^{d}\sum_{i=0}^{\infty}\frac{c_i(F^\vee-\mathrm{Sym}^2E^\vee)}{u_j^i}\right\}_{u^0},$$

where the term *fractions* stands for terms with at least one u_j variable in the denominator. Hence the formula further equals

$$-A^r_e c_1(F^\vee-\mathrm{Sym}^2E^\vee)-B^r_e c_1(E).$$

Using that $c_1(F^\vee-\mathrm{Sym}^2E^\vee)=c_1(F^\vee)-c_1(\mathrm{Sym}^2E^\vee)=-c_1(F)+(e+1)c_1(E)$, we obtain

$$[\overline{\Sigma^r_{e,f}}]=A^r_e c_1(F)-\left(A^r_e(e+1)+B^r_e\right)c_1(E).$$

Using the divisorial condition (11), this expression can be rewritten as (12). □

EXAMPLE 4.8. – We have

$$[\overline{\Sigma^1_{2,2}}] = -4c_1(E) + 2c_1(F), \qquad [\overline{\Sigma^1_{3,5}}] = -10c_1(E) + 3c_1(F),$$
$$[\overline{\Sigma^1_{4,9}}] = -18c_1(E) + 4c_1(F), \qquad [\overline{\Sigma^2_{3,3}}] = -8c_1(E) + 4c_1(F),$$
$$[\overline{\Sigma^2_{4,7}}] = -35c_1(E) + 10c_1(F), \qquad [\overline{\Sigma^2_{5,12}}] = -96c_1(E) + 20c_1(F).$$

5. Loci defined by discriminant

Let $e \geq 2$ and use the short hand notation $N = \binom{e+1}{2} - 2$. Let $E := \mathbb{C}^e$ be the standard representations of $GL_e(\mathbb{C})$. Consider the tautological exact sequence of $GL_e(\mathbb{C})$-equivariant bundles $0 \to S \to S^2E \to Q \to 0$ over the Grassmannian $\mathrm{Gr}(2, \mathrm{Sym}^2 E)$ of 2-planes in $\mathrm{Sym}^2 E$. Recall that we have introduced in Example 2.3 the $GL_e(\mathbb{C})$-invariant subset $\Sigma^1 \subseteq \mathrm{Sym}^2 E$ as the set of *degenerate* symmetric 2-forms. Define

$$\Phi_e := \Big\{ W \in \mathrm{Gr}(2, \mathrm{Sym}^2 E) : \mathbf{P}(W) \text{ is tangent to } \mathbf{P}(\Sigma^1) \Big\} \subseteq \mathrm{Gr}(2, \mathrm{Sym}^2 E).$$

Notice that we require $\mathbf{P}(W)$ to be tangent to $\mathbf{P}(\Sigma^1)$ (which is a smooth but not closed subvariety of $\mathbf{P}(\mathrm{Sym}^2 E)$), that is we require that the projective line $\mathbf{P}(W)$ intersect the smooth part of $\mathbf{P}(\overline{\Sigma^1})$, and the intersection be tangential. Our goal in this section is to calculate the equivariant fundamental class $[\overline{\Phi}_e] \in H^2(\mathrm{Gr}(2, \mathrm{Sym}^2 E))$.

Denote the $GL_e(\mathbb{C})$-equivariant Chern roots of S by γ_1, γ_2, those of E by $\alpha_1, \ldots, \alpha_e$, and those of Q by $\beta_1, \ldots, \beta_N$. The $GL_e(\mathbb{C})$-equivariant cohomology ring of $\mathrm{Gr}(2, \mathrm{Sym}^2 E)$ can be presented by one of

$$\mathbb{C}[\alpha_1, \ldots, \alpha_e, \gamma_1, \gamma_2]^{S_e \times S_2}/\text{relations} \qquad \text{or} \qquad \mathbb{C}[\alpha_1, \ldots, \alpha_e, \beta_1, \ldots, \beta_N]^{S_e \times S_N}/\text{relations}.$$

Since in each case the relations have degree > 2, the class $[\overline{\Phi}_e]$ is a well-defined linear polynomial $f(\alpha_1, \ldots, \alpha_e, \gamma_1, \gamma_2)$ in the α and γ variables, or a well-defined linear polynomial $g(\alpha_1, \ldots, \alpha_e, \beta_1, \ldots, \beta_N)$ in the α and β variables. Using the short exact sequence $0 \to S \to \mathrm{Sym}^2 E \to Q \to 0$, we obtain

$$\sum_{i=1}^{2} \gamma_i + \sum_{i=1}^{N} \beta_i = \sum_{1 \leq i \leq j \leq e} (\alpha_i + \alpha_j), \tag{13}$$

hence $f(\alpha, \gamma)$ and $g(\alpha, \beta)$ determine each other.

The polynomials $f(\alpha, \gamma)$ and $g(\alpha, \beta)$ have "degeneracy locus" interpretations as follows.

— Consider the $GL_2(\mathbb{C}) \times GL_e(\mathbb{C})$ representation $\mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^2 E)$ given by the following action $(A, B) \cdot \phi := \mathrm{Sym}^2 B \circ \phi \circ A^{-1}$, and the locus

$$\Phi'_e := \Big\{ \phi \in \mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^2 E) : \mathrm{rk}(\phi) = 2 \text{ and } \mathbf{P}(\mathrm{Im}(\phi)) \text{ is tangent to } \mathbf{P}(\Sigma^1) \Big\}.$$

Then

$$\begin{aligned}[\overline{\Phi'_e}] = f(\alpha, \gamma) &\in H^*_{GL_2(\mathbb{C}) \times GL_e(\mathbb{C})}(\mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^2 E)) \\ &= \mathbb{C}[\gamma_1, \gamma_2, \alpha_1, \ldots, \alpha_e]^{S_2 \times S_e}.\end{aligned}$$

— Consider the $GL_e(\mathbb{C}) \times GL_N(\mathbb{C})$ representation $\mathrm{Hom}(\mathrm{Sym}^2 E, \mathbb{C}^N)$ given by the following action $(A, B) \cdot \phi := B \circ \phi \circ \mathrm{Sym}^2 A^{-1}$, and the locus

$$\Phi''_e := \Big\{\phi \in \mathrm{Hom}(\mathrm{Sym}^2 E, \mathbb{C}^N) : \dim \mathrm{Ker}(\phi) = 2 \text{ and } \mathbf{P}(\mathrm{Ker}(\phi)) \text{ is tangent to } \mathbf{P}(\Sigma^1)\Big\}.$$

Then

$$\begin{aligned}[\overline{\Phi''_e}] = g(\alpha, \beta) &\in H^*_{GL_e(\mathbb{C}) \times GL_N(\mathbb{C})}\Big(\mathrm{Hom}(\mathrm{Sym}^2 E, \mathbb{C}^N)\Big) \\ &= \mathbb{C}[\alpha_1, \ldots, \alpha_e, \beta_1, \ldots, \beta_N]^{S_e \times S_N}.\end{aligned}$$

THEOREM 5.1. – *We have*

$$f(\alpha, \gamma) = (e-1)\left(4\sum_{i=1}^{n} \alpha_i - e\sum_{i=1}^{2} \gamma_i\right).$$

Proof. – For $\phi \in \mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^2 E)$ let $\phi((1,0)) = K$, $\phi((0,1)) = L$. The equation of the hypersurface $\overline{\Phi'_e}$ in terms of the entries of K and L is the *discriminant* of the polynomial $\det(\lambda K + L) = a_e(K, L)\lambda^e + a_{e-1}(K, L)\lambda^{e-1} + \cdots + a_0(K, L)$.

Consider the Sylvester matrix form of the discriminant

$$\frac{1}{a_e}\det\begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{e-1} & a_e & & & \\
 & a_0 & a_1 & & \cdots & a_{e-1} & a_e & & \\
 & & \ddots & \ddots & & & \ddots & \ddots & \\
 & & & a_0 & a_1 & & \cdots & a_{e-1} & a_e \\
a_1 & 2a_2 & \cdots & & ea_e & & & & \\
 & a_1 & 2a_2 & \cdots & & ea_e & & & \\
 & & \ddots & \ddots & & & \ddots & & \\
 & & & \ddots & \ddots & & & \ddots & \\
 & & & & a_1 & 2a_2 & \cdots & & ea_e
\end{pmatrix}_{((e-1)+e)\times((e-1)+e)}.$$

One of the terms of its expansion (the one coming from the main diagonal) is a non-zero constant times $(a_0 a_e)^{e-1}$. We have $a_e(K, L) = \det(K)$ and $a_0(K, L) = \det(L)$. Hence one of the monomials appearing in the discriminant is $(\prod_{i=1}^e K_{ii})^{e-1}(\prod_{i=1}^e L_{ii})^{e-1}$. The weight of this monomial is

$$(e-1)\left(\sum_{i=1}^{e}(2\alpha_i - \gamma_1)\right) + (e-1)\left(\sum_{i=1}^{e}(2\alpha_i - \gamma_2)\right). \tag{14}$$

Since $\overline{\Phi'_e}$ is invariant, all other terms must have the same weight, and this weight is the equivariant fundamental class of $\overline{\Phi'_e}$. Expression (14) simplifies to the formula in the theorem. □

REMARK 5.2. – Instead of the Sylvester matrix we could have used specializations of advanced equivariant formulas for more general discriminants, see for instance [21].

THEOREM 5.3. – *We have*

$$g(\alpha, \beta) = (e-1)\left(e\sum_{i=1}^{N} \beta_i - (e^2 + e - 4)\sum_{i=1}^{e} \alpha_i\right).$$

Proof. – The statement follows from Theorem 5.1 using relation (13). □

This completes the proof of Theorem 1.2.

6. The Petri divisor on the moduli space of curves

An immediate application of the Theorem 1.1 concerns the calculation of the class of the Petri divisor on $\overline{\mathcal{M}}_g$ consisting of genus g curves whose canonical model lies on a rank 3 quadric. We fix some notation. For $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$, let $\Delta_i \subseteq \overline{\mathcal{M}}_g$ be the boundary divisor of $\overline{\mathcal{M}}_g$ whose general point is a union of two smooth curves of genera i and $g-i$ meeting in one point. We denote by Δ_0 the closure of the locus of irreducible stable curves of genus g. As customary, we set $\delta_i = [\Delta_i]_{\mathbb{Q}} \in CH^1(\overline{\mathcal{M}}_g)$ for $i = 0, \ldots, \lfloor \frac{g}{2} \rfloor$ and denote by

$$\delta := \delta_0 + \delta_1 + \cdots + \delta_{\lfloor \frac{g}{2} \rfloor}$$

the class of the total boundary. Often we work with the partial compactification $\widetilde{\mathcal{M}}_g := \mathcal{M}_g \cup \Delta_0$, for which $CH^1(\widetilde{\mathcal{M}}_g) = \mathbb{Q}\langle \lambda, \delta_0 \rangle$.

Definition 6.1. – *For a projective variety X and a line bundle $L \in \mathrm{Pic}(X)$, for each integer $k \geq 0$ we denote by $I_{X,L}(k) := \mathrm{Ker}\{\mathrm{Sym}^k H^0(X, L) \to H^0(X, L^{\otimes k})\}$ and set $I_{X,L} := \bigoplus_{k \geq 0} I_{X,L}(k)$.*

We fix a smooth non-hyperelliptic curve C of genus g. From M. Noether's Theorem [4] the multiplication map $\mathrm{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$ is surjective. The space $I_C(2) = I_{C,\omega_C}(2)$ of quadrics containing the canonical curve $C \hookrightarrow \mathbf{P}^{g-1}$ has dimension

$$\dim I_C(2) = \binom{g-2}{2}.$$

We conclude that the locus $\mathcal{GP}_g$ of curves whose canonical model lies on a rank 3 quadric is expected to be a divisor. Via the Base Point Free Pencil Trick [4] p. 126, this expectation can be confirmed.

Proposition 6.2. – *The locus $\mathcal{GP}_g$ coincides set-theoretically with the divisor of curves $[C] \in \mathcal{M}_g$ having a pencil A such that the Petri map $\mu(A) : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^{\vee}) \to H^0(C, \omega_C)$ is not injective.*

Proof. – Let A be a line bundle on C with $h^0(C, A) = 2$. Denote by $F := \mathrm{bs}\,|A|$ its base locus and set $B := A(-F)$. Thus $H^0(C, B) \cong H^0(C, A)$. Applying the Base Point Free Pencil Trick, we obtain

$$\mathrm{Ker}(\mu(A)) \cong H^0(\omega_C \otimes A^{-2}(F)) \cong H^0(C, \omega_C \otimes B^{-2}(-F)).$$

Thus if $\mu(A)$ is not injective, by possibly enlarging the effective divisor F, we find there exists a base point free pencil B on C and an effective divisor F, such that $\omega_C = B^2(F)$.

Assume the canonical curve $C \subseteq \mathbf{P}^{g-1}$ lies on a rank 3 quadric Q. Denote by $F := C \cdot \mathrm{Sing}(Q)$, where $\mathrm{Sing}(Q) \cong \mathbf{P}^{g-4}$. Then if B is the pull back to C of the unique ruling of Q, we obtain the relation $\omega_C = \mathcal{O}_C(1) \cong B^2(F)$. Setting $A := B(F)$, we obtain that $\mu(A)$ is not injective.

To conclude that $\mathcal{GP}_g$ is a divisor in $\mathcal{M}_g$, we invoke the Gieseker-Petri Theorem which asserts that the Petri map $\mu(A)$ is *injective* for *every* line bundle A on a general curve C of genus g. $\square$

The divisor $\mathcal{GP}_g$ can be extended over $\overline{\mathcal{M}}_g$. Let $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ the universal curve of genus g. We denote by $\mathbb{E} := \pi_*(\omega_\pi)$ the Hodge bundle on $\overline{\mathcal{M}}_g$, having fibers $\mathbb{E}[C] := H^0(C, \omega_C)$. Let $\mathbb{F} := \pi_*(\omega_\pi^{\otimes 2})$. Both sheaves $\mathbb{E}$ and $\mathbb{F}$ are locally free over $\overline{\mathcal{M}}_g$ and denote by

$$\phi : \mathrm{Sym}^2(\mathbb{E}) \to \mathbb{F}$$

the morphism globalizing the multiplication maps $\phi_C : \mathrm{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$, as the curve $[C] \in \overline{\mathcal{M}}_g$ varies in moduli. Set

$$\widetilde{\mathcal{GP}}_g := \Big\{[C] \in \overline{\mathcal{M}}_g : \exists\, 0 \neq q \in \mathrm{Ker}(\phi_C),\ \ \mathrm{rk}(q) \leq 3\Big\}.$$

Clearly $\widetilde{\mathcal{GP}}_g$ is a divisor on $\overline{\mathcal{M}}_g$ and $\widetilde{\mathcal{GP}}_g \cap \mathcal{M}_g = \mathcal{GP}_g$. For a generic point $[C := C_1 \cup_p C_2] \in \Delta_i$, where C_1 and C_2 are smooth curves of genus i and $g-i$ respectively meeting at one point p, one has $H^0(C, \omega_C) \cong H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2})$, that is, every section from $H^0(C, \omega_C)$ vanishes at p. On the other hand,

$$H^0(C, \omega_C^2) \cong \mathrm{Ker}\big\{H^0(C_1, \omega_{C_1}^2(2p)) \oplus H^0(C_2, \omega_{C_2}^2(2p)) \to \mathbb{C}_p\big\},$$

that is, there exists quadratic differentials on C *not* vanishing at p. It follows that the multiplication map ϕ_C is not surjective, hence for dimension reasons $\mathrm{Ker}(\phi_C)$ contains quadrics of rank 3, whenever $[C] \in \Delta_i$. Thus $\Delta_i \subseteq \widetilde{\mathcal{GP}}_g$, for $i = 1, \ldots, \lfloor \frac{g}{2} \rfloor$. On the other hand, Δ_0 is not contained in $\widetilde{\mathcal{GP}}_g$. In fact, the generic g-nodal rational curve satisfies the Green-Lazarsfeld property $N_{\lfloor \frac{g-3}{2} \rfloor}$, that is, a much stronger property than projective normality, see [53]. Denoting by $\overline{\mathcal{GP}}_g$ the closure of the Petri divisor $\mathcal{GP}_g$ inside $\overline{\mathcal{M}}_g$, we thus have an equality of effective divisors on $\overline{\mathcal{M}}_g$

$$\widetilde{\mathcal{GP}}_g = \overline{\mathcal{GP}}_g + \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} b_i \Delta_i,$$

where $b_i \geq 1$, for all $i \geq 1$. The class of $\widetilde{\mathcal{GP}}_g$ can now be easily determined.

Proof of Theorem 1.6. – We apply Theorem 1.1 in the case of the morphism $\phi : \mathrm{Sym}^2(\mathbb{E}) \to \mathbb{F}$ over $\overline{\mathcal{M}}_g$ given by multiplication. We have $c_1(\mathbb{E}) = \lambda$, whereas by the Grothendieck-Riemann-Roch calculation carried out in [43] Theorem 5.10, one has $c_1(\mathbb{F}) = \lambda + \kappa_1 = 13\lambda - \delta$. $\square$

The Petri divisor decomposes into components depending on the degree of the pencil for which the Petri Theorem fails. For $\lfloor \frac{g+2}{2} \rfloor \leq k \leq g-1$, we denote by $D_{g,k}$ the locus of curves $[C] \in \mathcal{M}_g$ for which there exists a *base point free* pencil $A \in W^1_k(C)$ such that $\mu(A)$ is not injective. It is shown in [17] that $D_{g,k}$ has at least one divisorial component. In light of Proposition 6.2, we have the decomposition

$$\overline{\mathcal{GP}}_g = \sum_{k=\lfloor \frac{g+2}{2} \rfloor}^{g-1} a_{g,k} \overline{D}_{g,k}. \tag{15}$$

It is an interesting open question to determine the classes $[\overline{D}_{g,k}] \in CH^1(\overline{\mathcal{M}}_g)$ and their multiplicities $a_{g,k}$. For birational geometry application, it is more relevant to compute the slopes $s(\overline{D}_{g,k})$. Few of the individual divisors $D_{g,k}$ are well understood.

By the proof of Proposition 6.2, the divisor $D_{g,g-1}$ consists of curves with an even theta-characteristic $\vartheta \in \mathrm{Pic}^{g-1}(C)$ such that $h^0(C,\vartheta) \geq 2$. The class of its compactification in $\overline{\mathcal{M}}_g$ has been computed in [7] and we have:

$$[\overline{D}_{g,g-1}] = 2^{g-3}\Big((2^g+1)\lambda - 2^{g-3}\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor}(2^{g-i}-1)(2^i-1)\delta_i\Big). \tag{16}$$

When k is minimal, for odd $g = 2k-1$, the locus $D_{g,k}$ is the Hurwitz divisor of curves of gonality at most k. Its compactification is the image of the space of admissible covers $\overline{\mathcal{H}}_k$ defined in the Introduction. Harris and Mumford [30] computed its class, on their way to show that $\overline{\mathcal{M}}_g$ is of general type for large odd genus $g \geq 25$:

$$[\overline{D}_{2k-1,k}] = \frac{1}{(2k-2)(2k-3)}\binom{2k-2}{k-1}\Big(6(k+1)\lambda - k\delta_0 - \sum_{i=1}^{k-1}3i(2k-i-1)\delta_i\Big). \tag{17}$$

For even genus $g = 2k$, the divisor $\overline{D}_{2k,k+1}$ can be viewed as the branch map of the generically finite cover $\overline{\mathcal{H}}_{2k,k+1} \to \overline{\mathcal{M}}_{2k}$ from the space of admissible covers of degree $k+1$. The calculation of its class in [13] Theorem 2 has been instrumental in proving that $\overline{\mathcal{M}}_g$ is of general type for even genus $g \geq 24$:

$$[\overline{D}_{2k,k+1}] = \frac{2(2k-2)!}{(k-1)!(k+1)!}\Big((6k^2+13k+1)\lambda - k(k+1)\delta_0 - (2k-1)(3k+1)\delta_1 - \cdots\Big). \tag{18}$$

The only case when k is not extremal has been treated in [17] and it concerns the divisor $D_{2k-1,k+1}$. It is shown in [17] Corollary 0.6 that its slope equals

$$s(\overline{D}_{2k-1,k+1}) = \frac{6k^2+14k+3}{k(k+1)}. \tag{19}$$

In the range $g \leq 7$, these known cases exhaust all Gieseker-Petri divisors and we can compare Theorem 1.6 with the previously mentioned formulas (16), (17), (18). We denote by $\widetilde{D}_{g,k}$ the closure of $D_{g,k}$ in $\widetilde{\mathcal{M}}_g$. In order to determine the slope of $\overline{D}_{g,k}$, it suffices to compute the class $[\widetilde{D}_{g,k}] \in CH^1(\widetilde{\mathcal{M}}_g)$, for as in the case of $\overline{\mathcal{GP}}_g$, the δ_0-coefficient is smaller in absolute value than the higher boundary coefficients in the expansion of $[\overline{D}_{g,k}]$ in terms of the generators of $CH^1(\overline{\mathcal{M}}_g)$.

For $g = 4$, there is only one component and we obtain $[\widetilde{\mathcal{GP}}_4] = [\widetilde{D}_{4,3}] = 34\lambda - 4\delta_0 \in CH^1(\widetilde{\mathcal{M}}_4)$. For $g = 5$, we obtain $[\widetilde{\mathcal{GP}}_5] = [\widetilde{D}_{5,4}] + 4[\widetilde{D}_{5,3}] = 4(41\lambda - 5\delta_0)$, whereas for $g = 6$, we find

$$[\widetilde{\mathcal{GP}}_6] = [\widetilde{D}_{6,5}] + 4[\widetilde{D}_{6,4}] = 8(112\lambda - 14\delta_0) \in CH^1(\widetilde{\mathcal{M}}_6).$$

Finally, in the case $g = 7$, there are three Petri divisors and we obtain

$$[\widetilde{\mathcal{GP}}_7] = [\widetilde{D}_{7,6}] + 4[\widetilde{D}_{7,5}] + 16[\widetilde{D}_{7,4}] = 96(55\lambda - 7\delta_0) \in CH^1(\widetilde{\mathcal{M}}_7).$$

Based on this formulas for small genus, we make the following conjecture, though we admit that the evidence for it is rather moderate.

CONJECTURE 6.3. – *One has $a_{g,k} = 4^{g-1-k}$ for $\frac{g+2}{2} \le k \le g-1$, that is, the following holds:*

$$[\widetilde{\mathcal{GP}}_g] = \sum_{i=1}^{\lceil \frac{g-2}{2} \rceil} 4^{i-1}[\widetilde{D}_{g,g-i}] \in CH^1(\widetilde{\mathcal{M}}_g).$$

7. Effective divisors of small slope on $\overline{\mathcal{M}}_g$

We now present an infinite series of effective divisors on $\overline{\mathcal{M}}_g$ of slope less than $6 + \frac{12}{g+1}$, which recall, is the slope of all the Brill-Noether divisors. We fix integers $r \ge 3$ and $s \ge 1$ and set

$$g := rs + s \text{ and } d := rs + r.$$

Observe that $\rho(g, r, d) = g - (r+1)(g - d + r) = 0$, hence by general Brill-Noether Theory a general curve of genus g has a finite number of linear systems of type $\mathfrak{g}^r_d$. Let $\mathcal{M}^\sharp_g$ the open substack of $\mathcal{M}_g$ classifying smooth genus g curves C such that $W^r_{d-1}(C) = \emptyset$, $W^{r+1}_d(C) = \emptyset$ and furthermore $H^1(C, L^{\otimes 2}) = 0$, for every $L \in W^r_d(C)$. Then $\operatorname{codim}(\mathcal{M}_g - \mathcal{M}^\sharp_g, \mathcal{M}_g) \ge 2$. For codimension one calculation, one makes no difference between $\mathcal{M}_g$ and $\mathcal{M}^\sharp_g$. We denote by $\mathfrak{G}^r_{g,d}$ the stack parametrizing pairs $[C, L]$, with $[C] \in \mathcal{M}^\sharp_g$ and $L \in W^r_d(C)$ is a necessarily complete and base point free linear system. Let

$$\sigma : \mathfrak{G}^r_{g,d} \to \mathcal{M}^\sharp_g$$

be the natural projection. It is known from general Brill-Noether Theory that there exists a unique irreducible component of $\mathfrak{G}^r_{g,d}$ which maps dominantly onto $\mathcal{M}_g$.

We pick a general point $[C, L] \in \mathfrak{G}^r_{g,d}$ of the dominating component. It follows from the Maximal Rank Conjecture proved in this case in [16] or [38] Theorem 1.4, that the multiplication map

$$\phi_{C,L} : \operatorname{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is surjective. Since $H^1(C, L^{\otimes 2}) = 0$, by Riemann-Roch, the dimension of its kernel $I_{C,L}(2)$ equals

$$\dim I_{C,L}(2) = \binom{r+2}{2} - (2d + 1 - g).$$

We impose the condition that this number equal the codimension of the space

$$\Sigma^{r-a-1}_{r+1} \subseteq \operatorname{Sym}^2 H^0(C, L)$$

of quadrics of rank at most $a+2$ (that is, corank $r - a - 1$). Since $\operatorname{codim}(\Sigma^{r-a-1}_{r+1}) = \binom{r-a}{2}$, we obtain the following numerical constraint on s and r:

$$s = \frac{a(2r - 1 - a)}{2(r-1)}. \tag{20}$$

For each r and s such that the equation (20) is satisfied, we consider the locus

$$Z_{r,s} := \Big\{[C, L] \in \mathfrak{G}^r_{g,d} : \exists\, 0 \ne q \in I_{C,L}(2),\ \operatorname{rk}(q) \le a + 2\Big\}$$

and set $D_{r,s} := \sigma_*(Z_{r,s})$. Then $D_{r,s}$ is expected to be a divisor on $\mathcal{M}_g$, that is, either it is a divisor in which case there exists a smooth curve $[C] \in \mathcal{M}_g$ such that $I_{C,L}(2)$ contains no quadrics of rank at most $a+2$ for every $L \in W^r_d(C)$, or else $D_{r,s} = \mathcal{M}_g$. We shall determine the slope of the virtual class of its closure in $\overline{\mathcal{M}}_g$.

Before moving further, we discuss some solutions to equation (20). If $a = r-1$ (that is, when one considers quadrics of maximal rank), then $r = 2s$ and $g = s(2s+1)$. In this case $D_{2s,s}$ is the locus of curves $[C] \in \mathcal{M}_{s(2s+1)}$ for which there exists a linear series $L \in W^{2s}_{2s(s+1)}(C)$ such that the multiplication map $\phi_{C,L} : \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2})$ is not an isomorphism. This series of divisors has been studied in detail in [16] Theorem 1.5, as well as in [35] and shown to contradict the Harris-Morrison Slope Conjecture [29].

The first series of genuinely new examples is when for an integer $\ell \geq 1$, we take

$$s = 4\ell - 1, \;\; r = 9\ell - 2, \;\; a = 2(3\ell - 1), \;\; \text{and} \;\; g = (4\ell - 1)(9\ell - 1). \tag{21}$$

Specializing to the case $\ell = 1$, we obtain the following effective (virtual) divisor on $\mathcal{M}_{24}$:

$$D_{7,3} := \Big\{[C] \in \mathcal{M}_{24} : \exists\, L \in W^7_{28}(C),\ \exists\, 0 \neq q \in I_{C,L}(2),\ \ \mathrm{rk}(q) \leq 6\Big\}.$$

A second series of examples is when for an integer $\ell \geq 1$, we take the following values

$$s = 3\ell + 1, \; r = 8\ell + 3, \; a = 4\ell + 1, \;\; \text{and} \;\; g = 4(3\ell + 1)(2\ell + 1). \tag{22}$$

The first example in this series appears produces an effective (virtual) divisor on $\mathcal{M}_{48}$:

$$D_{11,4} := \Big\{[C] \in \mathcal{M}_{48} : \exists\, L \in W^{11}_{55}(C),\ \exists\, 0 \neq q \in I_{C,L}(2),\ \ \mathrm{rk}(q) \leq 7\Big\}.$$

We now describe the (virtual) divisor structure of $D_{r,s}$ and set up some notation that will help compute the class of their closure in $\overline{\mathcal{M}}_g$. We introduce the partial compactification $\widetilde{\mathcal{M}}^{\sharp}_g$ defined as the union of $\mathcal{M}^{\sharp}_g$ and the open substack $\Delta_0^{\sharp} \subseteq \Delta_0$ classifying 1-nodal irreducible genus g curves $C' = C/p \sim q$, where $[C,p,q] \in \mathcal{M}_{g-1,2}$ is a Brill-Noether general 2-pointed curve in the sense of [13] Theorem 1.1, together with all their degenerations consisting of unions of a smooth genus $g-1$ curve and a nodal rational curve. Note that $\widetilde{\mathcal{M}}_g$ and $\widetilde{\mathcal{M}}^{\sharp}_g$ agree outside a set of codimension 2 and we identify the Picard groups of the two stacks. We denote by $\widetilde{\mathfrak{G}}^r_{g,d}$ the parameter space of pairs $[C,L]$, where $[C] \in \mathcal{M}^{\sharp}_g$ and L is a torsion free sheaf of rank 1 and degree d on C such that $h^0(C,L) \geq r+1$. We still denote by $\sigma : \widetilde{\mathfrak{G}}^r_{g,d} \to \widetilde{\mathcal{M}}^{\sharp}_g$ the proper forgetful morphism.

We now consider the universal curve $\pi : \widetilde{\mathcal{M}}^{\sharp}_{g,1} \to \widetilde{\mathcal{M}}^{\sharp}_g$ and denote by $\mathcal{L}$ a universal bundle on the fiber product $\widetilde{\mathcal{M}}^{\sharp}_{g,1} \times_{\widetilde{\mathcal{M}}^{\sharp}_g} \widetilde{\mathfrak{G}}^r_{g,d}$. If

$$p_1 : \widetilde{\mathcal{M}}^{\sharp}_{g,1} \times_{\widetilde{\mathcal{M}}^{\sharp}_g} \widetilde{\mathfrak{G}}^r_{g,d} \to \widetilde{\mathcal{M}}^{\sharp}_{g,1} \quad \text{and} \quad p_2 : \widetilde{\mathcal{M}}^{\sharp}_{g,1} \times_{\widetilde{\mathcal{M}}^{\sharp}_g} \widetilde{\mathfrak{G}}^r_{g,d} \to \widetilde{\mathfrak{G}}^r_{g,d}$$

are the natural projections, then $\mathcal{E} := p_{2*}(\mathcal{L})$ and $\mathcal{F} := p_{2*}(\mathcal{L}^{\otimes 2})$ are locally free sheaves of ranks $r+1$ and $2d+1-g$ respectively. Finally, we denote by

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$$

the sheaf morphism given by multiplication of sections.

THEOREM 7.1. – *Set $r = 9\ell - 2$ and $s = 4\ell - 1$, therefore $g = (4\ell - 1)(9\ell - 1)$, where $\ell \geq 1$. Then the virtual class of the closure of the divisor $D_{9\ell-2,4\ell-1}$ inside $\overline{\mathcal{M}}_{(4\ell-1)(9\ell-1)}$ equals*

$$s(\overline{D}_{9\ell-2,4\ell-1}) = \frac{a}{b},$$

where

$$a := 15116544\ell^8 - 30233088\ell^7 + 26605584\ell^6 - 13594392\ell^5 + 4419720\ell^4 - 899433\ell^3 + 105656\ell^2 - 6101\ell + 122$$

and

$$b := 2(9\ell - 2)(9\ell - 1)(15552\ell^6 - 25920\ell^5 + 17484\ell^4 - 6102\ell^3 + 1181\ell^2 - 107\ell + 2).$$

In particular, $s(\overline{D}_{9\ell-2,4\ell-1}) < 6 + \frac{12}{g+1}$.

If we look at the difference between the slope of $\overline{D}_{9\ell-2,4\ell-1}$ and that of the Brill-Noether divisors we get a slightly simpler formula:

$$s(\overline{D}_{9\ell-2,4\ell-1}) = 6 + \frac{12}{g+1} - \frac{(13\ell-2)(36\ell-13)(27\ell^2-19\ell+2)(36\ell^2-13\ell-1)}{2(9\ell-2)(9\ell-1)(15552\ell^6-25920\ell^5+17484\ell^4-6102\ell^3+1181\ell^2-107\ell+2)(36\ell^2-13\ell+2)}.$$

We now record the slope of the effective divisors in the second series of examples:

THEOREM 7.2. – *Set $r = 8\ell + 3$ and $s = 3\ell + 1$, therefore $g = 4(3\ell + 1)(2\ell + 1)$. Then the virtual class of the closure of the divisor $D_{8\ell+3,3\ell+1}$ inside $\overline{\mathcal{M}}_{4(3\ell+1)(2\ell+1)}$ equals*

$$s(\overline{D}_{8\ell+3,3\ell+1}) = 6 + \frac{12}{g+1} - \frac{(11\ell+5)(2\ell-1)(12\ell^2+10\ell+1)(24\ell^2+20\ell+3)}{(3\ell+2)(8\ell+3)(2304\ell^6+4128\ell^5+2992\ell^4+1128\ell^3+248\ell^2+41\ell+5)(24\ell^2+20\ell+5)}.$$

Proof of Theorems 7.1 and 7.2. – We choose integers $r \geq 3$, $s, a \geq 1$ such that (20) holds. Recall that $d = rs + r$ and $g = rs + s$. We shall apply the techniques developed in [16] and [35] in the context of Theorem 4.7. Recall that we have defined the vector bundle morphism $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ over the parameter space $\widetilde{\mathfrak{G}}^r_{g,d}$. Applying Theorem 1.1, if $Z_{r,s}$ is a divisor on $\mathfrak{G}^r_{g,d}$, then the class of its closure $\widetilde{Z}_{r,s}$ inside $\widetilde{\mathfrak{G}}^r_{g,d}$ is given by the formula

$$[\widetilde{Z}^r_{g,d}] = \alpha\Big(c_1(\mathcal{F}) - \frac{2(2d+1-g)}{r+1}c_1(\mathcal{E})\Big). \tag{23}$$

We call the right hand side of the formula (23) the virtual class $[\widetilde{Z}^r_{g,d}]^{\mathrm{virt}}$ of the virtual divisor $\widetilde{Z}^r_{g,d}$. Following [16] we introduce the following tautological divisor classes on $\widetilde{\mathfrak{G}}^r_{g,d}$:

$$\mathfrak{a} := (p_2)_*\Big(c_1^2(\mathcal{L})\Big),\ \ \mathfrak{b} := (p_2)_*\Big(c_1(\mathcal{L}) \cdot c_1(\omega_{p_2})\Big) \text{ and } \mathfrak{c} := (p_2)_*\Big(c_1^2(\omega_{p_2})\Big) = \sigma^*(\kappa_1),$$

where we recall that $\kappa_1 = 12\lambda - \delta \in CH^1(\overline{\mathcal{M}}_g)$ is Mumford's class, see also [43].

Since $R^1(p_2)_*(\mathcal{L}^{\otimes 2}) = 0$, applying Grothendieck-Riemann-Roch to p_2, we compute

$$c_1(\mathcal{F}) = \sigma^*(\lambda) - \mathfrak{b} + 2\mathfrak{a}.$$

The push-forwards of the tautological classes $\mathfrak{a}, \mathfrak{b}$ and $c_1(E)$ under the generically finite proper morphism $\sigma : \widetilde{\mathfrak{G}}^r_{g,d} \to \widetilde{\mathcal{M}}_g^{\sharp}$ are determined in [16] Section 2 and [35] Theorem 2.11 and we summarize the results: There exists an explicit constant $\beta \in \mathbb{Z}_{>0}$ such that

$$\sigma_*(\mathfrak{a}) = \beta \frac{d}{(g-1)(g-2)}\Big((dg^2 - 2g^2 + 8d - 8g + 4)\lambda - (dg - 2g^2 + 4d - 3g + 2)\delta_0\Big),$$
$$\sigma_*(\mathfrak{b}) = \beta \frac{d}{g-1}\Big(6\lambda - \frac{\delta_0}{2}\Big)$$

and

$$\sigma_*(c_1(\mathcal{E})) = \beta\Big(-\frac{r(r+2)(r^2s^3 + 2rs^3 - r^2s + 6rs^2 + s^3 - 2rs + 6s^2 - 8r + 3s - 8)}{2(r+s+1)(rs+s-2)(rs+s-1)}\lambda + \frac{r(s-1)(s+1)(r+2)(r+1)(rs+s+4)}{12(r+s+1)(rs+s-2)(rs+s-1)}\delta_0\Big).$$

We substitute these formulas in (23) and we obtain a closed formula for $[\widetilde{Z}_{r,s}]$. Substituting the particular values in Theorems 7.1 and 7.2, we obtain the claimed formulas for the slopes. $\square$

We expect the virtual divisors constructed in Theorems 7.1 and 7.2 to be actual divisors for all ℓ. We can directly confirm this expectation for all bounded ℓ. We illustrate this in the case $\ell = 1$.

THEOREM 7.3. – *The locus $D_{7,3}$ is a divisor on $\mathcal{M}_{24}$, that is, for a general curve C of genus 24, the image curve $\varphi_L : C \hookrightarrow \mathbf{P}^7$ lies on no quadric of rank at most 6, for any linear system $L \in W^7_{28}(C)$.*

Proof. – By residuation, we have a birational isomorphism $\mathfrak{G}^7_{24,28} \cong \mathfrak{G}^2_{24,18}$ of parameter spaces over $\mathcal{M}_{24}$. The latter space is a quotient of the Severi variety of plane curves of genus 24 and degree 18 which is known to be irreducible [28], hence $\mathfrak{G}^7_{24,28}$ is an irreducible, generically finite cover of $\mathcal{M}_{24}$. To show that $D_{7,3}$ is a divisor, that is, $D_{7,3} \neq \mathcal{M}_{24}$, it suffices to produce *one* smooth curve $[C] \in \mathcal{M}_{24}$ and *one* very ample linear system $L \in W^7_{28}(C)$ such that the image curve $\varphi_L : C \hookrightarrow \mathbf{P}^7$ does not lie on any quadric of rank at most 6. The curve we construct lies on a rational surface X in $\mathbf{P}^7$ and has the property that all the quadrics containing C also contain X.

Precisely, we start with 16 general points $p_1, \ldots, p_{16} \in \mathbf{P}^2$. We embed the surface $X := \mathrm{Bl}_{16}(\mathbf{P}^2)$ obtained by blowing-up these points in the space $\mathbf{P}^7$ via the linear system

$$H = 9h - 3E_1 - 2\sum_{i=2}^{14} E_i - E_{15} - E_{16} \in \mathrm{Pic}(X),$$

where h is the hyperplane class and E_i is the exceptional divisor corresponding to the point p_i, for $i = 1, \ldots, 16$. By direct computation we find

$$h^0(X, \mathcal{O}_X(2)) = h^0\Big(X, \mathcal{O}_X\big(18h - 6E_1 - 4\sum_{i=2}^{14} E_i - 2E_{15} - 2E_{16}\big)\Big)$$
$$= \binom{20}{2} - \binom{7}{2} - 13\binom{5}{2} - 2\binom{3}{2} = 33.$$

By using *Macaulay*, we check that $|H|$ embeds X into $\mathbf{P}^7$, the natural multiplication map $\mathrm{Sym}^2 H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_X(2))$ is surjective, hence $\dim I_{X,\mathcal{O}_X(1)}(2) = 3$ and $H^1(\mathbf{P}^7, \mathcal{I}_{X/\mathbf{P}^7}(2)) = 0$. We check furthermore with *Macaulay* that $I_{X,\mathcal{O}_X(1)}(2) \cap \Sigma_8^2 = \emptyset$, that is, $X \subseteq \mathbf{P}^7$ lies on no quadric of rank at most 6.

We construct a curve $C \subseteq X$ as a general element of the linear system

$$C \in \Big|20h - 6E_1 - 5\sum_{i=2}^{13} E_i - 4E_{14} - 3E_{15} - 3E_{16}\Big|.$$

Then $C \cdot H = 28$ and we check by *Macaulay* that such a curve C is smooth. In particular, it follows that $g(C) = 1 + \frac{1}{2} C \cdot (C + K_X) = 24$. Furthermore, one has an exact sequence

$$0 \longrightarrow I_{X,\mathcal{O}_X(1)}(2) \longrightarrow I_{C,\mathcal{O}_C(1)}(2) \longrightarrow H^0(X, \mathcal{O}_X(2H - C)) \longrightarrow 0,$$

induced by the exact sequence $0 \to \mathcal{I}_{X/\mathbf{P}^7}(2) \to \mathcal{I}_{C/\mathbf{P}^7}(2) \to \mathcal{O}_X(2H - C) \to 0$, where we use once more that $H^1(\mathbf{P}^7, \mathcal{I}_{X/\mathbf{P}^7}(2)) = 0$. Since $H^0(X, \mathcal{O}_X(2H - C)) = 0$, this induces an isomorphism $I_{X,\mathcal{O}_X(1)}(2) \cong I_{C,\mathcal{O}_C(1)}(2)$. This shows that the smooth curve $C \subseteq \mathbf{P}^7$ lies on no quadric of rank at most 6, which finishes the proof[(1)]. $\square$

8. The slope of $\overline{\mathcal{M}}_{12}$

We explain in this section how using Theorems 1.2 and 5.3 one can construct an effective divisor on $\overline{\mathcal{M}}_{12}$ having slope less than $6 + \frac{12}{g+1}$.

A general curve $[C] \in \mathcal{M}_{12}$ has finitely many linear systems $L \in W^5_{15}(C)$. As already pointed out, the multiplication map $\phi_{C,L} : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ is surjective for each $L \in W^5_{15}(C)$, in particular $\mathbf{P}_L := \mathbf{P}\big(I_{C,L}(2)\big)$ is a pencil of quadrics in $\mathbf{P}^5$ containing the curve $\varphi_L : C \hookrightarrow \mathbf{P}^5$. By imposing the condition that the pencil $\mathbf{P}_L$ be degenerate, we produce a divisor on $\overline{\mathcal{M}}_{12}$, whose class we ultimately compute.

Proof of Theorem 1.10. – We retain the notation of the previous section and recall that $\sigma : \widetilde{\mathfrak{G}}^5_{12,15} \to \widetilde{\mathcal{M}}^{\sharp}_{12}$ denotes the proper forgetful morphism from the parameter space of generalized linear series $\mathfrak{g}^5_{15}$ onto (an open subset of) the moduli space of irreducible curves of genus 12. Furthermore, we retain the same notation for the tautological bundles $\mathcal{E}$ and $\mathcal{F}$ over $\widetilde{\mathfrak{G}}^5_{12,15}$, as well as for the vector bundle morphism $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$, globalizing the multiplication maps $\phi_{C,L}$, as $[C, L]$ varies over $\widetilde{\mathfrak{G}}^5_{12,15}$. In particular $\mathbf{P}_L \cong \mathbf{P}\big(\mathrm{Ker}(\phi_{C,L})\big)$, for every $[C, L] \in \widetilde{\mathfrak{G}}^5_{12,15}$. Noting that $\mathrm{rk}(\mathcal{E}) = 6$ and $\mathrm{rk}(\mathcal{F}) = 19$, we apply Proposition 5.3. The virtual class of the locus Z of pairs $[C, L] \in \widetilde{\mathfrak{G}}^5_{12,15}$ such that $\mathbf{P}_L$ is a degenerate pencil equals

$$[Z]^{\mathrm{virt}} = 10\Big(6c_1(\mathcal{F}) - 38c_1(\mathcal{E})\Big) \in CH^1(\widetilde{\mathfrak{G}}^5_{12,15}).$$

The pushforward classes $\sigma_*(c_1(\mathcal{E}))$ and $\sigma_*(c_1(\mathcal{F}))$ have been described in the proof of Theorems 7.1 and 7.2. After easy manipulations, we compute the class $[\overline{\mathfrak{D}\mathfrak{p}}_{12}]^{\mathrm{virt}} := \sigma_*([Z]^{\mathrm{virt}}) \in CH^1(\widetilde{\mathcal{M}}^{\sharp}_{12})$.

(1) The *Macaulay* file containing all the computations appearing in this proof can be found online at `https://www.mathematik.hu-berlin.de/~farkas/computations-gen24.m2`.

It remains to establish that Z is indeed a divisor inside $\widetilde{\mathfrak{G}}^5_{12,15}$. To that end, we observe that one has a birational isomorphism $\mathfrak{G}^5_{12,15} \cong \mathfrak{G}^1_{12,7}$. The latter being the Hurwitz space of degree 7 covers of $\mathbf{P}^1$, it is well-known to be irreducible, hence $\mathfrak{G}^5_{12,15}$ is irreducible as well. Therefore it suffices to exhibit one projectively normal smooth curve $C \subseteq \mathbf{P}^5$ of genus 12 and degree 15, such that $\mathbf{P}_{\mathcal{O}_C(1)}$ is non-degenerate. This is achieved in a way similar to the proof of Theorem 7.3, by choosing C to lie on a particular rational surface.

We pick 11 general points $p_1, \ldots, p_{11} \in \mathbf{P}^2$. We embed the surface $X := \mathrm{Bl}_{11}(\mathbf{P}^2)$ obtained by blowing-up these points in $\mathbf{P}^5$ via the linear system

$$H = 5h - 2E_1 - 2E_2 - \sum_{i=3}^{11} E_i \in \mathrm{Pic}(X),$$

where h is the hyperplane class and E_i is the exceptional divisor corresponding to the point p_i, for $i = 1, \ldots, 11$. We compute $h^0(X, \mathcal{O}_X(2)) = 19$ and $\dim I_{X,\mathcal{O}_X(1)}(2) = 2$. We check furthermore with *Macaulay* that the pencil $\mathbf{P}_{\mathcal{O}_X(1)}$ is non-degenerate.

We construct a curve $C \subseteq X$ as a general element of the following linear system on X

$$C \in \left|10h - 4E_1 - 4E_2 - 3E_3 - 3E_4 - 2\sum_{i=5}^{10} E_i - E_{11}\right|.$$

Then C is a smooth curve of genus 12 with $C \cdot H = 15$. Since $H^0(X, \mathcal{O}_X(2H - C)) = 0$, we have an isomorphism $I_{X,\mathcal{O}_X(1)}(2) \cong I_{C,\mathcal{O}_C(1)}(2)$, showing that the pencil $\mathbf{P}_{\mathcal{O}_C(1)}$ is non-degenerate. $\square$

9. Tautological classes on the moduli space of polarized $K3$ surfaces

For a positive integer g, we denote by $\mathcal{F}_g$ the moduli space of quasi-polarized $K3$ surfaces of genus g classifying pairs $[X, L]$, where X is a smooth $K3$ surface and $L \in \mathrm{Pic}(S)$ is a big and nef line bundle with $L^2 = 2g - 2$. Via the Torelli Theorem for $K3$ surfaces, one can realize $\mathcal{F}_g$ as the quotient Ω_g/Γ_g of a 19-dimensional symmetric domain Ω_g by an arithmetic subgroup Γ_g of $SO(3, 19)$.

We denote by $\pi : \mathcal{X} \to \mathcal{F}_g$ the universal polarized $K3$ surface of genus g and by $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ a universal polarization line bundle. Note that $\mathcal{L}$ is not unique, for it can be twisted by the pull-back of any line bundle coming from $\mathcal{F}_g$. Recall that the Hodge bundle on $\mathcal{F}_g$ is defined by

$$\lambda := \pi_*(\omega_\pi) \in \mathrm{Pic}(\mathcal{F}_g).$$

Following [39], for non-negative integers a, b we also consider the κ classes on $\mathcal{F}_g$, by setting

$$\kappa_{a,b} := \pi_*\Big(c_1(\mathcal{L})^a \cdot c_2(\mathcal{T}_\pi)^b\Big) \in CH^{a+2b-2}(\mathcal{F}_g).$$

We shall concentrate on the codimension 1 tautological classes, that is, on $\kappa_{3,0}$ and $\kappa_{1,1}$. Replacing $\mathcal{L}$ by $\widetilde{\mathcal{L}} := \mathcal{L} \otimes \pi^*(\alpha)$, where $\alpha \in \mathrm{Pic}(\mathcal{F}_g)$, the classes $\kappa_{3,0}$ and $\kappa_{1,1}$ change as follows:

$$\widetilde{\kappa}_{3,0} = \kappa_{3,0} + 6(g-1)\alpha \ \text{ and } \ \widetilde{\kappa}_{1,1} = \kappa_{1,1} + 24\alpha.$$

It follows that the following linear combination of κ classes

$$\gamma := \kappa_{3,0} - \frac{g-1}{4}\kappa_{1,1} \in CH^1(\mathcal{F}_g)$$

is well-defined and independent of the choice of a Poincaré bundle on $\mathcal{X}$.

9.1. $K3$ surfaces and rank 4 quadrics

Recall that in the Introduction we have introduced the Noether-Lefschetz divisors $D_{h,d}$ consisting of quasi-polarized $K3$ surfaces $[X, L] \in \mathcal{F}_g$ such that there exists a primitive embedding of a rank 2 lattice $\mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot D \subseteq \mathrm{Pic}(X)$, where $D \in \mathrm{Pic}(X)$ is a class with $D \cdot L = d$ and $D^2 = 2h-2$. In what follows, we fix a quasi-polarized $K3$ surface $[X, L] \in \mathcal{F}_g$ and consider the map

$$\varphi_L : X \to \mathbf{P}^g$$

induced by the polarization. We recall a few classical results on linear systems on $K3$ surfaces. Since L is big and nef, using [50] Proposition 2.6, we find that L is base point free unless there exists an elliptic curve $E \subseteq X$ with $E \cdot L = 1$. In this case, $L = gE + \Gamma$, where $\Gamma^2 = -2$ and $E \cdot \Gamma = 1$. This case corresponds to the NL divisor $D_{1,1}$. If L is base point free, then L is not very ample if and only if there is a divisor $E \in \mathrm{Pic}(X)$ with $E^2 = -2$ and $E \cdot L = 0$ (which corresponds to the NL divisor $D_{0,0}$), or there is a divisor $E \in \mathrm{Pic}(X)$ with $E^2 = 0$ and $E \cdot L = 2$, which corresponds to the NL divisor $D_{1,2}$.

When $[X, L] \in D_{0,0}$, the morphism φ_L contracts the smooth rational curve Γ. The NL divisor $D_{1,2}$ consists of *hyperelliptic* $K3$ surfaces, for in this case φ_L maps X with degree 2 onto a surface of degree $g-1$ in $\mathbf{P}^g$. Furthermore, for $[X, L] \in \mathcal{F}_g - \big(D_{0,0} \cup D_{1,1} \cup D_{1,2}\big)$, it is shown in [50] Theorem 6.1 that the multiplication map

$$\phi_{X,L} : \mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2})$$

is surjective. By Riemann-Roch, $h^0(X, L^{\otimes 2}) = \chi(X, \mathcal{O}_X) + 2L^2 = 4g - 2$ and we obtain

$$\dim I_{X,L}(2) = \binom{g+2}{2} - (4g-2) = \binom{g-2}{2} = \mathrm{codim}(\Sigma^{g-3}_{g+1}).$$

Recall that we have defined in the Introduction the locus $D^{\mathrm{rk}4}_g$ of quasi-polarized $K3$ surfaces $[X, L] \in \mathcal{F}_g$ such that the image $\varphi_L(X) \subseteq \mathbf{P}^g$ lies on a rank 4 quadric.

Proposition 9.1. – *The locus $D^{\mathrm{rk}4}_g$ is a Noether-Lefschetz divisor on $\mathcal{F}_g$. Set-theoretically, it consists of the quasi-polarized $K3$ surfaces $[X, L] \in \mathcal{F}_g$, for which there exists a decomposition $L = D_1 + D_2$ in* $\mathrm{Pic}(X)$*, with $h^0(X, D_i) \geq 2$, for $i = 1, 2$.*

Proof. – Suppose the embedded $K3$ surface $X \hookrightarrow \mathbf{P}^g$ lies on a quadric $Q \subseteq \mathbf{P}^g$ of rank at most 4. Assume $\mathrm{rk}(Q) = 4$, hence $\mathrm{Sing}(Q) \cong \mathbf{P}^{g-4}$. Then Q is isomorphic to the inverse image of $\mathbf{P}^1 \times \mathbf{P}^1$ under the projection $p_{\mathrm{Sing}(Q)} : \mathbf{P}^g \dashrightarrow \mathbf{P}^3$ with center $\mathrm{Sing}(Q)$. Accordingly, Q has two rulings which cut out line bundles D_1 and D_2 on X such that $h^0(X, D_i) \geq 2$ and $L = D_1 + D_2$. The argument is clearly reversible. $\square$

For $n \geq 1$, we introduce the following tautological bundles

$$\mathcal{U}_n := \pi_*(\mathcal{L}^{\otimes n})$$

on $\mathcal{F}_g$. Note that $R^i\pi_*(\mathcal{L}^{\otimes n}) = 0$ for $i = 1, 2$, hence $\mathcal{U}_n$ is locally free and $\mathrm{rk}(\mathcal{U}_n) = 2 + n^2(g-1)$.

Proposition 9.2. – *The following formula holds for every $n \geq 1$:*

$$c_1(\mathcal{U}_n) = \frac{n}{12}\kappa_{1,1} + \frac{n^3}{6}\kappa_{3,0} - \Big(\frac{n^2}{2}(g-1)+1\Big)\lambda \in CH^1(\mathcal{F}_g).$$

Proof. – We apply Grothendieck-Riemann-Roch to the universal $K3$ surface $\pi : \mathcal{X} \to \mathcal{F}_g$ and write:

$$\begin{aligned}\mathrm{ch}\big(\pi_!\mathcal{L}^{\otimes n}\big) = \pi_*\Big[&\Big(1 + nc_1(\mathcal{L}) + \frac{n^2}{2}c_1^2(\mathcal{L}) + \frac{n^3}{6}c_1^3(\mathcal{L}) + \cdots\Big)\\ &\times\Big(1 - \frac{1}{2}c_1(\omega_\pi) + \frac{1}{12}\big(c_1^2(\omega_\pi) + c_2(\Omega_\pi)\big) - \frac{1}{24}c_1(\omega_\pi)c_2(\Omega_\pi) + \cdots\Big)\Big].\end{aligned}$$

Note that $\kappa_{2,0} = \pi_*(c_1^2(\mathcal{L})) = 2g-2 \in CH^0(\mathcal{F}_g)$, hence by looking at degree 2 terms in this formula, we find $\kappa_{0,1} = 24$. We now consider degree 3 terms that get pushed forward under π, and use that $c_1(\Omega_\pi) = \pi^*(\lambda)$, hence $\pi_*\big(c_1(\mathcal{L})\cdot c_1^2(\omega_\pi)\big) = 0$. Collecting terms, we obtained the desired formula. □

We are now in a position to compute the class of the Noether-Lefschetz divisor $D_g^{\mathrm{rk}4}$.

Proof of Theorem 1.3. – On the moduli space $\mathcal{F}_g$ we consider the vector bundle morphism

$$\phi : \mathrm{Sym}^2(\mathcal{U}_1) \to \mathcal{U}_2.$$

The divisor $D_g^{\mathrm{rk}4}$ coincides with the locus where the kernel of ϕ contains a rank 4 quadric. Applying Theorem 4.7, we find the formula

$$[D_g^{\mathrm{rk}4}] = A_{g+1}^{g-3}\Big(c_1(\mathcal{U}_2) - \frac{8g-4}{g+1}c_1(\mathcal{U}_1)\Big).$$

In view of Proposition 9.2, $c_1(\mathcal{U}_1) = \frac{1}{12}\kappa_{1,1} + \frac{1}{6}\kappa_{3,0} - \frac{g+1}{2}\lambda$ and $c_1(\mathcal{U}_2) = \frac{1}{6}\kappa_{1,1} + \frac{4}{3}\kappa_{3,0} - (2g-1)\lambda$. Substituting, we obtain the claimed formula. □

9.2. Koszul cohomology of polarized $K3$ surfaces of odd genus

Theorem 1.3 shows that a certain linear combination of the classes λ and γ lies in the span of NL divisors. To conclude that both λ and γ are of NL-type, we find another linear combination of these two classes, that is guaranteed to be supported on NL divisors. To that end, for odd genus, we use Voisin's solution [52], [53] to the Generic Green's Conjecture on syzygies of canonical curves.

We fix a quasi-polarized $K3$ surface $[X, L] \in \mathcal{F}_g - D_{1,1}$, so that L is globally generated and we consider the induced morphism $\varphi_L : X \to \mathbf{P}^g$. We introduce the coordinate ring

$$\Gamma_X(L) := \bigoplus_{n\geq 0} H^0(X, L^{\otimes n}),$$

viewed as a graded module over the polynomial algebra $S := \mathrm{Sym}\, H^0(X,L)$. In order to describe the minimal free resolution of $\Gamma_X(L)$, for integers $p,q \geq 0$, we introduce the Koszul cohomology group

$$K_{p,q}(X,L) = \mathrm{Tor}^p_S(\Gamma_X(L),\mathbb{C})_{p+q}$$

of p-th order syzygies of weight q of the pair $[X,L]$. We set $b_{p,q}(X,L) := \dim K_{p,q}(X,L)$. For an introduction to Koszul cohomology in algebraic geometry, we refer to [26] and [3].

The graded minimal free S-resolution of $\Gamma_X(L)$ has the following shape:

$$0 \longleftarrow \Gamma_X(L) \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_{g-3} \longleftarrow F_{g-2} \longleftarrow 0,$$

where $F_p = \bigoplus_{q>0} S(-p-q) \otimes K_{p,q}(X,L)$, for all $p \leq g-2$.

The resolution is self-dual in the sense that $K_{p,q}(X,L)^\vee \cong K_{g-2-p,3-q}(X,L)$, see [26] Theorem 2.c.6. This shows that the *linear strand* of the Betti diagram of $[X,L]$ corresponding to the case $q=1$ is dual to the *quadratic strand* corresponding to the case $q=2$. In [53], in her course of proving Green's Conjecture for general curves [26], Voisin determined completely the shape of the minimal resolution of a generic quasi-polarized $K3$ surface $[X,H] \in \mathcal{F}_g$ of odd genus $g = 2i+3$. We summarize in the following table the relevant information contained in the rows of linear and quadratic syzygies of the Betti table.

TABLE 1. The Betti table of a general polarized $K3$ surface of genus $g = 2i+3$

1	2	$\cdots$	$i-1$	i	$i+1$	$i+2$	$\cdots$	$2i$
$b_{1,1}$	$b_{2,1}$	$\cdots$	$b_{i-1,1}$	$b_{i,1}$	0	0	$\cdots$	0
0	0	$\cdots$	0	0	$b_{i+1,2}$	$b_{i+2,2}$	$\cdots$	$b_{2i,2}$

The crux of Voisin's proof is showing $K_{i+1,1}(X,L) = 0$, which implies $K_{p,1}(X,L) = 0$ for $p > i$. Then by duality, the second row of the resolution has the form displayed above.

Our strategy is to treat this problem variationally and consider the locus of polarized $K3$ surfaces with extra syzygies, that is,

$$\mathfrak{Kosz}_g := \Big\{[X,L] \in \mathcal{F}_g : K_{i+1,1}(X,L) \neq 0\Big\}.$$

We shall informally refer to $\mathfrak{Kosz}_g$ as the *Koszul divisor* on $\mathcal{F}_g$, where $g = 2i+3$. It is shown in [3] Corollary 2.17 that the group $K_{i+1,1}(X,L)$ of linear syzygies has the following interpretation

$$K_{i+1,1}(X,L) \cong K_{i,2}\big(I_{X,L}, H^0(X,L)\big),$$

where $I_{X,L} := \bigoplus_k I_{X,L}(k)$ is the ideal of $X \subseteq \mathbf{P}^g$, cf. Definition 6.1, viewed as a graded $\mathrm{Sym}\, H^0(X,L)$-module. Thus, one has the following identification

$$\begin{aligned} K_{i+1,1}(X,L) &\cong \mathrm{Ker}\Big\{\bigwedge^i H^0(X,L) \otimes I_{X,L}(2) \to \bigwedge^{i-1} H^0(X,L) \otimes I_{X,L}(3)\Big\} \\ &\cong H^0\Big(\mathbf{P}^g, \Omega^i_{\mathbf{P}^g}(i+2) \otimes \mathcal{I}_{X/\mathbf{P}^g}\Big), \end{aligned} \tag{24}$$

where the map in question is given by the Koszul differential. The last identification in (24) is obtained by taking global sections in the exact sequence on $\mathbf{P}^g$

$$0 \longrightarrow \bigwedge^i M_{\mathbf{P}^g} \otimes \mathcal{I}_{X/\mathbf{P}^g}(2) \longrightarrow \bigwedge^i H^0(\mathbf{P}^g, \mathcal{O}_{\mathbf{P}^g}(1)) \otimes \mathcal{I}_{X/\mathbf{P}^g}(2) \longrightarrow \bigwedge^{i-1} M_{\mathbf{P}^g} \otimes \mathcal{I}_{X/\mathbf{P}^g}(3) \longrightarrow 0,$$

where $M_{\mathbf{P}^g} := \Omega_{\mathbf{P}^g}(1)$. More generally, we introduce the *Lazarsfeld bundle* of $[X, L]$ as the kernel of the evaluation map of global sections, that is,

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0. \tag{25}$$

Note that $M_L = \Omega_{\mathbf{P}^g|X}(1)$. Via (24), $[X, L] \in \mathfrak{Kosz}_g$ if and only the restriction map is not injective:

$$H^0\Big(\mathbf{P}^g, \bigwedge^i M_{\mathbf{P}^g}(2)\Big) \to H^0\Big(X, \bigwedge^i M_L \otimes L^2\Big). \tag{26}$$

The key observation is that the two spaces appearing in (26) have the same dimension, which leads to representing $\mathfrak{Kosz}_g$ as the degeneracy locus of a morphism between two vector bundles of the *same* rank over $\mathcal{F}_g$.

We collect a few technical results that will come up in the following calculations:

LEMMA 9.3. – *Let $[X, L] \in \mathcal{F}_{2i+3}$ be a quasi-polarized $K3$ surface such that L is base point free.*

(1) $H^1\big(X, \bigwedge^j M_L \otimes L^{i+2-j}\big) = 0$*, for* $j = 0, \dots, i$.

(2) $h^0\big(X, \bigwedge^i M_L \otimes L^2\big) = h^0\big(\mathbf{P}^{2i+3}, \bigwedge^i M_{\mathbf{P}^{2i+3}}(2)\big) = (i+1)\binom{2i+5}{i+2}$.

Proof. – It is proved in [9] Corollary 1 that under our assumption, the vector bundle M_L is μ_L-semistable. This implies that $\bigwedge^j M_L \otimes L^{2+i-j}$ is μ_L-semistable for all i and j as well. We take cohomology in the exact sequence

$$0 \longrightarrow \bigwedge^{j+1} M_L \otimes L^{i+1-j} \longrightarrow \bigwedge^{j+1} H^0(X, L) \otimes L^{i+1-j} \longrightarrow \bigwedge^{j} M_L \otimes L^{i+2-j} \longrightarrow 0.$$

Since $H^1(X, L^{i+1-j}) = 0$ and $H^2(X, L^{i+1-j}) = 0$ for $j \leq i$, we obtain the isomorphism

$$H^1\big(X, \bigwedge^j M_L \otimes L^{i+2-j}\big) \cong H^2\big(X, \bigwedge^{j+1} M_L \otimes L^{i+1-j}\big).$$

Since $\mathrm{rk}(M_L) = g$ and $c_1(M_L) = -L$, by standard Chern class calculation, we find

$$\mu_L\Big(\bigwedge^{j+1} M_L \otimes L^{i+1-j}\Big) = \frac{i+2}{2i+3}(2i - 2j + 1) > 0,$$

which establishes $H^2\big(X, \bigwedge^{j+1} M_L \otimes L^{i+1-j}\big) = 0$ by the stability of the vector bundle in question.

The fact that $h^0\big(\mathbf{P}^g, \bigwedge^i M_{\mathbf{P}^{2i+3}}(2)\big) = h^0\big(\mathbf{P}^g, \Omega_{\mathbf{P}^{2i+3}}(i+2)\big) = (i+1)\binom{2i+5}{i+2}$ follows directly from Bott's formula on the cohomology of spaces of twisted holomorphic forms on projective spaces, see e.g., [44] page 4. To compute the last quantity appearing, noting that $c_2(M_L) = 2g - 2$, after a Riemann-Roch calculation on X, we obtain

$$h^0\big(X, \bigwedge^i M_L \otimes L^2\big) = \chi\big(X, \bigwedge^i M_L \otimes L^2\big) = (i+1)\binom{2i+5}{i+2},$$

where we have used the standard formulas $c_1\big(\bigwedge^i M_L\big) = \binom{2i+2}{i-1} c_1(M_L)$ and

$$c_2\big(\bigwedge^i M_L\big) = \frac{1}{2}\binom{2i+2}{i-1}\Big(\binom{2i+2}{i-1} - 1\Big)c_1^2(M_L) + \binom{2i+1}{i-1} c_2(M_L). \qquad \square$$

Taking exterior powers in the short exact sequence (25) and using the first part of Lemma 9.3, for $j=0,\dots,i$, we obtain the exact sequences, valid for $[X,L]\in\mathcal{F}_g - D_{1,1}$:

$$0\longrightarrow H^0\big(X,\bigwedge^j M_L\otimes L^{i+2-j}\big)\longrightarrow\bigwedge^j H^0(X,L)\otimes H^0(X,L^{i+2-j})$$
$$\longrightarrow H^0\big(X,\bigwedge^{j-1} M_L\otimes L^{i+3-j}\big)\longrightarrow 0.$$

Globalizing these exact sequences over the moduli space, for $j = 0,\dots,i$, we define inductively the vector bundles $\mathcal{G}_{j,i+2-j}$ over $\mathcal{F}_g$ via the exact sequences

$$0\longrightarrow\mathcal{G}_{j,i+2-j}\longrightarrow\bigwedge^j\mathcal{U}_1\otimes\mathcal{U}_{i+2-j}\longrightarrow\mathcal{G}_{j-1,i+3-j}\longrightarrow 0,\tag{27}$$

starting from $\mathcal{G}_{0,i+2}:=\mathcal{U}_{i+2}$.

Similarly, taking exterior powers in the Euler sequence on $\mathbf{P}^g$, we find the exact sequences

$$0\longrightarrow H^0\big(\bigwedge^j M_{\mathbf{P}^g}(i+2-j)\big)\longrightarrow\bigwedge^j H^0(\mathcal{O}_{\mathbf{P}^g}(1))\otimes H^0(\mathcal{O}_{\mathbf{P}^g}(i+2-j))$$
$$\longrightarrow H^0\big(\bigwedge^{j-1} M_{\mathbf{P}^g}(i+3-j)\big)\longrightarrow 0,$$

which can also be globalizes to exacts sequences over $\mathcal{F}_g$. We define inductively the vector bundles $\mathcal{H}_{j,i+2-j}$ for $j=0,\dots,i$, starting from $\mathcal{H}_{0,i+2}:=\mathrm{Sym}^{i+2}(\mathcal{U}_1)$ and then via the exact sequences

$$0\longrightarrow\mathcal{H}_{j,i+2-j}\longrightarrow\bigwedge^j\mathcal{U}_1\otimes\mathrm{Sym}^{i+2-j}(\mathcal{U}_1)\longrightarrow\mathcal{H}_{j-1,i+3-j}\longrightarrow 0.\tag{28}$$

In particular, there exist restriction morphisms $\mathcal{H}_{j,i+2-j}\to\mathcal{G}_{j,i+2-j}$ for all $j=0,\dots,i$. Setting $j=i$, we observe that the second part of Lemma 9.3 yields $\mathrm{rk}(\mathcal{H}_{i,2})=\mathrm{rk}(\mathcal{G}_{i,2})$, and the degeneracy locus of the morphism

$$\phi:\mathcal{H}_{i,2}\to\mathcal{G}_{i,2}$$

is precisely the locus $\mathfrak{Kosz}_g$ of quasi-polarized $K3$ surfaces having extra syzygies.

PROPOSITION 9.4. – *The locus $\mathfrak{Kosz}_g$ is an effective divisor on $\mathcal{F}_g$ of NL type.*

Proof. – Let $[X,L]\in\mathcal{F}_g$ be a quasi-polarized $K3$ surface with $\mathrm{Pic}(X)=\mathbb{Z}\cdot L$ and choose a general curve $C\in|L|$. Using the Koszul duality $K_{i,2}(X,L)\cong K_{i+1,1}(X,L)^\vee$, in order to conclude, it suffices to show that $K_{i,2}(X,L)=0$. Using the main result of [53], we have that $K_{i,2}(X,L)\cong K_{i,2}(C,\omega_C)=0$, for the genus g curve $C\in|L|$ is known to be Brill-Noether general, in particular it has maximal Clifford index $\mathrm{Cliff}(C)=i+1$. □

In what follows, we shall repeatedly use that if E is a vector bundle of rank r on a stack X, then

$$c_1\big(\bigwedge^n E\big)=\binom{r-1}{n-1}c_1(E)\quad\text{and}\quad c_1\big(\mathrm{Sym}^n(E)\big)=\binom{r+n-1}{r}c_1(E).\tag{29}$$

THEOREM 9.5. – *Set $g = 2i + 3$. The class of the Koszul divisor of $K3$ surfaces with extra syzygies is given by*

$$[\mathfrak{Kosz}_g] = \frac{2}{i+2}\binom{2i-1}{i}\Big(2(i+1)(i+5)\lambda + \gamma\Big) + \alpha \cdot [D_{1,1}] \in CH^1(\mathcal{F}_g),$$

for some coefficient $\alpha \in \mathbb{Z}$.

Proof. – As explained, off the divisor $D_{1,1}$, the locus $\mathfrak{Kosz}_g$ is the degeneracy locus of the morphism $\phi : \mathcal{H}_{i,2} \to \mathcal{G}_{i,2}$, therefore $[\mathfrak{Kosz}_g] = c_1(\mathcal{G}_{i,2}) - c_1(\mathcal{H}_{i,2}) + \alpha \cdot [D_{1,1}]$, for a certain integral coefficient α. Using repeatedly the exact sequences (27) and the formulas for the ranks of the vector bundles $\mathcal{U}_{2+j}$, we find

$$\begin{aligned}
c_1(\mathcal{G}_{i,2}) &= \sum_{j=0}^{i}(-1)^j c_1\Big(\bigwedge^{i-j}\mathcal{U}_1 \otimes \mathcal{U}_{2+j}\Big)\\
&= \sum_{j=0}^{i}(-1)^j\Big(\big(2+(j+2)^2(g-1)\big)\binom{g}{i-j-1}c_1(\mathcal{U}_1) + \binom{g+1}{i-j}c_1(\mathcal{U}_{2+j})\Big).
\end{aligned}$$

Similarly, in order to compute the first Chern class of $\mathcal{H}_{i,2}$, we use the exact sequences (28):

$$\begin{aligned}
c_1(\mathcal{H}_{i,2}) &= \sum_{j=0}^{i}(-1)^j c_1\Big(\bigwedge^{i-j}\mathcal{U}_1 \otimes \mathrm{Sym}^{j+2}\mathcal{U}_1\Big)\\
&= \sum_{j=0}^{i}(-1)^j\Big(\binom{g+j+2}{g}\binom{g}{i-j-1} + \binom{g+1}{i-j}\binom{g+j+2}{g+1}\Big)c_1(\mathcal{U}_1)\\
&= \frac{i+1}{2}\binom{2i+5}{i+2}c_1(\mathcal{U}_1).
\end{aligned}$$

Substituting in these formulas the Chern classes computed in Proposition 9.2, after some manipulations we obtain the claimed formula for $[\mathfrak{Kosz}_g]$. □

10. Lazarsfeld-Mukai bundles on $K3$ surfaces of even genus and tautological classes

For even genus, in order to obtain a Noether-Lefschetz relation between the classes λ and γ which is different than the one in Theorem 1.3, we use the geometry of the rank 2 Lazarsfeld-Mukai vector bundle one associates to a sufficiently general polarized $K3$ surface. We denote by $D_{\mathrm{NL}} \subseteq \mathcal{F}_g$ the Noether-Lefschetz divisor consisting of $K3$ surfaces $[X, L]$ of genus g, such that $L = \mathcal{O}_X(D_1 + D_2)$, with both D_1 and D_2 being non-trivial effective divisors on X. We set $\mathcal{F}_g^\sharp := \mathcal{F}_g - D_{\mathrm{NL}}$ and slightly abusing notation, we denote by $\pi : \mathcal{X}^\sharp \to \mathcal{F}_g^\sharp$ the corresponding restriction of the universal $K3$ surface. Throughout this subsection we fix an even genus $g = 2i$, with $i \geq 4$. Our aim is to show that the restriction of both classes λ and γ to $\mathcal{F}_g^\sharp$ is trivial. The geometric source of such a relation lies in the geometry of Lazarsfeld-Mukai vector bundles that have proved to be instrumental in Lazarsfeld's proof [37] of the Petri Theorem.

Definition 10.1. – *For a polarized* $K3$ *surface* $[X, L] \in \mathcal{F}_g^{\sharp}$, *we denote by* E_L *the unique stable rank* 2 *vector bundle on* X, *satisfying* $\det(E_L) = L$, $c_2(E_L) = i+1$ *and* $h^0(X, E_L) = i+2$.

The vector bundle $E := E_L$, which we refer to as the *Lazarsfeld-Mukai vector bundle* of $[X, L]$ has been first considered in [42] and [37]. In order to construct it, one chooses a smooth curve $C \in |L|$ and a pencil of minimal degree $A \in W^1_{i+1}(C)$. By Lazarsfeld [37], it is known that C verifies the Brill-Noether Theorem, in particular $\mathrm{gon}(C) = i+1$. We define the dual Lazarsfeld-Mukai bundle via the following exact sequence on X

$$0 \longrightarrow E_L^{\vee} \longrightarrow H^0(C, A) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} \iota_* A \longrightarrow 0, \tag{30}$$

where $\iota : C \hookrightarrow X$ denotes the inclusion map. Dualizing the previous sequence, we obtain the short exact sequence

$$0 \longrightarrow H^0(C, A)^{\vee} \otimes \mathcal{O}_X \longrightarrow E_L \longrightarrow \omega_C \otimes A^{\vee} \longrightarrow 0.$$

We summarize the properties of this vector bundle and refer to [37] for proofs:

Proposition 10.2. – *Let* $[X, L] \in \mathcal{F}_g^{\sharp}$ *and* $E = E_L$ *be the corresponding rank* 2 *Lazarsfeld-Mukai bundle.* 1.1.

1. E is globally generated and $H^1(X, E) = H^2(X, E) = 0$.
2. $h^0(X, E) = h^0(C, \omega_C \otimes A^{\vee}) + h^0(C, A) = i+2$.
3. E is μ_L-stable, in particular $h^0(X, E \otimes E^{\vee}) = 1$ as well as rigid, that is, $H^1(X, E \otimes E^{\vee}) = 0$.
4. The vector bundle E is independent of the choice of C and of that of the pencil $A \in W^1_{i+1}(C)$.

In particular, Proposition 10.2 implies that E is the only μ_L-semistable sheaf on X having Mukai vector $v = v(E) = (2, L, i)$. We denote by $\det : \bigwedge^2 H^0(X, E) \to H^0(X, L)$ the determinant map.

Let $\mathcal{E}$ be the universal rank 2 Lazarsfeld-Mukai vector bundle over $\mathcal{X}^{\sharp}$, that is, $\mathcal{E}|_X = E_L$, for every $[X, L] \in \mathcal{F}_g^{\sharp}$. In this case $\mathcal{L} := \det(\mathcal{E})$ can be taken to be the polarization line bundle, and apart from the classes $\kappa_{1,1} = \pi_*\big(c_1(\mathcal{E}) \cdot c_2(\mathcal{T}_{\pi})\big)$ and $\kappa_{3,0} = \pi_*\big(c_1(\mathcal{E})^3\big)$, we also have a third tautological class

$$\vartheta := \pi_*\big(c_1(\mathcal{E}) \cdot c_2(\mathcal{E})\big).$$

To show that both classes λ and $\gamma = \kappa_{1,1} - \frac{g-1}{4}\kappa_{3,0}$ are of NL type, we need *two* further sources of geometric relations in terms of Lazarsfeld-Mukai bundles. These provide two relations involving $\lambda, \kappa_{3,0}, \kappa_{1,1}$ and ϑ, hence by eliminating ϑ, one relation between λ and γ, which turns out to be different than the one given by Theorem 1.3.

10.1. The Chow form of the Grassmannian and Lazarsfeld-Mukai bundles.

One such source of relations is provided by the recent work [2], where among other things a Schubert-theoretic description of the Cayley-Chow form of the Grassmannian $\mathrm{Gr}(2,n)$ of lines is provided.

Let V be an n-dimensional complex vector space and choose a linear subspace $K \subseteq \bigwedge^2 V$. Then $K^{\perp} \subseteq \bigwedge^2 V^{\vee}$. It is shown in [2] Theorem 3.1 that the condition $\mathbf{P}(K^{\perp}) \cap \mathrm{Gr}(V,2) = \emptyset$, the intersection being taken inside $\mathbf{P}(\bigwedge^2 V^{\vee})$, is equivalent to the exactness of the complex

$$K \otimes \mathrm{Sym}^{n-3}(V) \xrightarrow{\delta_2} V \otimes \mathrm{Sym}^{n-2}(V) \xrightarrow{\delta_1} \mathrm{Sym}^{n-1}(V),$$

where $\delta_i : \bigwedge^i V \otimes \mathrm{Sym}(V) \to \bigwedge^{i-1} V \otimes \mathrm{Sym}(V)$ denotes the Koszul differential, for $i = 1, 2$.

We apply this result for polarized $K3$ surfaces, when we take $V := H^0(X,E)^{\vee}$ and

$$K^{\perp} := \mathrm{Ker}\Big\{\det : \bigwedge^2 H^0(X,E) \to H^0(X,L)\Big\}$$

is the kernel of the determinant map. Note that det does not vanish on any element of rank 2, see [53] page 380, for the existence of an element $0 \neq s_1 \wedge s_2 \in \bigwedge^2 H^0(X,E)$ such that $\det(s_1 \wedge s_2) = 0$, would imply a splitting of L as a sum of two pencils. By dualizing, we conclude that the complex

$$\mathrm{Sym}^{i+1} H^0(X,E) \longrightarrow H^0(X,E) \otimes \mathrm{Sym}^i H^0(X,E) \xrightarrow{\beta} H^0(X,L) \otimes \mathrm{Sym}^{i-1} H^0(X,E) \tag{31}$$

is exact for every point $[X,L] \in \mathcal{F}_g^{\sharp}$. The map β is obtained by composing the (dual) Koszul differential

$$H^0(E) \otimes \mathrm{Sym}^i H^0(X,E) \to \bigwedge^2 H^0(X,E) \otimes \mathrm{Sym}^{i-1} H^0(X,E)$$

with the map $\det \otimes \mathrm{id}_{\mathrm{Sym}^{i-1} H^0(X,E)} : \bigwedge^2 H^0(X,E) \otimes \mathrm{Sym}^{i-1} H^0(X,E) \to H^0(X,L) \otimes \mathrm{Sym}^{i-1} H^0(X,E)$.

We globalize this geometric fact. For $n \geq 1$, we introduce the vector bundle $\mathcal{V}_n := \pi_*(\mathrm{Sym}^n \mathcal{E})$ on $\mathcal{F}_g^{\sharp}$, where we observe that $R^i\pi_*(\mathrm{Sym}^n \mathcal{E}) = 0$, for $i = 1, 2$. We shall make use of the following formulas:

Proposition 10.3. – *The following formulas hold in $CH^1(\mathcal{F}_{2i}^{\sharp})$:*

$$c_1(\mathcal{V}_1) = \frac{1}{12}\kappa_{1,1} + \frac{1}{6}\kappa_{3,0} - \frac{i+2}{2}\lambda - \frac{1}{2}\vartheta \quad \textit{and} \quad c_1(\mathcal{V}_2) = \frac{1}{4}\kappa_{1,1} + \frac{3}{2}\kappa_{3,0} - \frac{6i-3}{2}\lambda - 4\vartheta.$$

Proof. – We only discuss the calculation of $c_1(\mathcal{V}_2)$. For any $[X,L] \in \mathcal{F}_g^{\sharp}$, observe that

$$h^0\big(X, \mathrm{Sym}^2(E)\big) = \chi\big(X, \mathrm{Sym}^2(E)\big) = \frac{c_1^2\big(\mathrm{Sym}^2(E)\big)}{2} - c_2(\mathrm{Sym}^2(E)) + 3\chi(X, \mathcal{O}_X) = 6i - 3,$$

where we use the formulas $c_1\big(\mathrm{Sym}^2(E)\big) = 3c_1(E)$ and $c_2\big(\mathrm{Sym}^2(E)\big) = 2c_1^2(E) + 4c_2(E)$. Applying Grothendieck-Riemann-Roch to the universal family $\pi : \mathcal{X}^\sharp \to \mathcal{F}_g^\sharp$, we find:

$$\begin{aligned} c_1(\mathcal{V}_2) &= c_1\big(\pi_!\big(\mathrm{Sym}^2 \mathcal{E}\big)\big) \\ &= \pi_*\Big[\Big(3 + 3c_1(\mathcal{E}) + \frac{5c_1^2(\mathcal{E}) - 8c_2(\mathcal{E})}{2} + \frac{9c_1^3(\mathcal{E}) - 24c_1(\mathcal{E})c_2(\mathcal{E})}{6}\Big) \\ &\quad \times \Big(1 - \frac{1}{2}c_1(\omega_\pi) + \frac{1}{12}\big(c_1^2(\omega_\pi) + c_2(\Omega_\pi)\big) - \frac{1}{24}c_1(\omega_\pi)c_2(\Omega_\pi) + \cdots\Big)\Big]_2. \end{aligned}$$

Expanding the product and using again that $\pi_*(c_2(\Omega_\pi)) = 24$, we obtain the claimed formula. $\square$

In order to treat the complex (31) variationally, we consider the following vector bundles over $\mathcal{F}_{2i}^\sharp$

$$\mathcal{A} := \frac{\mathcal{V}_1 \otimes \mathrm{Sym}^i(\mathcal{V}_1)}{\mathrm{Sym}^{i+1}(\mathcal{V}_1)} \quad \text{and} \quad \mathcal{B} := \mathcal{U}_1 \otimes \mathrm{Sym}^{i-1}(\mathcal{V}_1).$$

Note that $\mathrm{rk}(\mathcal{A}) = (i+2)\binom{2i+1}{i} - \binom{2i+2}{i+1} = (2i+1)\binom{2i}{i-1} = \mathrm{rk}(\mathcal{B})$ and there is a sheaf morphism

$$\beta : \mathcal{A} \to \mathcal{B},$$

which over a point $[X, L] \in \mathcal{F}_g^\sharp$ is precisely the map

$$\beta_{X,L} : \frac{H^0(X, E) \otimes \mathrm{Sym}^i H^0(X, E)}{\mathrm{Sym}^{i+1} H^0(X, E)} \to H^0(X, L) \otimes \mathrm{Sym}^{i-1} H^0(X, E)$$

induced by (31). As explained, the morphism β is everywhere non-degenerate over $\mathcal{F}_g^\sharp$.

Theorem 10.4. – *One has the following formula*

$$\vartheta = \frac{i}{8i+4}\kappa_{1,1} + \frac{i}{4i+2}\kappa_{3,0} - \frac{i+2}{2}\lambda.$$

Proof. – The morphism $\beta : \mathcal{A} \to \mathcal{B}$ being everywhere non-degenerate, we find that $c_1(\mathcal{A}) = c_1(\mathcal{B})$. Applying systematically the formulas (29), we write:

$$\begin{aligned} c_1(\mathcal{A}) &= \left(\binom{2i+1}{i} + (i+2)\binom{2i+1}{i+2} - \binom{2i+2}{i+2}\right)c_1(\mathcal{V}_1), \\ c_1(\mathcal{B}) &= \binom{2i}{i-1}c_1(\mathcal{U}_1) + (2i+1)\binom{2i}{i+2}c_1(\mathcal{V}_1), \end{aligned}$$

hence, after manipulations

$$0 = c_1(\mathcal{B} - \mathcal{A}) = \binom{2i}{i-1}\Big(c_1(\mathcal{U}_1) - \frac{4i+2}{i+2}c_1(\mathcal{V}_1)\Big).$$

We then replace $c_1(\mathcal{U}_1)$ and $c_1(\mathcal{V}_1)$ with their respective expressions provided by Propositions 9.2 and 10.3, clear denominators (our Chow groups are with $\mathbb{Q}$-coefficients), then conclude. $\square$

10.2. Lazarsfeld-Mukai bundles and rank 6 quadrics

A second source of relations between the classes $\lambda, \kappa_{1,1}, \kappa_{3,0}$ and ϑ is obtained by studying the kernel of the multiplication map

$$\mu_E : \operatorname{Sym}^2 H^0(X, E) \to H^0\big(X, \operatorname{Sym}^2(E)\big)$$

associated to the Lazarsfeld-Mukai bundle $E = E_L$ corresponding to an element $[X, L] \in \mathcal{F}^{\sharp}_{2i}$. We assume throughout that $i \geq 4$.

LEMMA 10.5. – *One has $H^i\big(X, \operatorname{Sym}^2(E)\big) = 0$, for $i = 1, 2$.*

Proof. – We choose a general curve $C \in |L|$ and a minimal pencil $A \in W^1_{i+1}(C)$. Tensoring the exact sequence (30) by $E^\vee$ and taking global sections implies that $H^2\big(X, E \otimes E\big) \cong H^0(X, E^\vee \otimes E^\vee) = 0$. Similarly, we can prove that $H^1(X, E \otimes E) = 0$, which implies that $H^1\big(X, \operatorname{Sym}^2(E)\big) = 0$. We tensor again (30) by $E^\vee$ and take cohomology. The vanishing of $H^1(X, E^\vee \otimes E^\vee) \cong H^1(X, E \otimes E)^\vee$ is implied by $H^0(C, A \otimes E_{|C}^\vee) = 0$, which follows because $E_{|C}$ is stable on C and $\mu\big(A \otimes E_{|C}^\vee\big) = 4 - 2i < 0$. □

Using Lemma 10.5, we compute $h^0\big(X, \operatorname{Sym}^2(E)\big) = 6i - 3$, then observe that

$$\dim \operatorname{Sym}^2 H^0(X, E) - h^0\big(X, \operatorname{Sym}^2(E)\big) = \binom{i+3}{2} - (6i - 3) = \binom{i-3}{2},$$

that is, the locus

$$D^{\mathrm{rk}6}_{2i} := \Big\{[X, L] \in \mathcal{F}^{\sharp}_{2i} : \exists\, 0 \neq q \in \operatorname{Ker}(\mu_E),\ \operatorname{rk}(q) \leq 6\Big\},$$

is expected to be a divisor on $\mathcal{F}^{\sharp}_{2i}$. We confirm this expectation in a very precise form.

THEOREM 10.6. – *For a polarized $K3$ surface $[X, L] \in \mathcal{F}^{\sharp}_{2i}$, the kernel of the map μ_E contains no non-zero elements of rank at most 6, that is, $D^{\mathrm{rk}6}_{2i} = \emptyset$.*

Proof. – We start with a $K3$ surface $[X, L] \in \mathcal{F}^{\sharp}_{2i}$ and assume we have an element $0 \neq q \in \operatorname{Ker}(\mu_E)$, where $\operatorname{rk}(q) = n \leq 6$. We write $q = s_1^2 + \cdots + s_n^2$, where $s_i \in H^0(X, E)$. Denoting by $\mathbf{P}(E) \to X$ the projective bundle associated to E, we have the canonical identifications

$$H^0(X, E) \cong H^0\big(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(1)\big) \ \text{ and } \ H^0\big(X, \operatorname{Sym}^2(E)\big) \cong H^0\big(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(2)\big).$$

Let $V := \langle s_1, \ldots, s_n \rangle \subseteq H^0(X, E)$. Since $H^0(X, E(-L)) = 0$, the cokernel of the evaluation map $V \otimes \mathcal{O}_X \to E$ is supported along finitely many points and we denote by

$$\varphi_V : X \dashrightarrow \operatorname{Gr}(V, 2) \subseteq \mathbf{P}\Big(\bigwedge^2 V^\vee\Big)$$

the induced rational map. Note that $\varphi_V^*(\mathcal{O}(1)) = L$. We further denote by $Q \subseteq \mathbf{P}\big(V^\vee\big)$ the quadric given by the equation $q = 0$. The condition $q \in \operatorname{Ker}(\mu_E)$ can be interpreted geometrically as saying that the image of $\mathbf{P}(E)$ under the composition

$$\mathbf{P}(E) \xrightarrow{|\mathcal{O}(1)|} \mathbf{P}\big(H^0(E)^\vee\big) \dashrightarrow \mathbf{P}(V^\vee)$$

lies on the quadric Q. This in turn, amounts to saying that $\varphi_V(X)$ is contained in orthogonal Grassmannian $\operatorname{Gr}_Q \subseteq \operatorname{Gr}(V, 2)$ of lines in $\mathbf{P}\big(V^\vee\big)$ contained in Q. The essential observation is that for $n \leq 6$, the pull-back $\mathcal{O}_{\mathrm{Gr}_Q}(1)$ splits *non-trivially* as the sum of two effective line bundles, which in turn, induces a decomposition of the polarization class L on X, contradicting the assumption that $[X, L]$ is NL-general.

We discuss in detail the case $n = 6$, the situation for $n \leq 5$ being similar. Thus $Q \subseteq \mathbf{P}(V^\vee) = \mathbf{P}^5$ is a rank 6 quadric and we may assume that $Q = \mathrm{Gr}(2, U) \subseteq \mathbf{P}(\bigwedge^2 U) \cong \mathbf{P}(V^\vee)$ is the Grassmannian of lines in $\mathbf{P}^3 \cong \mathbf{P}(U)$, where U is a 4-dimensional complex vector space such that $\bigwedge^2 U \cong V^\vee$. Then every line inside Gr_Q is of the form $L_{\ell,H} := \{\Pi \in \mathrm{Gr}(2, U) : \ell \subseteq \Pi \subseteq H\}$, where $\ell \subseteq U$ is 1-dimensional and $H \subseteq U$ is a 3-dimensional subspace. Accordingly, one has an isomorphism between Gr_Q and the incidence correspondence $\Sigma \subseteq \mathbf{P}(U) \times \mathbf{P}(U^\vee)$, assigning to the pair $(\ell, H) \in \Sigma$ with $\ell \subseteq H$ the line $L_{\ell,H}$ defined above. Denoting by

$$\mathbf{P}(U) \xleftarrow{\pi_1} \mathrm{Gr}_Q \cong \Sigma \subseteq \mathbf{P}(U) \times \mathbf{P}(U^\vee) \xrightarrow{\pi_2} \mathbf{P}(U^\vee)$$

the two projections, we have $\mathcal{O}_{\mathrm{Gr}_Q}(1) \cong \pi_1^*\big(\mathcal{O}_{\mathbf{P}(U)}(1)\big) \otimes \pi_2^*\big(\mathcal{O}_{\mathbf{P}(U^\vee)}(1)\big)$. Let $q_1 := \pi_1 \circ \varphi_V : X \dashrightarrow \mathbf{P}(U)$ and $q_2 := \pi_2 \circ \varphi_V : X \dashrightarrow \mathbf{P}(U^\vee)$. It is now enough to observe that $h^0(X, q_i^*(\mathcal{O}(1)) \geq 2$, for $i = 1, 2$. Indeed, else the image of one of the maps q_i, say q_1, is a point, hence there exists $\ell_0 \in \mathbf{P}(U)$, such that $\mathrm{Im}(\varphi_V) \subseteq \pi_1^{-1}(\ell_0)$. But $\pi_1^{-1}(\ell_0) \cong \mathbf{P}^2 \subseteq \mathbf{P}(\bigwedge^2 U)$, that is, $\varphi_V(X) \subseteq \mathrm{Gr}(2, 3)$, which is impossible, for V is 6-dimensional. We conclude that $L = q_1^*\big(\mathcal{O}_{\mathbf{P}(U)}(1)\big) \otimes q_2^*\big(\mathcal{O}_{\mathbf{P}(U^\vee)}(1)\big)$ is NL special.

We briefly mention the cases $n \leq 5$. For $n = 4$, we have $Q \subseteq \mathbf{P}^3$ and the variety of lines Gr_Q consists of two copies of $\mathbf{P}^1$. For $n = 5$, when $Q \subseteq \mathbf{P}^4$ is a rank 5 quadric, the variety of lines Gr_Q is identified with $\mathbf{P}^3$ in its second Veronese embedding $\mathbf{P}^3 \hookrightarrow \mathbf{P}^9 \cong \mathbf{P}(\bigwedge^2 V^\vee)$. The assumption that there exist $0 \neq q \in \mathrm{Ker}(\mu_E)$ with $\mathrm{rk}(q) \in \{4, 5\}$ implies that L is non-primitive, a contradiction. □

THEOREM 10.7. – *The relation* $\frac{1}{2i+1}\gamma + (i+2)\lambda = 0$ *holds in* $CH^1(\mathcal{F}_{2i}^\sharp)$.

Proof. – We first use the fact that the divisor $D_{2i}^{\mathrm{rk}6}$ is of Noether-Lefschetz type, that is, by applying Theorem 1.1 coupled with Proposition 10.3, we obtain the relation

$$0 = [D_{2i}^{\mathrm{rk}6}] = c_1(\mathcal{V}_2) - \frac{2(6i-3)}{i+2} c_1(\mathcal{V}_1) = \frac{3}{2}(2i-1)\lambda + \frac{2i-11}{i+2}\vartheta - \frac{i-8}{2i+4}\kappa_{3,0} - \frac{3i-4}{4i+8}\kappa_{1,1}.$$

Substituting ϑ obtained in this way in the formula provided by Theorem 10.4, we obtained the desired relation between λ and γ. □

Proof of Theorem 1.4 for even g. – Theorem 1.3 provides the relation $(4i-1)\lambda + \frac{2}{2i+1}\gamma = 0 \in CH^1(\mathcal{F}_{2i}^\sharp)$. Coupled with Theorem 10.7, we conclude that both classes λ and γ vanish on $\mathcal{F}_{2i}^\sharp$, hence they are of Noether-Lefschetz type on $\mathcal{F}_{2i}$. □

11. Semistability of the second Hilbert point of a polarized $K3$ surface

A simple and somewhat surprising application of the techniques developed in Subsection 9.1 concerns the GIT semistability of polarized $K3$ surfaces. We fix once and for all a vector space $V \cong \mathbb{C}^{g+1}$.

DEFINITION 11.1. – *For a polarized $K3$ surface $[X, L] \in \mathcal{F}_g$ such that* $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$, *we define its* second Hilbert point *to be the quotient*

$$[X, H]_2 := \Big[\mathrm{Sym}^2 H^0(X, L) \longrightarrow H^0(X, L^{\otimes 2}) \longrightarrow 0\Big] \in \mathrm{Gr}\Big(\mathrm{Sym}^2 H^0(X, H), 4g-2\Big).$$

The group $SL(V)$ acts linearly on the Grassmannian

$$\mathrm{Gr}\big(\mathrm{Sym}^2 V, 4g-2\big) \subseteq \mathbf{P}\Big(\bigwedge^{4g-2} \mathrm{Sym}^2 V^\vee\Big),$$

where the last inclusion is given by the Plücker embedding. Let $\overline{\mathrm{Hilb}}_g$ be the closure inside the quotient

$$\mathbf{P}\Big(\bigwedge^{4g-2} \mathrm{Sym}^2 V^\vee\Big) /\!/ SL(V)$$

of the locus of semistable second Hilbert points $[X, H]_2$ of quasi-polarized $K3$ surfaces of genus g. Then the GIT quotient

$$\overline{\mathcal{F}}_g := \overline{\mathrm{Hilb}}_g^{\mathrm{ss}} /\!/ SL(V)$$

is a projective birational model of the moduli space $\mathcal{F}_g$, provided the locus $\overline{\mathrm{Hilb}}_g^{\mathrm{ss}}$ of semistable second Hilbert points is not empty.

THEOREM 11.2. – *Let $[X, L] \in \mathcal{F}_g$ be a polarized $K3$ surface with* $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot L$. *Then the second Hilbert point $[X, H]_2$ is semistable. In particular, $\overline{\mathcal{F}}_g$, defined as above, exists.*

Proof. – By definition of semistability, since the Grassmannian $\mathrm{Gr}(\mathrm{Sym}^2 V, 4g-2)$ has Picard numer 1, it suffices to construct an $SL(V)$-invariant effective divisor $\mathcal{D}$ of $\mathrm{Gr}(\mathrm{Sym}^2 V, 4g-2)$ such that $[X, L]_2 \notin \mathcal{D}$. Theorem 1.3 provides such a divisor. We take $\mathcal{D}$ to be the locus of $(4g-2)$-dimensional quotients $\phi : \mathrm{Sym}^2 V \twoheadrightarrow Q$ such that $\mathrm{Ker}(\phi)$ contains a quadric of rank at most 4. The parameter count from Subsection 9.1 shows that $\mathcal{D}$ is indeed a divisor. If $[X, H] \notin D_g^{\mathrm{rk}4}$, then $I_{X,L}(2)$ contains no quadrics of rank at most 4, in particular $[X, L]_2 \notin \mathcal{D}$, hence its second Hilbert point is semistable. □

REMARK 11.3. – By the analogy with the much studied case of the moduli space of curves [32], we expect that, apart from smooth $K3$ surfaces, $\overline{\mathcal{F}}_g$ also parametrizes degenerate $K3$ surfaces with various singularities. It is also likely that the 2nd Hilbert point of NL special smooth K3 surfaces is not semistable, that is, the natural map $\mathcal{F}_g \dashrightarrow \overline{\mathcal{F}}_g$ might not be regular along NL special loci.

12. The geometry of the Hurwitz spaces of admissible covers

In what follows, for a Deligne-Mumford stack M, we shall denote by $\mathcal{M}$ its coarse moduli space. If $X \subseteq \mathcal{M}$ is an irreducible subvariety, we denote by $[X] \in CH^*_{\mathbb{Q}}(\mathcal{M})$ its class in the stack sense, that is, we divide by the order of the automorphism group of a general element in X.

We denote by $\mathcal{H}_k^{\mathrm{o}}$ the Hurwitz space of degree k covers $f : C \to \mathbf{P}^1$ with simple ramification from a smooth curve C of genus $2k-1$, together with an *ordering* $(p_1, \ldots, p_{6k-4})$ of its branch points. Let $\overline{\mathcal{H}}_k^{\mathrm{o}}$ denote the compactification of $\mathcal{H}_k^{\mathrm{o}}$ by admissible covers. By [1], the stack $\overline{H}_k^{\circ}$ (whose coarse moduli space is precisely $\overline{\mathcal{H}}_k^{\circ}$) is isomorphic to the stack of *twisted stable* maps into the classifying stack $\mathcal{B}\mathfrak{S}_k$ of the symmetric group $\mathfrak{S}_k$, that is,

$$\overline{H}_k^{\circ} := \overline{M}_{0,6k-4}\Big(\mathcal{B}\mathfrak{S}_k\Big).$$

Points of $\overline{\mathcal{H}}_k^{\circ}$ are admissible covers $[f : C \to R, p_1, \dots, p_{6k-4}]$, where C and R are nodal curves of genus $2k-1$ and 0 respectively, and $p_1, \dots, p_{6k-4} \in R_{\mathrm{reg}}$ are the branch points of f. Let $\mathfrak{b} : \overline{\mathcal{H}}_k^{\circ} \to \overline{\mathcal{M}}_{0,6k-4}$ be the *branch* morphism. The symmetric group $\mathfrak{S}_{6k-4}$ acts on $\overline{\mathcal{H}}_k^{\circ}$ by permuting the branch points of each cover. Denoting by

$$\overline{\mathcal{H}}_k := \overline{\mathcal{H}}_k^{\circ}/\mathfrak{S}_{6k-4}$$

the quotient parametrizing admissible covers without an ordering of the branch points, the projection $q : \overline{\mathcal{H}}_k^{\circ} \to \overline{\mathcal{H}}_k$ is a principal $\mathfrak{S}_{6k-4}$-bundle. We denote by $\sigma : \overline{\mathcal{H}}_k \to \overline{\mathcal{M}}_{2k-1}$ the map assigning to an admissible cover the stable model of its source curve. We shall use throughout the isomorphism $CH^1_{\mathbb{Q}}(\overline{\mathcal{H}}_k) \cong H^2(\overline{\mathcal{H}}_k, \mathbb{Q})$, see [10] Theorem 5.1 and Proposition 2.2.

For $i = 0, \dots, 3k-2$, let B_i be the boundary divisor of $\overline{\mathcal{M}}_{0,6k-4}$ whose general point is the union of two smooth rational curves meeting at one point, such that precisely i of the marked points lie on one component. The boundary divisors of $\overline{\mathcal{H}}_k^{\circ}$ are parametrized by the following combinatorial data:

1. A partition $I \sqcup J = \{1, \dots, 6k-4\}$, such that $|I| \geq 2$, $|J| \geq 2$.
2. Transpositions $\{w_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ in $\mathfrak{S}_k$, with $\prod_{i \in I} w_i = u$, $\prod_{j \in J} w_j = u^{-1}$, for some $u \in \mathfrak{S}_k$.

To this data, we associate the locus of admissible covers with labeled branch points

$$\big[f : C \to R,\ p_1, \dots, p_{6k-4}\big] \in \overline{\mathcal{H}}_k^{\circ},$$

where $[R = R_1 \cup_p R_2, p_1, \dots, p_{6k-4}] \in B_{|I|} \subseteq \overline{\mathcal{M}}_{0,6k-4}$ is a pointed union of two smooth rational curves R_1 and R_2 meeting at the point p. The marked points lying on R_1 are precisely those labeled by the set I. Let $\mu := (\mu_1, \dots, \mu_\ell) \vdash k$ be the partition induced by $u \in \mathfrak{S}_k$ and denote by $E_{i:\mu}$ the boundary divisor on $\overline{\mathcal{H}}_k^{\circ}$ classifying twisted stable maps with underlying admissible cover as above, with $f^{-1}(p)$ having partition type μ, and precisely i of the points $p_1, \dots, p_{6k-4}$ lying on R_1. We denote by $D_{i:\mu}$ the reduced boundary divisor of $\overline{\mathcal{H}}_k$ pulling back to $E_{i:\mu}$ under the map q.

For $i = 2, \dots, 3k-2$, we have the following relation, see [30] p. 62, as well as [25] Lemma 3.1:

$$\mathfrak{b}^*(B_i) = \sum_{\mu \vdash k} \mathrm{lcm}(\mu) E_{i:\mu}. \tag{32}$$

The class of the Hodge class $\lambda := (\sigma \circ q)^*(\lambda)$ on $\overline{\mathcal{H}}_k^{\circ}$ has been determined in [36] and [25]:

$$\lambda = \sum_{i=2}^{3k-2} \sum_{\mu \vdash k} \mathrm{lcm}(\mu)\Big(\frac{i(6k-4-i)}{8(6k-5)} - \frac{1}{12}\Big(k - \sum_{j=1}^{\ell(\mu)} \frac{1}{\mu_j}\Big)\Big)[E_{i:\mu}] \in CH^1(\overline{\mathcal{H}}_k^{\circ}). \tag{33}$$

The sum is taken over partitions μ that correspond to permutations that can be written as products of i transpositions. Furthermore, $\ell(\mu)$ denotes the length of the partition μ and $\mathrm{lcm}(\mu)$ is the lowest common multiple of the parts of μ.

We now discuss in detail the divisors of $\overline{\mathcal{H}}_k^{\circ}$ lying over the boundary divisor B_2. We pick a cover

$$[f : C = C_1 \cup C_2 \to R = R_1 \cup_p R_2,\ p_1, \dots, p_{6k-4}] \in \mathfrak{b}^*(B_2),$$

where $C_i = f^{-1}(R_i)$. We assume $I = \{1, \dots, 6k-6\}$, thus $p_1, \dots, p_{6k-6} \in R_1$ and $p_{6k-5}, p_{6k-4} \in R_2$.

We denote by $E_{2:(1^k)}$ the closure in $\overline{\mathcal{H}}_k^{\circ}$ of the set of admissible covers for which the transpositions w_{6k-5} and w_{6k-4} corresponding to the points p_{6k-5} and p_{6k-4} are equal. Let E_0 be the component of $E_{2:(1^k)}$ corresponding to the case when C_1 is connected, which happens precisely when $\langle w_1, \dots, w_{6k-6}\rangle = \mathfrak{S}_k$. Clearly E_0 is irreducible. When w_{6k-5} and w_{6k-4} are distinct but not disjoint then $\mu = (3, 1^{k-3}) \vdash k$ and one is led to the boundary divisor $E_{2:(3,1^{k-3})}$. We denote by E_3 the (irreducible) subdivisor of $E_{2:(3,1^{k-3})}$ corresponding to the case $\langle w_1, \dots, w_{6k-6}\rangle = \mathfrak{S}_k$. Finally, the case when w_{6k-5} and w_{6k-4} are disjoint corresponds to the boundary divisor $E_{2:(2,2,1^{k-4})}$ and we denote by E_2 the irreducible component of $E_{2:(2,2,1^{k-4})}$ parametrizing covers for which $\langle w_1, \dots, w_{6k-6}\rangle = \mathfrak{S}_k$.

We discuss the behavior of the divisors E_0, E_2 and E_3 under the map q and we have

$$q^*(D_0) = 2E_0,\ q^*(D_2) = E_2 \text{ and } q^*(D_3) = 2E_3.$$

Indeed the general point of both E_0 and E_3 has no automorphism that fixes all branch points, but admits an automorphism of order two that fixes C_1 and permutes the branch points p_{6k-4} and p_{6k-5}. The general admissible cover in E_2 has an automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (each of the two components of C_2 mapping $2:1$ onto R_2 has an automorphism of order 2). In the stack $\overline{H}_k^{\circ}$ we have two points lying over this admissible cover and each of them has an automorphism group of order 2. In particular the map $\overline{H}_k^{\circ} \to \overline{\mathcal{H}}_k^{\circ}$ from the stack to its coarse moduli space is simply ramified along E_2.

The Hurwitz formula applied to the finite ramified cover $\mathfrak{b} : \overline{\mathcal{H}}_k^{\circ} \to \overline{\mathcal{M}}_{0,6k-4}$, coupled with the expression $K_{\overline{\mathcal{M}}_{0,6k-4}} = \sum_{i=2}^{3k-2}\big(\frac{i(6k-4-i)}{6k-5} - 2\big)[B_i]$ for the canonical class of $K_{\overline{\mathcal{M}}_{0,6k-4}}$, yields the following formula (on the stack!):

$$K_{\overline{H}_k^{\circ}} = \mathfrak{b}^* K_{\overline{\mathcal{M}}_{0,6k-4}} + \mathrm{Ram}(\mathfrak{b}), \tag{34}$$

where $\mathrm{Ram}(\mathfrak{b}) = \sum_{i,\mu}(\mathrm{lcm}(\mu) - 1)E_{i:\mu}$.

12.1. A partial compactification of the Hurwitz space

It turns out to be convenient to work with a partial compactification of $\mathcal{H}_k$. We denote by $\widetilde{\mathcal{H}}_k$ the (quasi-projective) parameter space of pairs $[C, A]$, where C is an irreducible nodal curve of genus $2k-1$ and A is a base-point-free locally-free sheaf of degree k on C with $h^0(C, A) = 2$. There exists a rational map $\overline{\mathcal{H}}_k \dashrightarrow \widetilde{\mathcal{H}}_k$, well-defined on the general point of each of the divisors D_0, D_2 and D_3 respectively. Under this map, to the general point $[f : C_1 \cup C_2 \to R_1 \cup_p R_2]$ of D_3 (respectively D_2) we assign the pair $[C_1, A_1 := f^*\mathcal{O}_{R_1}(1)] \in \widetilde{\mathcal{H}}_k$. Note that C_1 is a smooth curve of genus $2k-1$ and A_1 is a pencil with a triple point (respectively with two ramification points in the fiber over p). The two partial compactifications $\mathcal{H}_k \cup D_0 \cup D_2 \cup D_3$ and $\widetilde{\mathcal{H}}_k$ differ outside a set of codimension at least 2 and for divisor class calculations they will be identified. Using this, formula (33) has the following translation at the level of $\widetilde{\mathcal{H}}_k$:

$$\lambda = \frac{3(k-1)}{4(6k-5)}[D_0] - \frac{1}{4(6k-5)}[D_2] + \frac{3k-7}{12(6k-5)}[D_3] \in CH^1(\widetilde{\mathcal{H}}_k). \tag{35}$$

We now record the formula for the canonical class of $\widetilde{\mathcal{H}}_k$:

Proposition 12.1. – *One has* $K_{\widetilde{\mathcal{H}}_k} = 8\lambda + \frac{1}{6}[D_3] - \frac{3}{2}[D_0]$.

Proof. – We combine the equation (34) with the Hurwitz formula applied to $q : \overline{\mathcal{H}}_k^{\circ} \dashrightarrow \widetilde{\mathcal{H}}_k$ and write:

$$q^*(K_{\widetilde{\mathcal{H}}_k}) = K_{\overline{H}_k^{\circ}} - [E_0] - [E_2] - [E_3] = q^*\Big(-\frac{2}{6k-5}[D_2] - \frac{6k-3}{2(6k-5)}[D_0] + \frac{6k-11}{2(6k-5)}[D_3]\Big).$$

The divisors E_0 and E_3 lie in the ramification locus of q, so they are subtracted from $K_{\overline{H}_k^{\circ}}$. Furthermore, the morphism $\overline{H}_k^{\circ} \to \overline{\mathcal{H}}_k^{\circ}$ is simply ramified along E_2, so this divisor has to be subtracted as well. We now use (35) to express $[D_2]$ in terms of λ, $[D_0]$ and $[D_3]$ and reach the claimed formula. $\square$

Let $f : \mathcal{C}_k \to \widetilde{\mathcal{H}}_k$ be the universal curve and we choose a universal degree k line bundle $\mathcal{L}$ on $\mathcal{C}_k$, that is, satisfying $\mathcal{L}_{|f^{-1}[C,A]} = A$, for every $[C, A] \in \widetilde{\mathcal{H}}_k$. Just like in Section 7, we define the following codimension one tautological classes:

$$\mathfrak{a} := f_*\big(c_1^2(\mathcal{L})\big) \text{ and } \mathfrak{b} := f_*\big(c_1(\mathcal{L}) \cdot c_1(\omega_f)\big) \in CH^1(\widetilde{\mathcal{H}}_k).$$

Note that $\mathcal{V} := f_*\mathcal{L}$ is a vector bundle of rank two on $\widetilde{\mathcal{H}}_k$. Although $\mathcal{L}$ is not unique, the class

$$\gamma := \mathfrak{b} - \frac{2k-2}{k}\mathfrak{a} \in CH^1(\widetilde{\mathcal{H}}_k) \tag{36}$$

is well-defined and independent of the choice of $\mathcal{L}$.

Proposition 12.2. – *We have that* $\mathfrak{a} = kc_1(\mathcal{V}) \in CH^1(\widetilde{\mathcal{H}}_k)$.

Proof. – Recall that $\widetilde{\mathcal{H}}_k$ has been defined to consist of pairs $[C, A]$ such that A is a base point free pencil of degree k. In particular, the image under f of the codimension 2 locus in $\mathcal{C}_k$ where the morphism of vector bundles $f^*(\mathcal{V}) \to \mathcal{L}$ is not surjective is empty, hence by Porteous' formula

$$0 = f_*\Big(c_2(f^*\mathcal{V}) - c_1(f^*\mathcal{V}) \cdot c_1(\mathcal{L}) + c_1^2(\mathcal{L})\Big) = -kc_1(\mathcal{V}) + \mathfrak{a}. \qquad \square$$

We now introduce the following locally free sheaves on $\widetilde{\mathcal{H}}_k$:

$$\mathcal{E} := f_*(\omega_f \otimes \mathcal{L}^{\vee}) \text{ and } \mathcal{F} := f^*(\omega_f^2 \otimes \mathcal{L}^{-2})$$

Proposition 12.3. – *The following formulas hold*

$$c_1(\mathcal{E}) = \lambda - \frac{1}{2}\mathfrak{b} + \frac{k-2}{2k}\mathfrak{a} \textit{ and } c_1(\mathcal{F}) = 13\lambda + 2\mathfrak{a} - 3\mathfrak{b} - [D_0].$$

Proof. – We apply Grothendieck-Riemann-Roch twice to f. Since $R^1 f_*(\omega_f^2 \otimes \mathcal{L}^{-2}) = 0$, we write:

$$c_1(\mathcal{F}) = f_*\Big[\Big(1 + 2c_1(\omega_f) - 2c_1(\mathcal{L}) + 2(c_1(\omega_f) - c_1(\mathcal{L})^2\Big) \cdot \Big(1 - \frac{c_1(\omega_f)}{2} + \frac{c_1^2(\Omega_f^1) + c_2(\Omega_f^1)}{12}\Big)\Big]_2.$$

Now use Proposition 12.2 as well as $f_*\big(c_1^2(\Omega_f^1) + c_2(\Omega_f^1)\big) = 12\lambda$, see [30] p. 49, in order to conclude. To compute the first Chern class of $\mathcal{E}$, note that $c_1\big(R^1 f_*(\omega_f \otimes \mathcal{L}^\vee)\big) = -c_1(\mathcal{V})$, hence applying again Grothendieck-Riemann-Roch together with Proposition 12.2, we write:

$$\begin{aligned} c_1(\mathcal{E}) &= -kc_1(\mathcal{V}) \\ &\quad + f_*\Big[\Big(1 + c_1(\omega_f) - c_1(\mathcal{L}) + \frac{(c_1(\omega_f) - c_1(\mathcal{L})^2}{2}\Big)\Big(1 - \frac{c_1(\omega_f)}{2} + \frac{c_1^2(\Omega_f^1) + c_2(\Omega_f^1)}{12}\Big)\Big]_2, \end{aligned}$$

which quickly leads to the claimed formula. □

We summarize the relation between the class γ and the classes $[D_0], [D_2]$ and $[D_3]$ as follows:

PROPOSITION 12.4. – *One has the formula* $[D_3] = 6\gamma + 24\lambda - 3[D_0]$.

Proof. – We form the fiber product of the universal curve $f : \mathcal{C}_k \to \widetilde{\mathcal{H}}_k$ together with its projections:

$$\mathcal{C}_k \xleftarrow{\pi_1} \mathcal{C}_k \times_{\widetilde{\mathcal{H}}_k} \mathcal{C}_k \xrightarrow{\pi_2} \mathcal{C}_k.$$

For $\ell \geq 1$, we introduce the sheaf of principal parts $\mathcal{P}_f^\ell(\mathcal{L}) := (\pi_2)_*\Big(\pi_1^*(\mathcal{L}) \otimes \mathcal{J}_{(\ell+1)\Delta}\Big)$. Observe that $\mathcal{P}_f^\ell(\mathcal{L})$ is not locally free along the codimension 2 locus in $\mathcal{C}_k$ where f is not smooth. The jet bundle $J_f^\ell(\mathcal{L}) := \big(\mathcal{P}_f^\ell(\mathcal{L})\big)^{\vee\vee}$, viewed as a *locally free* replacement of $\mathcal{P}_f^\ell(\mathcal{L})$, sits in a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Omega_f^{\otimes \ell} \otimes \mathcal{L} \arrow[r] \arrow[d] & \mathcal{P}_f^\ell(\mathcal{L}) \arrow[r] \arrow[d] & \mathcal{P}_f^{\ell-1}(\mathcal{L}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \omega_f^{\otimes \ell} \otimes \mathcal{L} \arrow[r] & J_f^\ell(\mathcal{L}) \arrow[r] & J_f^{\ell-1}(\mathcal{L}) \arrow[r] & 0. \end{tikzcd}$$

We introduce the vector bundle morphism $\nu_2 : f^*(\mathcal{V}) \to J_f^2(\mathcal{L})$, which for points $[C, A, p] \in \mathcal{C}_k$ such that $p \in C_{\text{reg}}$ is just the evaluation morphism $H^0(C, A) \to H^0(A \otimes \mathcal{O}_{3p})$. We consider the codimension 2 locus $Z \subseteq \mathcal{C}_k$ where $\nu_2 : f^*(\mathcal{V}) \to J_f^2(\mathcal{L})$ is not injective. Over the locus of smooth curves, D_3 is the set-theoretic image of Z. A simple local analysis shows that the morphism ν_2 is simply degenerate for each point $[C, A, p]$, where $p \in C_{\text{sing}}$, that is, the divisor D_0 also appears (with multiplicity 1) in the degeneracy locus of ν_2. Assuming this fact for a moment, via the Porteous formula we obtain:

$$[D_3] = f_*c_2\left(\frac{J_f^2(\mathcal{L})}{f^*(\mathcal{V})}\right) - [D_0] \in CH^1(\widetilde{\mathcal{H}}_k).$$

One computes $c_1(J_f^2(\mathscr{L})) = 3c_1(\mathscr{L}) + 3c_1(\omega_f)$ and $c_2(J_f^2(\mathscr{L})) = 3c_1^2(\mathscr{L}) + 6c_1(\mathscr{L}) \cdot c_1(\omega_f) + 2c_1^2(\omega_f)$, therefore

$$f_*c_2\left(\frac{J_f^2(\mathscr{L})}{f^*(\mathscr{V})}\right) = 3\mathfrak{a} + 6\mathfrak{b} - 3(5k-4)c_1(\mathscr{V}) + 2\kappa_1 = 6\gamma + 2\kappa_1.$$

Recalling that $\kappa_1 = 12\lambda - [D_0] \in CH^1(\widetilde{\mathscr{H}}_k)$, the claimed formula follows by substitution.

We are left with concluding that D_0 appears with multiplicity 1 in the degeneracy locus Z. Let $F : \mathscr{X} \to B$ be a family of curves of genus $2k-1$ over a 1-dimensional base B, with $\mathscr{X}$ smooth, such that there is a point $b_0 \in B$ with $X_b := F^{-1}(b)$ smooth for $b \in B \setminus \{b_0\}$, whereas $X_{b_0} := F^{-1}(b_0)$ has a unique node $u \in \mathscr{X}$. Assume we are given $\mathscr{A} \in \mathrm{Pic}(\mathscr{X})$ such that $A_b := \mathscr{A}|_{X_b} \in W_k^1(X_b)$, for each $b \in B$. We pick a parameter $t \in \mathcal{O}_{B,b_0}$ and $x, y \in \mathcal{O}_{\mathscr{X},u}$, such that $xy = t$ represents the local equation of $\mathscr{X}$ around u. Then ω_F is locally generated by the meromorphic differential τ given by $\frac{dx}{x}$ outside the divisor $x = 0$ and by $-\frac{dy}{y}$ outside the divisor $y = 0$. We choose a $\mathbb{C}[[t]]$-basis (s_1, s_2) of $H^0(\mathscr{X}, A)$, where $s_1(u) \neq 0$, whereas s_2 vanishes with order 1 at the node u of X_{b_0}, along both its branches. Passing to germs at u, we may assume that $s_{2,u} = (x+y)s_{1,u}$. Denoting by $\partial : \mathcal{O}_{\mathscr{X}} \to \omega_F$ the canonical derivation, we note that $\partial(x) = x\tau$ and $\partial(y) = -y\tau$. Then Z is locally given by the 2×2 minors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ x+y & x-y & x+y \end{pmatrix},$$

which proves our claim, that D_0 appears with multiplicity 1. □

12.2. The divisor $\mathfrak{H}_k^{\mathrm{rk}4}$

We fix a cover $[f : C \to \mathbf{P}^1] \in \mathscr{H}_k$ and set $A := f^*(\mathcal{O}_{\mathbf{P}^1}(1))$. First we observe that by choosing $[f]$ outside a subset of codimension 2 in $\mathscr{H}_k$, we may assume that $\omega_C \otimes A^\vee$ is very ample. Otherwise by Riemann-Roch there exist points $p, q \in C$ such that $A(p+q) \in W_{k+1}^2(C)$. The Brill-Noether number of this linear series equals $\rho(2k-1, 2, k+1) = -1-k < -3$ and it follows from [11] that the locus of curves $[C] \in \mathscr{M}_{2k-1}$ possessing such a linear system has codimension at least 3 in moduli, which establishes the claim.

THEOREM 12.5. – *Fix a general point $[C, A] \in \mathscr{H}_k$. Then the embedded curve $\varphi_{\omega_C \otimes A^\vee} : C \hookrightarrow \mathbf{P}^{k-1}$ lies on no quadrics of rank 4 or less. It follows that $\mathfrak{H}_k^{\mathrm{rk}4}$ is a divisor on $\mathscr{H}_k$.*

Proof. – We choose a polarized $K3$ surface X such that $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot E$, where $L^2 = 4k-4$, the curve E is elliptic with $E^2 = 0$ and $L \cdot E = k$. First we observe that X contains no (-2)-curves, hence an effective line bundle $\alpha \in \mathrm{Pic}(X)$ must necessarily be nef and satisfy $\alpha^2 \geq 0$.

Since $(L - 2E)^2 = -4$, we compute $\chi(X, L(-2E)) = 0$. Furthermore, as we have just pointed out $H^0(X, L(-2E)) = 0$, whereas obviously $H^2(X, L(-2E)) = 0$, which implies that $H^1(X, L(-2E)) = 0$, as well. We choose a general curve $C \in |L|$ and set $A := \mathcal{O}_C(E) \in W_k^1(C)$. By taking cohomology in the exact sequence

$$0 \longrightarrow L(-2E) \longrightarrow L^{\otimes 2}(-2E) \longrightarrow \omega_C^{\otimes 2}(-2A) \longrightarrow 0,$$

we obtain an isomorphism $H^0\big(X, L^{\otimes 2}(-2E)\big) \cong H^0\big(C, \omega_C^{\otimes 2}(-2A)\big)$. Since clearly, the isomorphism $H^0(X, L(-E)) \cong H^0(C, \omega_C(-A))$ also holds, we obtain

$$I_{X,L}(2) \cong I_{C,\omega_C(-A)}(2),$$

so it suffices to show that the embedded surface $\varphi_{L(-E)} : X \hookrightarrow \mathbf{P}^{k-1}$ lies on no quadric of rank 4. This amounts to showing that one cannot have a decomposition $L - E = D_1 + D_2$, where D_1 and D_2 are divisor classes on X with $h^0(X, D_i) \geq 2$, for $i = 1, 2$. Assume we have such a decomposition and write $D_i = x_i C + y_i E$, where $x_1 + x_2 = 1$ and $y_1 + y_2 = -1$. Since E is nef, we obtain that both x_1 and x_2 have to be non-negative and we assume $x_1 = 0$ and $x_2 = 1$. Then $D_1 \equiv y_1 E$, therefore $y_1 \geq 1$, yielding $y_2 \leq -2$, which implies that $h^0(X, D_2) \leq h^0(X, L(-2E)) = 0$, which leads to a contradiction. □

The divisor $\mathfrak{H}_k^{\mathrm{rk}4}$ decomposes into components, depending on the degrees of the pencils A_1 and A_2 for which the decomposition (2) holds. For instance, when $\deg(A_1) = \deg(A) = k$, we obtain the component denoted in [18] by $\mathfrak{BN}$ and which consists of pairs $[C, A] \in \mathcal{H}_k$, such that C carries a second pencil of degree k. It is shown in [18] that $\mathfrak{BN}$ has a syzygy-theoretic incarnation that makes reference only to the canonical bundle, being equal to the Eagon-Northcott divisor on $\mathcal{H}_k$ of curves for which $b_{k-1,1}(C, \omega_C) \geq k$. It is an interesting question whether the remaining components of $\mathfrak{H}_k^{\mathrm{rk}4}$ have a similar intrinsic realization.

We now compute the class of the closure of $\mathfrak{H}_k^{\mathrm{rk}4}$ inside $\widetilde{\mathcal{H}}_k$:

THEOREM 12.6. – *The following formula holds:* $[\overline{\mathfrak{H}}_k^{\mathrm{rk}4}] = A_k^{k-4}\Big(\frac{5k+12}{k}\lambda + \frac{k-6}{k}\gamma - [D_0]\Big)$.

Proof. – We are in a position to apply Theorem 1.1 and then $[\overline{\mathfrak{H}}_k^{\mathrm{rk}4}] = A_k^{k-4}\Big(c_1(\mathcal{F}) - \frac{4}{k}(2k-3)c_1(\mathcal{E})\Big)$ and we substitute these Chern classes with the formulas provided in Proposition 12.3. □

The proof of Theorem 1.7 from the Introduction now follows. We substitute the formula for the class γ obtained from Theorem 12.6 in the expression provided by Proposition 12.4, then compare it to the formula for $K_{\widetilde{\mathcal{H}}_k}$.

Proof of Theorem 1.8. – It is enough to observe that for $k \geq 12$, the class $7\lambda - \delta_0$ is big on $\overline{\mathcal{M}}_{2k-1}$ and there exists an effective divisor of this slope that does not contain $\mathrm{Im}(\sigma) = \overline{\mathcal{M}}^1_{2k-1,k}$ as a component. This follows from results in [17] Corollary 0.6, where it is proved that the divisor $\overline{D}_{2k-1,k+1}$ on $\overline{\mathcal{M}}_{2k-1}$ already considered in (19), has support distinct from that of $\overline{\mathcal{M}}^1_{2k-1,k}$ and slope $\frac{6k^2+14k+3}{k(k+1)} < 7$. □

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PSEUDOSPECTRAL AND SPECTRAL BOUNDS FOR THE OSEEN VORTICES OPERATOR

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ABSTRACT. – In this paper, we solve Gallay's conjecture on the spectral lower bound and pseudospectral bound for the linearized operator of the Navier-Stokes equations in $\mathbb{R}^2$ around rapidly rotating Oseen vortices. This shows that the linearized operator becomes highly non-selfadjoint in the fast rotating limit, and the fast rotation has a strong stabilizing effect on vortices. The main difficulty is to handle the nonlocal part of the linearized operator. By introducing the polar coordinate, the linearized operator can be reduced to a family of one-dimensional operators $\widetilde{\mathcal{H}}_k$ for $|k| \geq 1$. For the case of $|k| \geq 2$, the nonlocal part could be treated as a perturbation by establishing some sharp coercive estimates. The case of $|k| = 1$ is critical in some sense. For this case, the nonlocal part is eliminated by constructing a wave operator. After these reductions, the resolvent estimates can be proved by using the multiplier method.

RÉSUMÉ. – Dans cet article, nous résolvons la conjecture de Gallay sur la borne inférieure spectrale et la borne pseudospectrale pour l'opérateur linéarisé des équations de Navier-Stokes dans $\mathbb{R}^2$ autour des tourbillons Oseen en rotation rapide. Cela montre que l'opérateur linéarisé devient très non autojointif dans la limite de rotation rapide et que la rotation rapide a un fort effet stabilisant sur les tourbillons. La principale difficulté est de gérer la partie non locale de l'opérateur linéarisé. En introduisant la coordonnée polaire, l'opérateur linéarisé peut être réduit à une famille d'opérateurs unidimensionnels $\widetilde{\mathcal{H}}_k$ pour $|k| \geq 1$. Pour le cas de $|k| \geq 2$, la partie non locale pourrait être traitée comme une perturbation en établissant des estimations coercitives précises. Le cas de $|k| \geq 1$ est critique dans un certain sens. Dans ce cas, la partie non locale est éliminée en construisant un opérateur d'onde. Après ces réductions, les estimations de résolution peuvent être prouvées en utilisant la méthode du multiplicateur.

1. Introduction

In this paper, we consider the Navier-Stokes equations in $\mathbb{R}^2$

$$
\left\{
\begin{array}{l}
\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla p = 0,\\
\operatorname{div} v = 0,\\
v(0,x) = v_0(x),
\end{array}
\right.
\tag{1}
$$

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where $v(t,x)$ denotes the velocity, $p(t,x)$ denotes the pressure and $\nu > 0$ is the viscosity coefficient. Let $\omega(t,x) = \partial_2 v^1 - \partial_1 v^2$ be the vorticity. The vorticity formulation of (1) takes

$$\partial_t \omega - \nu\Delta\omega + v\cdot\nabla\omega = 0, \quad \omega(0,x) = \omega_0(x). \tag{2}$$

Given the vorticity ω, the velocity can be recovered by the Biot-Savart law

$$v(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(t,y)dy = K_{BS} * \omega. \tag{3}$$

It is well known that the Navier-Stokes Equation (2) has a family of self-similar solutions called Lamb-Oseen vortices of the form

$$\omega(t,x) = \frac{\alpha}{\nu t}\mathcal{G}\Big(\frac{x}{\sqrt{\nu t}}\Big), \quad v(t,x) = \frac{\alpha}{\sqrt{\nu t}} v^G\Big(\frac{x}{\sqrt{\nu t}}\Big), \tag{4}$$

where the vorticity profile and the velocity profile are given by

$$\mathcal{G}(\xi) = \frac{1}{4\pi}e^{-|\xi|^2/4}, \quad v^G(\xi) = \frac{1}{2\pi}\frac{\xi^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big).$$

It is easy to see that $\int_{\mathbb{R}^2}\omega(t,x)dx = \alpha$ for any $t > 0$. The parameter $\alpha \in \mathbb{R}$ is called the circulation Reynolds number.

To investigate the long-time behavior of (2), it is convenient to introduce the self-similar variables

$$\xi = \frac{x}{\sqrt{\nu t}}, \quad \tau = \log t,$$

and the rescaled vorticity w and the rescaled velocity u

$$\omega(t,x) = \frac{1}{t}w\Big(\log t, \frac{x}{\sqrt{\nu t}}\Big), \quad v(t,x) = \sqrt{\frac{\nu}{t}}u\Big(\log t, \frac{x}{\sqrt{\nu t}}\Big).$$

Then (w,u) satisfies

$$\partial_\tau w + u\cdot\nabla w = Lw, \tag{5}$$

where the linear operator L is given by

$$L = \Delta + \frac{\xi}{2}\cdot\nabla + 1. \tag{6}$$

For any $\alpha \in \mathbb{R}$, the Lamb-Oseen vortex $\alpha\mathcal{G}(\xi)$ is a steady solution of (5). Gallay and Wayne [14, 15] proved that for any integrable initial vorticity, the long-time behavior of the 2-D Navier-Stokes equations can be described by the Lamb-Oseen vortex. More precisely, for any initial data $w_0 \in L^1(\mathbb{R}^2)$, the solution of (5) satisfies

$$\lim_{\tau\to+\infty}\big\|w(\tau) - \alpha\mathcal{G}\big\|_{L^1(\mathbb{R}^2)} = 0, \quad \alpha = \int_{\mathbb{R}^2} w_0(\xi)d\xi.$$

This result suggests that $\alpha\mathcal{G}$ is a stable equilibrium of (5) for any $\alpha \in \mathbb{R}$. This situation is very similar to the Couette flow $(y,0)$ in a finite channel, which is stable for any Reynolds number [9]. Recently, there are many important works [2, 3, 1, 20, 30] devoted to the study of long-time behavior of the Navier-Stokes(Euler) equations around the Couette flow.

To study the stability of αG, it is natural to consider the linearized equation of (5) around $\alpha\mathcal{G}(\xi)$, which takes as follows

$$\partial_\tau w = (L - \alpha\Lambda)w, \tag{7}$$

where Λ is a nonlocal linear operator defined by

$$\Lambda w = v^G \cdot \nabla w + u \cdot \nabla \mathcal{G} = \Lambda_1 w + \Lambda_2 w, \quad u = K_{BS} * w. \tag{8}$$

The operator $L - \alpha\Lambda$ in the weighted space $Y = L^2(\mathbb{R}^2, \mathcal{G}^{-1}dx)$ defined in Section 2 has a compact resolvent. Thus, the spectrum of $L - \alpha\Lambda$ in Y is a sequence of eigenvalues $\{\lambda_n(\alpha)\}_{n\in\mathbb{N}}$. Moreover, $L - \alpha\Lambda$ is also dissipative, so $\mathrm{Re}\lambda_n(\alpha) \le 0$ for any n, α(see [13] for example). A very important problem is to study how the spectrum changes as $|\alpha| \to +\infty$, which corresponds to the high Reynolds number limit(the most relevant regime for turbulent flows).

The eigenvalues corresponding to the eigenfunctions in the kernel of Λ do not change as α varies. Thus, we introduce two operators $L_\perp$ and $\Lambda_\perp$, which are the restriction of the operators L and Λ to the orthogonal complement of $\ker\Lambda$ in Y respectively. We define the spectral lower bound

$$\Sigma(\alpha) = \inf\Big\{\mathrm{Re}\, z : z \in \sigma\big(-L_\perp + \alpha\Lambda_\perp\big)\Big\} \tag{9}$$

and pseudospectral bound

$$\Psi(\alpha) = \Big(\sup_{\lambda\in\mathbb{R}} \|\big(L_\perp - \alpha\Lambda_\perp - i\lambda\big)^{-1}\|_{Y\to Y}\Big)^{-1}. \tag{10}$$

It is easy to see that $\Sigma(\alpha) \ge \Psi(\alpha)$ for any $\alpha \in \mathbb{R}$. For selfadjoint operators, the spectral and pseudospectral bounds are the same. Since $L - \alpha\Lambda$ is a non-selfadjoint operator, $\Sigma(\alpha)$ and $\Psi(\alpha)$ could be different. Let us mention that the pseudospectra has become an important concept in understanding the hydrodynamic stability [27]. The spectral theory of non-selfadjoint operator is also a current active topic [5, 6, 25, 26].

There are many works devoted to studying $\Sigma(\alpha)$ and $\Psi(\alpha)$. Maekawa [22] proved that $\Sigma(\alpha)$ and $\Psi(\alpha)$ tend to infinity as $|\alpha| \to +\infty$. However, the proof does not provide explicit bounds on $\Sigma(\alpha)$ and $\Psi(\alpha)$. Numerical calculations performed by Prochazka and Pullin [23, 24] indicate that $\Sigma(\alpha) = O(|\alpha|^{\frac{1}{2}})$ as $|\alpha| \to +\infty$. For the rigorous analysis, the main difficulty comes from the nonlocal part Λ_2 of the linearized operator. In [10], Gallagher, Gallay and Nier considered the following toy operator(see also Villani [29, 28]):

$$H_\alpha = -\partial_x^2 + x^2 + i\alpha f(x).$$

Let $\Sigma(\alpha)$ be the infimum of the real part of $\sigma(H_\alpha)$ and $\Psi(\alpha)^{-1}$ be the supremum of the norm of the resolvent of H_α along the imaginary axis. Under the appropriate conditions on f, they proved that $\Sigma(\alpha)$ and $\Psi(\alpha)$ go to infinity as $|\alpha| \to +\infty$, and presented the precise estimate of the growth rate of $\Psi(\alpha)$. Their proof used the hypocoercive method, localization techniques, and semiclassical subelliptic estimates.

For the simplified linearized operator $L - \alpha\Lambda_1$, Deng [7] proved that $\Psi(\alpha) = O(|\alpha|^{\frac{1}{3}})$. The same result was proved by Deng [8] for the full linearized operator restricted to a smaller subspace than $\ker(\Lambda)^\perp$ by using the multiplier method and the Weyl calculus [18].

In this paper, we proved the following conjecture proposed by Gallay [11].

Theorem 1.1. – *There exists $C > 0$ independent of α so that as $|\alpha| \to +\infty$,*

$$\Sigma(\alpha) \ge C^{-1}|\alpha|^{\frac{1}{2}}, \quad C^{-1}|\alpha|^{\frac{1}{3}} \le \Psi(\alpha) \le C|\alpha|^{\frac{1}{3}}.$$

This result shows that the linearized operator $L - \alpha\Lambda$ becomes highly non-selfadjoint in the fast rotating limit, and the fast rotation has a strong stabilizing effect on vortices. This effect is related to a well-known experiment fact in 2-D viscous flows, where the isolated vortices relax to axisymmetry in a relatively shorter time than the diffusive time. Bernoff and Lingevitch [4] proposed the following *mixing hypothesis:*

If the vorticity distribution is subject to a nonaxisymmetric linear perturbation that preserves the first momentum of vorticity, this perturbation will decay on a time scale $O(Re^{1/3})$, here Re is the Reynolds number.

Recently, Gallay [12] made significant progress on this hypothesis, and proved that the vortex relaxes to axisymmetry in a time scale $Re^{2/3}$. The key point is to derive the enhanced dissipation rate of the semigroup $e^{t(L-\alpha\Lambda)}$ from the resolvent estimate of $L - \alpha\Lambda$. The pseudospectral bound may be helpful to understand other related problems such as the stability of Burgers vortices [16, 21].

The hypocoercive method introduced by Villani [29] seems efficient for the linearized operator without the nonlocal part. In this paper, we introduce two methods to handle the nonlocal part of the linearized operator. One method is to eliminate the nonlocal part by constructing a wave operator, which is motivated by the scattering theory. Another one is to establish the sharp coercive estimates so that the nonlocal part could be treated as a perturbation in some situations. After making these reductions, we can use the multiplier method to establish the resolvent estimates. In [31] and [19], these two methods are applied to study the linearized Navier-Stokes equations around the Kolmogorov flow.

2. Spectral analysis of the linearized operator

In this section, we recall some facts about the spectrum of the linearized operator $L-\alpha\Lambda$ from [14, 15, 11, 13]. Although these facts will not be used in our proof, they will be helpful to understand this spectral problem.

Let $\rho(\xi)$ be a nonnegative function. We introduce the weighted L^2 space

$$L^2(\mathbb{R}^2, \rho d\xi) = \Big\{ w \in L^2(\mathbb{R}^2) : \|w\|^2_{L^2(\rho)} = \int_{\mathbb{R}^2} |w(\xi)|^2 \rho(\xi) d\xi < +\infty \Big\},$$

which is a (real) Hilbert space equipped with the scalar product

$$\langle w_1, w_2 \rangle_{L^2(\rho)} = \int_{\mathbb{R}^2} w_1(\xi) w_2(\xi) \rho(\xi) d\xi.$$

In this paper, we will choose the Gaussian weight $\rho = \mathcal{G}^{-1}$ and denote $Y = L^2(\mathbb{R}^2, \mathcal{G}^{-1} d\xi)$. Let us refer to [11] for more discussions on the choice of the weight.

LEMMA 2.1. – *It holds that*

1. *the operator L is selfadjoint in Y with compact resolvent and purely discrete spectrum*

$$\sigma(L) = \Big\{ -\frac{n}{2} : n = 0, 1, 2, \dots \Big\}. \tag{11}$$

2. *the operator Λ is skew-symmetric in Y.*

The first fact follows from the following observation:

$$\mathscr{L} = -\mathcal{G}^{-\frac{1}{2}} L \mathcal{G}^{\frac{1}{2}} = -\Delta + \frac{|\xi|^2}{16} - \frac{1}{2} \tag{12}$$

is a two-dimensional harmonic oscillator, which is self-adjoint in $L^2(\mathbb{R}^2)$ with compact resolvent and discrete spectrum given by $-\sigma(L)$. Furthermore, we know that

1. $\lambda_0 = 0$ is a simple eigenvalue of L with the eigenfunction $\mathcal{G}$;
2. $\lambda_1 = -\frac{1}{2}$ is an eigenvalue of L of multiplicity two with the eigenfunctions $\partial_1 \mathcal{G}$ and $\partial_2 \mathcal{G}$;
3. $\lambda_1 = -1$ is an eigenvalue of L of multiplicity three with the eigenfunctions $\Delta \mathcal{G}$, $(\partial_1^2 - \partial_2^2)\mathcal{G}$ and $\partial_1 \partial_2 \mathcal{G}$.

Now we consider the spectrum of $L - \alpha\Lambda$ in Y for any fixed $\alpha \in \mathbb{R}$. Since Λ is a relatively compact perturbation of L in Y, $L - \alpha\Lambda$ has a compact resolvent in Y by the classical perturbation theory [17]. So, the spectrum of $L - \alpha\Lambda$ is a sequence of eigenvalues $\{\lambda_n(\alpha)\}_{n\in\mathbb{N}}$. Using the fact that

$$\Lambda w = 0 \quad \text{for} \quad w = \mathcal{G}, \partial_1 \mathcal{G}, \partial_2 \mathcal{G}, \Delta \mathcal{G},$$

we deduce that $0, -\frac{1}{2}, -1$ are also eigenvalues of $L - \alpha\Lambda$ for any $\alpha \in \mathbb{R}$. Let us introduce the following subspaces of X:

$$Y_0 = \Big\{ w \in Y : \int_{\mathbb{R}^2} w(\xi) d\xi = 0 \Big\} = \{\mathcal{G}\}^{\perp}, \tag{13}$$

$$Y_1 = \Big\{ w \in Y_0 : \int_{\mathbb{R}^2} \xi w(\xi) d\xi = 0 \Big\} = \{\mathcal{G}, \partial_1 \mathcal{G}, \partial_2 \mathcal{G}\}^{\perp}, \tag{14}$$

$$Y_2 = \Big\{ w \in Y_1 : \int_{\mathbb{R}^2} |\xi|^2 w(\xi) d\xi = 0 \Big\} = \{\mathcal{G}, \partial_1 \mathcal{G}, \partial_2 \mathcal{G}, \Delta \mathcal{G}\}^{\perp}. \tag{15}$$

These spaces are invariant under the linear evolution generated by $L - \alpha\Lambda$.

The following proposition shows that the Oseen vortex $\alpha\mathcal{G}$ is spectrally stable in Y for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

PROPOSITION 2.2 ([15]). – *For any $\alpha \in \mathbb{R}$, $\alpha \neq 0$, the spectrum of $L - \alpha\Lambda$ satisfies*

$$\begin{aligned}
\sigma(L - \alpha\Lambda) &\subset \big\{ z \in \mathbb{C} : Re\, z \le 0 \big\} && \text{in} \quad Y,\\
\sigma(L - \alpha\Lambda) &\subset \big\{ z \in \mathbb{C} : Re\, z \le -\tfrac{1}{2} \big\} && \text{in} \quad Y_0,\\
\sigma(L - \alpha\Lambda) &\subset \big\{ z \in \mathbb{C} : Re\, z \le -1 \big\} && \text{in} \quad Y_1,\\
\sigma(L - \alpha\Lambda) &\subset \big\{ z \in \mathbb{C} : Re\, z < -1 \big\} && \text{in} \quad Y_2.
\end{aligned}$$

The operator $L - \alpha\Lambda$ is invariant under rotations with respect to the origin. Thus, it is natural to introduce the polar coordinates (r, θ) in $\mathbb{R}^2$. Let us decompose

$$Y = \bigoplus_{n\in\mathbb{N}} X_n, \tag{16}$$

where X_n denotes the subspace of all $w \in Y$ so that

$$w(r\cos\theta, r\sin\theta) = a(r)\cos(n\theta) + b(r)\sin(n\theta)$$

for some radial functions $a, b : \mathbb{R}^+ \to \mathbb{R}$.

LEMMA 2.3 ([22]). – *ker* $\Lambda = X_0 \oplus \{\alpha\partial_1 \mathcal{G} + \beta\partial_2\mathcal{G}\}$. *In particular, ker* $\Lambda^\perp \subset \bigoplus_{n>0} X_n$.

3. Reduction to one-dimensional operators

Following Deng's work [8], we reduce the linearized operator to a family of one-dimensional operators.

We conjugate the linearized operator $L - \alpha\Lambda$ with $\mathcal{G}^{\frac{1}{2}}$, and then obtain a linear operator $\mathcal{H}_\alpha$ in $L^2(\mathbb{R}^2, d\xi)$:

$$\mathcal{H}_\alpha = -\mathcal{G}^{-\frac{1}{2}} L\, \mathcal{G}^{\frac{1}{2}} + \alpha\, \mathcal{G}^{-\frac{1}{2}} \Lambda\, \mathcal{G}^{\frac{1}{2}} = \mathcal{L} + \alpha\, \mathcal{M}, \tag{17}$$

where $\mathcal{L}$ is defined by (12) and $\mathcal{M}$ is defined by

$$\mathcal{M}w = v^G \cdot \nabla w - \frac{1}{2}\mathcal{G}^{\frac{1}{2}}\xi \cdot \big(K_{BS} * (\mathcal{G}^{\frac{1}{2}} w)\big).$$

Let us introduce some notations:

$$\mathcal{K}_k[h] = \frac{1}{2|k|}\int_0^{+\infty} \min(\frac{r}{s}, \frac{s}{r})^{|k|} s h(s) ds, \tag{18}$$

$$\sigma(r) = \frac{1 - e^{-r^2/4}}{r^2/4}, \quad g(r) = e^{-r^2/8}. \tag{19}$$

Then for $w = \sum_{k\in\mathbb{Z}^*} w_k(r)e^{ik\theta}$, we have

$$\big((\mathcal{H}_\alpha - i\lambda)w\big)(r\cos\theta, r\sin\theta) = \sum_{k\in\mathbb{Z}^*} (\mathcal{H}_{\alpha,k,\lambda} w_k)(r)e^{ik\theta},$$

where the operator $\mathcal{H}_{\alpha,k,\lambda}$ acts on $L^2(\mathbb{R}_+, rdr)$ and is given by

$$\mathcal{H}_{\alpha,k,\lambda} = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2} + \frac{r^2}{16} - \frac{1}{2} + i\beta_k(\sigma(r) - \nu_k) - i\beta_k g\, \mathcal{K}_k[g\cdot], \tag{20}$$

where

$$\beta_k = \frac{\alpha k}{8\pi}, \quad \lambda = \beta_k \nu_k \in \mathbb{R}. \tag{21}$$

Without loss of generality, we assume $|\beta_k| \geq 1$ for any $|k| \geq 1$.

We introduce the operator

$$\widetilde{\mathcal{H}}_{\alpha,k,\lambda} = r^{\frac{1}{2}}\, \mathcal{H}_{\alpha,k,\lambda} r^{-\frac{1}{2}} := \widetilde{\mathcal{H}}_k. \tag{22}$$

Then $\widetilde{\mathcal{H}}_k$ acts on $L^2(\mathbb{R}_+, dr)$ and is given by

$$\widetilde{\mathcal{H}}_k = -\partial_r^2 + \frac{k^2 - \frac{1}{4}}{r^2} + \frac{r^2}{16} - \frac{1}{2} + i\beta_k(\sigma(r) - \nu_k) - i\beta_k g\, \widetilde{\mathcal{K}}_k[g\cdot], \tag{23}$$

$$\widetilde{\mathcal{K}}_k[h] = \frac{1}{2|k|}\int_0^{+\infty} \min(\frac{r}{s}, \frac{s}{r})^{|k|} (rs)^{\frac{1}{2}} h(s) ds, \tag{24}$$

and $C_0^\infty(\mathbb{R}_+)$ is a core of the operator $\widetilde{\mathcal{H}}_k$ with domain

$$D(\widetilde{\mathcal{H}}_k) = \big\{w \in H^2_{\mathrm{loc}}(\mathbb{R}_+, dr) \cap L^2(\mathbb{R}_+, dr) : \widetilde{\mathcal{H}}_k w \in L^2(\mathbb{R}_+, dr)\big\}. \tag{25}$$

That is,

$$D = D(\widetilde{\mathcal{H}}_k) = \Big\{w \in L^2(\mathbb{R}_+, dr) : \partial_r^2 w, \frac{w}{r^2}, r^2 w \in L^2(\mathbb{R}_+, dr)\Big\} \quad |k| \geq 2,$$

$$D_1 = D(\widetilde{\mathcal{H}}_k) = \Big\{w \in L^2(\mathbb{R}_+, dr) : r^{\frac12}\partial_r^2(w/r^{\frac12}), r^{\frac12}\partial_r(w/r^{\frac32}), r^2 w \in L^2(\mathbb{R}_+, dr)\Big\} \quad |k| = 1.$$

Then the resolvent estimate is reduced to the following estimate

$$\|\widetilde{\mathcal{H}}_k u\|_{L^2(\mathbb{R}_+,dr)} \gtrsim |\beta_k|^{\frac13}\|u\|_{L^2(\mathbb{R}_+,dr)}.$$

We also write

$$\widetilde{\mathcal{H}}_k = \widetilde{A}_k + i\beta_k \widetilde{B}_k - i\lambda, \tag{26}$$

where

$$\widetilde{A}_k = -\partial_r^2 + \frac{k^2 - \frac14}{r^2} + \frac{r^2}{16} - \frac12,$$
$$\widetilde{B}_k = \sigma(r) - g\widetilde{\mathcal{K}}_k[g\cdot].$$

It is easy to see that

$$\mathrm{Ker}(\widetilde{B}_1) = \mathrm{span}\big\{r^{\frac32} g(r)\big\}, \quad \mathrm{Ker}(\widetilde{B}_k) = \big\{0\big\} \quad \text{for } |k| \geq 2. \tag{27}$$

Thus, $L - \alpha\Lambda|_{(\ker\Lambda)^\perp}$ is unitary equivalent to $\bigoplus_{|k|=1} \widetilde{\mathcal{H}}_k|_{(\ker \widetilde{B}_1)^\perp} \oplus \bigoplus_{|k|\geq 2} \widetilde{\mathcal{H}}_k$.

In the sequel, we denote by $\langle\cdot,\cdot\rangle$ the $L^2(\mathbb{R}_+, dr)$ inner product, and by $\|\cdot\|$ the norm of $L^2(\mathbb{R}_+, dr)$, $\|\cdot\|_{L^p}$ the norm of $L^p(\mathbb{R}_+, dr)$. The notation $a \gtrsim b$ or $a \lesssim b$ means that there exists a constant $C > 0$ independent of α, k, λ so that

$$a \geq C^{-1} b \quad \text{or} \quad a \leq Cb.$$

4. Sketch of the proof and ideas

With the notations in Section 2 and Section 3, we are in a position to present some key ideas in this paper. Recall that

$$\widetilde{\mathcal{H}}_k = \widetilde{A}_k + i\beta_k \widetilde{B}_k - i\lambda, \tag{28}$$

where $\beta_k = \frac{\alpha k}{8\pi}$, $\lambda = \beta_k \nu_k \in \mathbb{R}$ and

$$\widetilde{A}_k = -\partial_r^2 + \frac{k^2 - \frac14}{r^2} + \frac{r^2}{16} - \frac12, \qquad \widetilde{B}_k = \sigma(r) - g\widetilde{\mathcal{K}}_k[g\cdot].$$

4.1. Resolvent estimate of $\widetilde{\mathcal{H}}_k$, $|k| \geq 2$

First of all, we can establish the following key coercive estimates: for any $|k| \geq 1$ and $w \in L^2(\mathbb{R}_+; dr)$

$$\langle (I - \widetilde{B}_k)w, w\rangle \geq \int_0^{+\infty} (1 - \sigma(r))|w|^2 dr,$$

$$\langle \widetilde{B}_k w, w\rangle \geq (1 - \frac{1}{|k|}) \int_0^{+\infty} \sigma(r)|w|^2 dr.$$

In particular, when $|k| \geq 2$, this implies that

$$\frac{1}{2}\sigma(r) \leq \widetilde{B}_k \leq \sigma(r).$$

This fact ensures that the nonlocal operator $g\widetilde{\mathcal{K}}_k[g\cdot]$ could be treated as a perturbation of $\sigma(r)$ for $|k| \geq 2$. Thus, we do not need to use the wave operator for $|k| \geq 2$.

According to the sign of $\sigma(r) - \nu_k$, the resolvent estimate is split into two cases. When $\nu_k \geq 1$ or $\nu_k \leq 0$, the sign of $\sigma(r)-\nu_k$ does not change. The resolvent estimate of $\widetilde{\mathcal{H}}_k (k \geq 2)$ could be proved by using the coercive estimates of $\widetilde{A}_k$ and $\widetilde{B}_k$.

The case of $0 < \nu_k < 1$ is difficult. We will use the multiplier method to handle it. We know that $\sigma(r) - \nu_k$ has a unique zero point r_k. Then we divide the proof into two subcases:

$$|\beta_k| \leq \max\big(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6\big) \quad \text{and} \quad |\beta_k| \geq \max\big(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6\big).$$

The first case is simple. In fact, we know that $|\langle\widetilde{\mathcal{H}}_k w, w\rangle| \geq \langle\widetilde{A}_k w, w\rangle$ and $\widetilde{A}_k \gtrsim \frac{k^2}{r^2} + r^2 \gtrsim |k|$, thus the case of $|\beta_k| \leq |k|^3$ is trivial. When $|k|^2 \leq |\beta_k| \leq \max\big(\frac{|k|^3}{r_k^4}, r_k^6\big)$, we need to use the coercive estimate of $\widetilde{B}_k$ and some simple properties of $\sigma(r)$.

The second case is highly nontrivial. It is natural to divide the integral interval into $(0, r_k)$ and $(r_k, +\infty)$. We introduce the truncated integral operator $\widetilde{\mathcal{K}}_k^{(r_k)}$(see (51)). For this, we have the following important coercive estimate:

$$\mathrm{Re}\int_0^{r_k} g(r)\widetilde{\mathcal{K}}_k^{(r_k)}[gw](r)\overline{w}(r)dr \leq \frac{2}{|k|+1}\int_0^{r_k}(\sigma(s)-\nu_k)|w(s)|^2 ds.$$

This is another key point why the case of $|k| \geq 2$ is relatively simple. First of all, we choose the multiplier like $i\,\mathrm{sgn}(\beta_k)(\chi_{(0,r_k)} - \chi_{(r_k,+\infty)})$ and make the following estimate

$$\mathrm{Re}\big\langle\widetilde{\mathcal{H}}_k w, i\,\mathrm{sgn}(\beta_k)(\chi_{(0,r_k)} - \chi_{(r_k,+\infty)})w\big\rangle,$$

from which and by integration by parts, we can obtain a good term like $|\beta_k|\int_0^{+\infty}|\sigma - \nu_k||w|^2 dr$. To handle the case when $|\sigma - \nu_k|$ is small, it is natural to try another multiplier $i\,\mathrm{sgn}(\beta_k)\frac{\chi_{\mathbb{R}_+}}{\sigma-\nu_k}$, which is however singular at r_k. Thus, we use the multiplier $i\,\mathrm{sgn}(\beta_k)\frac{\chi_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)}}{\sigma-\nu_k}$. On the other hand, we just control $\|w\|^2_{L^2(B(r_k,\delta))}$ by $\delta\|w\|^2_{L^\infty}$, and control $\|w\|^2_{L^\infty}$ in the following way

$$\|w\|^2_{L^\infty} \leq \|w'\|\|w\|, \quad \|w'\|^2 \leq \|w\|\|\widetilde{\mathcal{H}}_k w\|.$$

The terms $\delta\|w\|^2_{L^\infty}$, $\delta^2\|w'\|^2$, $\delta^2\|w\|\|\widetilde{\mathcal{H}}_k w\|$ and $\delta^4\|\widetilde{\mathcal{H}}_k w\|^2$ are of the same order. This motivates us to introduce the energy functional

$$\mathcal{J}(w) = \delta\|w\|^2_{L^\infty} + \delta^2\|w'\|^2 + \delta^2\|w\|\|\widetilde{\mathcal{H}}_k w\| + \delta^4\|\widetilde{\mathcal{H}}_k w\|^2. \tag{29}$$

It can be proved that

$$\|w\|^2 \leq C\,\mathcal{J}(w) \leq C(\delta^2\|\widetilde{\mathcal{H}}_k w\|)^2.$$

Then the resolvent estimate follows by a suitable choice of δ.

4.2. Resolvent estimate of $\widetilde{\mathscr{H}}_k$, $|k| = 1$ and wave operator

From the above arguments, the case of $|k| = 1$ is critical. To handle this case, we need to use the wave operator method. More precisely, we will construct a partial unitary operator T which satisfies the following important properties:

$$T\widetilde{B}_1 = \sigma(r)T, \quad [T, \widetilde{A}_1]w = T\widetilde{A}_1 w - \widetilde{A}_1 T w = f(r)Tw$$

for some $f(r) \geq 0$. Then we conjugate $\widetilde{\mathscr{H}}_1$ with T to obtain

$$T\widetilde{\mathscr{H}}_1 T^{-1}u = \widetilde{A}_1 u + f(r)u + i\beta_1\sigma(r)u - i\lambda u = \mathscr{L}_1 u.$$

The operator $\mathscr{L}_1$ is similar to the model operator. The resolvent estimate of $\mathscr{L}_1$ can be obtained by following similar arguments as in the case of $|k| \geq 2$.

This idea is motivated by the scattering theory. Let A, B be two selfadjoint operators in the Hilbert space H. Let $U(t) = e^{itA}$ and $V(t) = e^{itB}$ be the strongly continuous groups of unitary operators. The wave operator is defined by

$$W_\pm = \lim_{t\to\pm\infty} W(t), \quad W(t) = U(-t)V(t).$$

Here the limit is understood as the strong convergence. Then it holds that

$$AW_\pm = W_\pm B. \tag{30}$$

In fact, we have

$$e^{isA}e^{-itA}e^{itB} = e^{-i(t-s)A}e^{i(t-s)B}e^{isB},$$

which gives by taking $t \to \pm\infty$ that

$$e^{isA}W_\pm = W_\pm e^{isB}.$$

Then the identity (30) follows by taking the derivative in s at $s = 0$.

Here the construction of the wave operator will follow a different procedure. The intuitive origin of our construction comes from our work on the inviscid damping [30]. Let us present some key details.

Let Ω be a simply connected domain including the spectrum $\sigma(\widetilde{B}_1) = [0, 1]$ of $\widetilde{B}_1$. We have the following representation formula of the semigroup $e^{it\widetilde{B}_1}$:

$$e^{it\widetilde{B}_1}w = \frac{1}{2\pi i}\int_{\partial\Omega} e^{itc}(c - \widetilde{B}_1)^{-1}w\,dc,$$
$$\widetilde{\mathscr{K}}_1[ge^{it\widetilde{B}_1}w] = \frac{1}{2\pi i}\int_{\partial\Omega} e^{itc}\widetilde{\mathscr{K}}_1[g(c - \widetilde{B}_1)^{-1}w]dc.$$

Let $\Phi \in \dot{H}^1(\mathbb{R}_+)$ be a solution of the inhomogeneous Rayleigh equation:

$$\partial_r^2\Phi - \frac{3\Phi}{4r^2} + \frac{g^2}{\sigma - c}\Phi = \frac{gw}{\sigma - c}.$$

Then we find that

$$\widetilde{\mathscr{K}}_1[g(c - \widetilde{B}_1)^{-1}w](y) = \Phi(y, c). \tag{31}$$

Here we used the formula $\big(-\partial_r^2+\frac{k^2-\frac14}{r^2}\big)\widetilde{\mathscr{K}}_k[w](r)=w(r)$ in (48). Moreover, the Rayleigh equation can be written as

$$\partial_r\left((\sigma-c)^2r^3\partial_r\left(\frac{\Phi}{(\sigma-c)r^{3/2}}\right)\right)=r^{\frac32}gw,$$

and hence the solution is given by

$$\Phi(r,c)=-(\sigma(r)-c)r^{\frac32}\int_r^{+\infty}\frac{\int_0^u s^{\frac32}g(s)w(s)ds}{(\sigma(u)-c)^2u^3}du\quad\text{for } c\in\mathbb{C}\setminus[0,1].$$

The next step is to establish the limiting absorption principle: as $\varepsilon\to 0+$,

$$\Phi(r,c\pm i\varepsilon)\to\Phi_\pm(r,c)\quad\text{for } c\in(0,1).$$

In fact, for $c\in(0,1)$, let $r_c\in\mathbb{R}_+$ be so that $\sigma(r_c)=c$. Then we find that

$$\Phi_\pm(r,c)=-(\sigma(r)-c)r^{\frac32}\int_r^{+\infty}\frac{\int_0^u s^{\frac32}g(s)w(s)ds}{(\sigma(u)-c)^2u^3}du\quad\text{for } r>r_c.$$

The expression of $\Phi_\pm(r,c)$ for $0<r<r_c$ is more complicated. We write $c_\varepsilon=c+i\varepsilon$ and

$$\begin{aligned}\Phi(r,c_\varepsilon)=&-(\sigma(r)-c_\varepsilon)r^{\frac32}\int_r^{+\infty}\frac{\int_{r_c}^u s^{\frac32}g(s)w(s)ds}{(\sigma(u)-c_\varepsilon)^2u^3}du\\&-(\sigma(r)-c_\varepsilon)r^{\frac32}\int_0^{r_c}s^{\frac32}g(s)w(s)ds\int_r^{+\infty}\frac{du}{(\sigma(u)-c_\varepsilon)^2u^3},\end{aligned}$$

where

$$\int_r^{+\infty}\frac{du}{(\sigma(u)-c_\varepsilon)^2u^3}=\frac{1}{r_c^3\sigma'(r_c)}\left(\int_r^{+\infty}\frac{\sigma'(u)du}{(\sigma(u)-c_\varepsilon)^2}-\int_r^{+\infty}\frac{u^3\sigma'(u)-r_c^3\sigma'(r_c)}{(\sigma(u)-c_\varepsilon)^2u^3}du\right).$$

Using the method in [30, 31], we can prove the following limit (since σ is decreasing)

$$\lim_{\varepsilon\to0\pm}\int_r^{+\infty}\frac{F(u)du}{(\sigma(u)-c_\varepsilon)^2}=\text{p.v.}\int_r^{+\infty}\frac{F(u)du}{(\sigma(u)-c)^2}\mp i\pi\frac{F'(r_c)}{\sigma'(r_c)^2},\ \text{for } r\in(0,r_c),\ F(r_c)=0.$$

Then it follows that for $r\in(0,r_c)$,

$$\begin{aligned}\Phi_\pm(r,c)=&-(\sigma(r)-c)r^{\frac32}\text{p.v.}\int_r^{+\infty}\frac{\int_{r_c}^u s^{\frac32}g(s)w(s)ds}{(\sigma(u)-c)^2u^3}du\\&\pm i\pi(\sigma(r)-c)r^{\frac32}\frac{g(r_c)w(r_c)}{r_c^{\frac32}\sigma'(r_c)^2}-\frac{(\sigma(r)-c)r^{\frac32}}{r_c^3\sigma'(r_c)}I_1[w](r_c)\\&\times\left(\frac{1}{\sigma(r)-c}+\frac1c-\text{p.v.}\int_r^{+\infty}\frac{u^3\sigma'(u)-r_c^3\sigma'(r_c)}{(\sigma(u)-c)^2u^3}du\pm i\pi\frac{(r_c^3\sigma')'(r_c)}{r_c^3\sigma'(r_c)^2}\right),\end{aligned}$$

here $I_1[w](r_c)=\int_0^{r_c}s^{\frac32}g(s)w(s)ds$. Using this, we find that

$$\Phi_-(r,c)-\Phi_+(r,c)=\mu(c)f(r,c),$$

where

$$f(r,c) = -2i\pi \frac{(\sigma(r)-c)r^{3/2}}{r_c^{3/2}\sigma'(r_c)^2} \text{ for } r \in (0,r_c), \quad f(r,c) = 0 \text{ for } r > r_c,$$

$$\mu(c) = \mu[w](c) = g(r_c)w(r_c) + I_1[w](r_c)\frac{g(r_c)^2}{r_c^{3/2}\sigma'(r_c)},$$

here we used the fact that $\partial_r(r^3\partial_r\sigma) = -r^3 g$. Moreover, the operator $w \mapsto \mu[w]$ satisfies $\mu[e^{it\widetilde{B}_1}w](c) = e^{itc}\mu[w](c)$. Now the operator $w \mapsto \mu[w] \circ \sigma$ plays a role of wave operator. The operator T can be defined as $Tw(r) = g(r)^{-1}\mu[w](\sigma(r))$. See also [31], where we construct a wave operator by a similar procedure to handle the linearized Navier-Stokes equations around the Kolmogorov flow.

5. Resolvent estimate of $\widetilde{\mathcal{H}}_1$

As $\overline{\widetilde{\mathcal{H}}_{-1}w} = \widetilde{\mathcal{H}}_1\overline{w}$, it is enough to prove the following resolvent estimate for $\widetilde{\mathcal{H}}_1 = \widetilde{\mathcal{H}}_{\alpha,1,\lambda}$.

Theorem 5.1. – *For any $\lambda \in \mathbb{R}$ and $w \in \{r^{\frac{3}{2}}g(r)\}^\perp \cap D_1$, we have*

$$\|\widetilde{\mathcal{H}}_{\alpha,1,\lambda}w\| \gtrsim |\beta_1|^{\frac{1}{3}}\|w\|.$$

Moveover, there exist $\lambda \in \mathbb{R}$ and $v \in \{r^{\frac{3}{2}}g(r)\}^\perp \cap D_1$ so that

$$\|\widetilde{\mathcal{H}}_{\alpha,1,\lambda}v\| \lesssim |\beta_1|^{\frac{1}{3}}\|v\|.$$

5.1. Reduction to the model operator $\mathcal{L}_1$

Let us introduce the operator T defined by

$$Tw(r) = w(r) + \frac{I_1[w](r)g(r)}{\sigma'(r)r^{\frac{3}{2}}}, \tag{32}$$

where

$$I_1[w](r) = \int_0^r s^{\frac{3}{2}}g(s)w(s)ds. \tag{33}$$

It is easy to check that T is a bounded linear operator in $L^2(\mathbb{R}_+,dr)$. The adjoint operator T^* is also a bounded linear operator in $L^2(\mathbb{R}_+,dr)$ given by

$$T^*w(r) = w(r) + r^{\frac{3}{2}}g(r)\int_r^{+\infty}\frac{w(s)g(s)}{s^{\frac{3}{2}}\sigma'(s)}ds. \tag{34}$$

Lemma 5.2. – *It holds that*

1. $\|Tw\|^2 = \|w\|^2 - \frac{\langle w, r^{\frac{3}{2}}g\rangle^2}{\|r^{\frac{3}{2}}g\|^2}$;
2. $T^*T = P$, *where P is the projection to $\{r^{\frac{3}{2}}g(r)\}^\perp \cap L^2(\mathbb{R}_+,dr)$;*
3. $TT^* = I_{L^2(\mathbb{R}_+,dr)}$.

Proof. – The first one is equivalent to the second one. Thanks to

$$-(r^3\sigma'(r))' = r^3 g(r)^2 \quad \text{and} \quad I_1[s^{\frac{3}{2}}g(s)](r) = \int_0^r s^3 g(s)^2 ds = -r^3\sigma'(r),$$

we find that

$$T(r^{\frac{3}{2}}g)(r) = r^{\frac{3}{2}}g(r) + \frac{I_1[s^{\frac{3}{2}}g(s)](r)g(r)}{\sigma'(r)r^{\frac{3}{2}}} = 0.$$

Thus, it suffices to check that for any $w \in \{r^{\frac{3}{2}}g(r)\}^{\perp} \cap L^2(\mathbb{R}_+, dr)$, $u \in C_0^{\infty}(\mathbb{R}_+)$,

$$\langle T^*Tw, u\rangle = \langle w, u\rangle,$$

which is equivalent to verifying that

$$\Big\langle \frac{I_1[w](r)g(r)}{\sigma'(r)r^{\frac{3}{2}}}, u\Big\rangle + \Big\langle w, \frac{I_1[u](r)g(r)}{\sigma'(r)r^{\frac{3}{2}}}\Big\rangle + \Big\langle \frac{I_1[w](r)g(r)}{\sigma'(r)r^{\frac{3}{2}}}, \frac{I_1[u](r)g(r)}{\sigma'(r)r^{\frac{3}{2}}}\Big\rangle = 0.$$

Using the facts that for $w \in \big\{r^{\frac{3}{2}}g(r)\big\}^{\perp}$,

$$I_1[w](0) = \lim_{r\to+\infty} I_1[w](r) = 0,$$

and $(r^3\sigma'(r))' = -r^3 g(r)^2$, we get by integration by parts that

$$\begin{aligned}
\Big\langle \frac{I_1[w](r)g(r)}{\sigma'(r)r^{\frac{3}{2}}}, u\Big\rangle &= \int_0^{+\infty} \frac{I_1[w]}{\sigma'(r)r^3} dI_1[\overline{u}] \\
&= -\int_0^{+\infty} I_1[\overline{u}]\Big(\frac{I_1[w]}{\sigma'(r)r^3}\Big)' dr \\
&= -\int_0^{+\infty} I_1[\overline{u}]\frac{r^{\frac{3}{2}}g(r)w(r)\sigma'(r)r^3 - I_1[w](\sigma'(r)r^3)'}{(\sigma'(r)r^3)^2} dr \\
&= -\int_0^{+\infty} w(r)\frac{I_1[\overline{u}]g(r)}{\sigma'(r)r^{\frac{3}{2}}} dr - \int_0^{+\infty} I_1[w]I_1[\overline{u}]\frac{g^2(r)}{(\sigma'(r)r^{\frac{3}{2}})^2} dr.
\end{aligned}$$

This shows that $T^*T = P$.

On the other hand, we have

$$|T^*w|^2 - |w|^2 = -\partial_r(r^3\sigma'(r)|f_1|^2), \quad f_1(r) = \int_r^{+\infty} \frac{w(s)g(s)}{s^{\frac{3}{2}}\sigma'(s)} ds,$$

which gives $\|T^*w\|^2 = \|w\|^2$, thus $TT^* = I$. □

We have the following important relationship between T and $\widetilde{B}_1$.

LEMMA 5.3. – *It holds that*

$$T\widetilde{B}_1 = \sigma(r)T.$$

Proof. – Direct calculation gives

$$\begin{aligned}
T\widetilde{B}_1 w &= \widetilde{B}_1 w + \frac{I_1[\widetilde{B}_1 w]g(r)}{\sigma'(r)r^{\frac{3}{2}}} \\
&= \sigma(r)w - g\widetilde{\mathscr{K}}_1[gw] + \frac{I_1[\sigma w]g(r)}{\sigma'(r)r^{\frac{3}{2}}} - \frac{I_1[g\widetilde{\mathscr{K}}_1[gw]]g(r)}{\sigma'(r)r^{\frac{3}{2}}}.
\end{aligned}$$

Thus, it suffices to show that

$$I_1[\sigma w] = \sigma'(r) r^{\frac{3}{2}} \widetilde{\mathscr{K}}_1[gw] + I_1[g\widetilde{\mathscr{K}}_1[gw]] + \sigma(r) I_1[w]. \tag{35}$$

Direct calculation shows that

$$\begin{aligned}
\sigma'(r) r^{\frac{3}{2}} \widetilde{\mathscr{K}}_1[gw] &= r^{\frac{1}{2}}(e^{-\frac{r^2}{4}} - \sigma(r)) \int_0^{+\infty} \min(\frac{r}{s}, \frac{s}{r})(rs)^{\frac{1}{2}} g(s)w(s)ds \\
&= r^{\frac{1}{2}}(e^{-\frac{r^2}{4}} - \sigma(r))[\int_0^r r^{-\frac{1}{2}} s^{\frac{3}{2}} g(s)w(s)ds + \int_r^{+\infty} r^{\frac{3}{2}} s^{-\frac{1}{2}} g(s)w(s)ds] \\
&= (e^{-\frac{r^2}{4}} - \sigma(r)) I_1[w] + r^2(e^{-\frac{r^2}{4}} - \sigma(r)) \int_r^{+\infty} s^{-\frac{1}{2}} g(s)w(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
I_1[g\widetilde{\mathscr{K}}_1[gw]](r) &= \int_0^r s^{\frac{3}{2}} g^2(s) \widetilde{\mathscr{K}}_1[gw](s)ds \\
&= \int_0^r s^{\frac{3}{2}} g^2(s)ds \frac{1}{2} \int_0^{+\infty} \min(\frac{t}{s}, \frac{s}{t})(ts)^{\frac{1}{2}} g(t)w(t)dt \\
&= \frac{1}{2}\int_0^r se^{-\frac{s^2}{4}} ds \int_0^s t^{\frac{3}{2}} g(t)w(t)dt + \frac{1}{2}\int_0^r s^3 e^{-\frac{s^2}{4}} ds \int_s^{+\infty} t^{-\frac{1}{2}} g(t)w(t)dt \\
&= \frac{1}{2}\int_0^r t^{\frac{3}{2}} g(t)w(t)dt \int_t^r se^{-\frac{s^2}{4}} ds + \frac{1}{2}\int_0^r s^3 e^{-\frac{s^2}{4}} ds \int_s^r t^{-\frac{1}{2}} g(t)w(t)dt \\
&\quad + \frac{1}{2}\int_0^r s^3 e^{-\frac{s^2}{4}} ds \int_r^{+\infty} t^{-\frac{1}{2}} g(t)w(t)dt \\
&= \int_0^r t^{\frac{3}{2}} g(t)w(t)(e^{-\frac{t^2}{4}} - e^{-\frac{r^2}{4}})dt + \frac{1}{2}\int_0^r t^{-\frac{1}{2}} g(t)w(t)dt \int_0^t s^3 e^{-\frac{s^2}{4}} ds \\
&\quad + [4(1 - e^{-\frac{r^2}{4}}) - r^2 e^{-\frac{r^2}{4}}] \int_r^{+\infty} t^{-\frac{1}{2}} g(t)w(t)dt \\
&= -e^{-\frac{r^2}{4}} I_1[w] + I_1[\sigma w] + r^2(\sigma(r) - e^{-\frac{r^2}{4}}) \int_r^{+\infty} t^{-\frac{1}{2}} g(t)w(t)dt,
\end{aligned}$$

which give (35). □

LEMMA 5.4. – *It holds that*

$$[T, \widetilde{A}_1]w = T\widetilde{A}_1 w - \widetilde{A}_1 T w = f(r) T w,$$

where

$$f(r) = 2\frac{g(r)^4}{(\sigma'(r)^2)} + \frac{g(r)^2}{\sigma'(r)}\big(\frac{6}{r} - r\big) \geq 0.$$

Proof. – First of all, we have

$$[T, \widetilde{A}_1]w = \frac{I_1[(-\partial_r^2 + \frac{3}{4}\frac{1}{r^2} + \frac{r^2}{16})w]g(r)}{\sigma'(r) r^{\frac{3}{2}}} - \Big(-\partial_r^2 + \frac{3}{4}\frac{1}{r^2} + \frac{r^2}{16}\Big)\Big(\frac{I_1[w]g(r)}{\sigma'(r) r^{\frac{3}{2}}}\Big).$$

Using the facts that

$$I_1[-\partial_r^2 w] = -r^{\frac{3}{2}} g(r) w'(r) + (r^{\frac{3}{2}} g(r))' w(r) - \int_0^r (s^{\frac{3}{2}} g(s))'' w(s) ds,$$

$$(-\partial_r^2 + \frac{3}{4}\frac{1}{r^2} + \frac{r^2}{16}) r^{\frac{3}{2}} g(r) = r^{\frac{3}{2}} g(r),$$

we deduce that

$$I_1\big[(-\partial_r^2 + \frac{3}{4}\frac{1}{r^2} + \frac{r^2}{16}) w\big] = -r^{\frac{3}{2}} g(r) w'(r) + (r^{\frac{3}{2}} g(r))' w(r) + I_1[w],$$

Direct calculation gives

$$\partial_r^2\Big(\frac{I_1[w] g(r)}{\sigma'(r) r^{\frac{3}{2}}}\Big) = w'(r)\frac{g^2(r)}{\sigma'(r)} + w(r)\big(\frac{g^2(r)}{\sigma'(r)}\big)' + r^{\frac{3}{2}} g(r) w(r)\big(\frac{g(r) r^{\frac{3}{2}}}{\sigma'(r) r^3}\big)' + I_1[w]\big(\frac{g(r) r^{\frac{3}{2}}}{\sigma'(r) r^3}\big)''.$$

Let $F = r^{\frac{3}{2}} g(r)$ and $G(r) = \sigma'(r) r^3$. We have

$$F' = (\frac{3}{2}\cdot\frac{1}{r} - \frac{r}{4}) F, \quad F'' = (\frac{3}{4}\cdot\frac{1}{r^2} - 1 + \frac{r^2}{16}) F, \quad G' = -F^2.$$

Summing up, we obtain

$$\begin{aligned}[T, \widetilde{A}_1] w &= \big[\frac{F'F}{G} + (\frac{F^2}{G})' + F(\frac{F}{G})'\big] w \\ &\quad + \big(1 - \frac{3}{4}\cdot\frac{1}{r^2} - \frac{r^2}{16}\big)\frac{I_1[w] F}{G} + I_1[w](\frac{F}{G})''.\end{aligned}$$

On the other hand, we have

$$\big(\frac{F}{G}\big)'' = \big(\frac{3}{4}\cdot\frac{1}{r^2} - 1 + \frac{r^2}{16}\big)\frac{F}{G} + \frac{F}{G}\big(4\frac{F'F}{G} + 2\frac{(G')^2}{G^2}\big),$$

$$\frac{F'F}{G} + (\frac{F^2}{G})' + F\big(\frac{F}{G}\big)' = 4\frac{F'F}{G} + 2\frac{(G')^2}{G^2},$$

Then we infer that

$$\begin{aligned}[T, \widetilde{A}_1] w &= \big(4\frac{F'F}{G} + 2\frac{(G')^2}{G^2}\big)(w + \frac{I_1[w] F}{G}) \\ &= \big(2\frac{g^4}{(\sigma')^2} + \frac{g^2}{\sigma'(r)}(\frac{6}{r} - r)\big) T w = f(r) T w.\end{aligned}$$

It remains to prove that $f(r) \geq 0$. We have

$$2\frac{g^4}{(\sigma')^2} + \frac{g^2}{\sigma'(r)}(\frac{6}{r} - r) = \frac{r^6 + r^2(6 - r^2)(r^2 + 4 - 4e^{\frac{r^2}{4}})}{32(\frac{r^2}{4} + 1 - e^{\frac{r^2}{4}})^2},$$

while by Taylor expansion, we have

$$\begin{aligned}r^6 + r^2(6 - r^2)(r^2 + 4 - 4e^{\frac{r^2}{4}}) &= 2r^4 + 24r^2 - 24r^2 e^{\frac{r^2}{4}} + 4r^4 e^{\frac{r^2}{4}} \\ &= 2r^2\Big[r^2 + 2r^2\sum_{n=0}^{+\infty}\frac{1}{n!}(\frac{r^2}{4})^n - 12\sum_{n=1}^{+\infty}\frac{1}{n!}(\frac{r^2}{4})^n\Big] \\ &= 2r^2\Big[8\sum_{n=2}^{+\infty}\frac{1}{(n-1)!}(\frac{r^2}{4})^n - 12\sum_{n=2}^{+\infty}\frac{1}{n!}(\frac{r^2}{4})^n\Big] \geq 0.\end{aligned}$$

This completes the proof. □

It follows from Lemma 5.3 and Lemma 5.2 that for $w \in \{r^{\frac{3}{2}}g(r)\}^{\perp} \cap D_1$,

$$\begin{aligned} T\widetilde{\mathcal{H}}_1 w &= T\widetilde{A}_1 w + i\beta_1 T\widetilde{B}_1 w - i\lambda T w \\ &= T\widetilde{A}_1 T^* T w + i\beta_1 \sigma(r) T w - i\lambda T w. \end{aligned}$$

Lemma 5.2 ensures that $T : \{r^{\frac{3}{2}}g(r)\}^{\perp} \to L^2(\mathbb{R}_+, dr)$ is invertible and $T^{-1} = T^*$. Let $w = T^{-1}u$. We infer from Lemma 5.4 that

$$\begin{aligned} T\widetilde{\mathcal{H}}_1 T^{-1} u &= T\widetilde{A}_1 T^{-1} u + i\beta_1 \sigma(r) u - i\lambda u \\ &= \widetilde{A}_1 u + f(r) u + i\beta_1 \sigma(r) u - i\lambda u = \mathcal{L}_1 u, \end{aligned} \tag{36}$$

where

$$f(r) = 2\frac{g(r)^4}{(\sigma(r)')^2} + \frac{g(r)^2}{\sigma(r)'}\big(\frac{6}{r} - r\big). \tag{37}$$

So, the operator T plays a role of wave operator. Let

$$D(\mathcal{L}_1) = \big\{w \in H^2_{\mathrm{loc}}(\mathbb{R}_+, dr) \cap L^2(\mathbb{R}_+, dr) : \mathcal{L}_1 w \in L^2(\mathbb{R}_+, dr)\big\}.$$

Then $u \in D(\mathcal{L}_1) \Leftrightarrow T^*u \in D(\mathcal{H}_1) \cap \big\{r^{\frac{3}{2}}g(r)\big\}^{\perp}$, and $D(\mathcal{L}_1) = D(\widetilde{\mathcal{H}}_3) = D$.

Moreover, we have

$$\langle \widetilde{\mathcal{H}}_1 w, w\rangle = \langle \widetilde{\mathcal{H}}_1 T^{-1} u, T^* u\rangle = \langle \mathcal{L}_1 u, u\rangle.$$

On the other hand, $\|w\| = \|Tw\| = \|u\|$ for any $w \in \{r^{\frac{3}{2}}g(r)\}^{\perp} \cap D_1$. Thus, we reduce the resolvent estimate of $\widetilde{\mathcal{H}}_1$ to one of the model operator $\mathcal{L}_1$.

5.2. Coercive estimates

LEMMA 5.5. – *The operator $\widetilde{A}_1$ can be represented as*

$$\big(\widetilde{A}_1 - \frac{1}{2}\big)w = -r^{-\frac{3}{2}}g^{-1}\partial_r\big[r^3 g^2 \partial_r(r^{-\frac{3}{2}}g^{-1}w)\big]. \tag{38}$$

In particular, we have

$$\widetilde{A}_k \geq \frac{1}{2} \quad \textit{for} \quad |k| \geq 1. \tag{39}$$

Proof. – Let $F(r) = r^{\frac{3}{2}}g(r)$. Then we have

$$\begin{aligned} -r^{-\frac{3}{2}}g^{-1}\partial_r\big[r^3 g^2 \partial_r(r^{-\frac{3}{2}}g^{-1}w)\big] &= -F^{-1}\partial_r\big[F^2\partial_r(F^{-1}w)\big] = \big(-\partial_r^2 + \frac{F''}{F}\big)w \\ &= (-\partial_r^2 + \frac{3}{4}\cdot\frac{1}{r^2} - 1 + \frac{r^2}{16})w = \big(\widetilde{A}_1 - \frac{1}{2}\big)w, \end{aligned}$$

here we used $F'' = (\frac{3}{4}\cdot\frac{1}{r^2} - 1 + \frac{r^2}{16})F$.

Then for any $w \in D$, we have

$$\big\langle(\widetilde{A}_1 - \frac{1}{2})w, w\big\rangle = -\big\langle F^{-1}\partial_r\big[F^2\partial_r(F^{-1}w)\big], w\big\rangle = \big\|F\partial_r(F^{-1}w)\big\|^2 \geq 0.$$

This shows that $\widetilde{A}_k \geq \widetilde{A}_1 \geq \frac{1}{2}$. □

LEMMA 5.6. – *It holds that*

$$\widetilde{A}_1 + f(r) \gtrsim \frac{1}{r^2} + r^2. \tag{40}$$

Proof. – By the proof of Lemma 5.4, we know that

$$f(r) = r^2 \frac{\left\{\sum\limits_{n=2}^{+\infty}(\frac{2}{(n-1)!} - \frac{3}{n!})(\frac{r^2}{4})^n\right\}}{4(\frac{r^2}{4}+1-e^{r^2/4})^2} \geq r^2 \frac{\sum\limits_{n=2}^{+\infty}\frac{1}{n!}(\frac{r^2}{4})^n}{4(\frac{r^2}{4}+1-e^{r^2/4})^2} \geq \frac{r^2}{4(e^{\frac{r^2}{4}}-1-\frac{r^2}{4})}.$$

Let $h(r) = \frac{3}{4}\cdot\frac{1}{r^2} + \frac{r^2}{16} - \frac{1}{2} + \frac{r^2}{4(e^{\frac{r^2}{4}}-1-\frac{r^2}{4})}$. Then there exists $\varepsilon_0 \in (0,1)$ so that $h(r) \gtrsim \frac{1}{r^2}$ for $r < \varepsilon_0$ and $h(r) \gtrsim r^2$ for $r > \frac{1}{\varepsilon_0}$, and $h(r)$ can attain its minimum. Thus, if $h(r) > 0$, $h(r)$ has a positive lower bound. For this, let $u = \frac{r^2}{4}$. Then by Taylor's expansion, we get

$$\begin{aligned} h(r) &= \frac{3}{16}\cdot\frac{1}{u} + \frac{1}{4}u - \frac{1}{2} + \frac{u}{e^u-1-u} \\ &= \frac{\frac{3}{16}\sum\limits_{n=2}^{+\infty}\frac{1}{n!}u^n + \frac{1}{4}u^2\sum\limits_{n=2}^{+\infty}\frac{1}{n!}u^n - \frac{1}{2}u\sum\limits_{n=2}^{+\infty}\frac{1}{n!}u^n + u^2}{u(e^u-1-u)} = \frac{\sum\limits_{n=2}^{+\infty}a_n u^n}{u(e^u-1-u)}, \end{aligned}$$

where

$$\begin{aligned} &a_2 = \frac{35}{32}, \quad a_3 = -\frac{7}{32}, \quad a_4 = \frac{19}{384}, \quad 2\sqrt{a_2 a_4} > |a_3|, \\ &a_n = \frac{1}{n!}(\frac{3}{16} + \frac{n(n-1)}{4} - \frac{n}{2}) > 0 (n \geq 5). \end{aligned}$$

Hence, there exists $c_0 > 0$ such that $h(r) \geq c_0$. So, there exists $C > 0$ such that for any $r \in [\varepsilon_0, \frac{1}{\varepsilon_0}]$, we have $h(r) \geq C(\frac{1}{r^2} + r^2)$.

Summing up, we conclude that

$$\widetilde{A}_1 + f(r) \geq \frac{3}{4r^2} + \frac{r^2}{16} - \frac{1}{2} + f(r) \geq h(r) \gtrsim \frac{1}{r^2} + r^2.$$

The proof is completed. □

5.3. Resolvent estimate of $\mathcal{L}_1$

In this subsection, we prove Theorem 5.1. It suffices to show that for any $u = Tw, w \in \{r^{\frac{3}{2}}g(r)\}^{\perp} \cap D_1$,

$$\|\mathcal{L}_1 u\| \gtrsim |\beta_1|^{\frac{1}{3}}\|u\|. \tag{41}$$

The proof is split into three cases.

Case 1. $\nu_1 \geq 1$. – Be Lemma 5.6, we get

$$\begin{aligned}|\langle \mathcal{L}_1 u, u\rangle| &\sim \langle (\tilde{A}_1 + f)u, u\rangle + |\beta_1| \langle (\nu_1 - \sigma(r))u, u\rangle \\ &\gtrsim \langle (\frac{1}{r^2} + r^2)u, u\rangle + |\beta_1| \langle (1 - \sigma(r))u, u\rangle.\end{aligned}$$

Using the fact that

$$1 - \sigma(r) = 1 - \frac{1 - e^{-\frac{r^2}{4}}}{r^2/4} \sim r^2 (r \to 0), \quad \lim_{r\to\infty} 1 - \sigma(r) = 1,$$

we deduce that

$$\int_0^1 \big[\frac{1}{r^2} + |\beta_1|(1 - \sigma(r))\big]|u|^2 dr \gtrsim \int_0^1 (\frac{1}{r^2} + |\beta_1| r^2)|u|^2 dr \gtrsim \int_0^1 |\beta_1|^{\frac{1}{2}} |u|^2 dr,$$

$$\int_1^{+\infty} \big[\frac{1}{r^2} + r^2 + |\beta_1|(1 - \sigma(r))\big]|u|^2 dr \gtrsim \int_1^{+\infty} (1 + |\beta_1|)|u|^2 dr \gtrsim \int_1^{+\infty} |\beta_1|^{\frac{1}{2}} |u|^2 dr,$$

which show that for $\nu_1 \geq 1$,

$$|\langle \mathcal{L}_1 u, u\rangle| \gtrsim |\beta_1|^{\frac{1}{2}} \|u\|^2. \tag{42}$$

Case 2. $\nu_1 \leq 0$. – In this case, we have by Lemma 5.6 that

$$\begin{aligned}|\langle \mathcal{L}_1 u, u\rangle| &\sim \langle (\tilde{A}_1 + f)u, u\rangle + |\beta_1| \langle (\sigma(r) - \nu_1)u, u\rangle \\ &\gtrsim \langle (\frac{1}{r^2} + r^2)u, u\rangle + |\beta_1| \langle \sigma(r)u, u\rangle.\end{aligned}$$

Thanks to $\lim\limits_{r\to 0} \sigma(r) = 1$ and $\sigma(r) \sim \frac{1}{r^2}(r \to \infty)$, we infer that

$$\int_0^1 \big[\frac{1}{r^2} + r^2 + |\beta_1|\sigma(r)\big]|u|^2 dr \gtrsim \int_0^1 (1 + |\beta_1|)|u|^2 dr \gtrsim \int_0^1 |\beta_1|^{\frac{1}{2}} |u|^2 dr,$$

$$\int_1^{+\infty} (r^2 + |\beta_1|\sigma(r))|u|^2 dr \gtrsim \int_1^{+\infty} (r^2 + \frac{1}{r^2}|\beta_1|)|u|^2 dr \gtrsim \int_1^{+\infty} |\beta_1|^{\frac{1}{2}} |u|^2 dr,$$

which shows that for $\nu_1 \leq 0$,

$$|\langle \mathcal{L}_1 u, u\rangle| \gtrsim |\beta_1|^{\frac{1}{2}} \|u\|^2. \tag{43}$$

Case 3. $0 < \nu_1 < 1$. – Let $\nu_1 = \sigma(r_1)$ for some $r_1 > 0$. We split this case into two subcases:

$$|\beta_1| \leq \max\big(\frac{1}{r_1^4}, r_1^6\big) \quad \text{and} \quad |\beta_1| \geq \max\big(\frac{1}{r_1^4}, r_1^6\big).$$

LEMMA 5.7. – *If* $|\beta_1| \leq \max\big(\frac{1}{r_1^4}, r_1^6\big)$*, then we have*

$$\|\mathcal{L}_1 u\| \gtrsim |\beta_1|^{\frac{1}{3}} \|u\|.$$

Proof. – If $|\beta_1| \leq 1$, then $\frac{1}{r^2} + r^2 \geq 1 \geq |\beta_1|^{\frac{1}{3}}$. Lemma 5.6 gives

$$|\langle \mathcal{L}_1 u, u\rangle| \gtrsim \big\langle (r^2 + \frac{1}{r^2})u, u\big\rangle \geq |\beta_1|^{\frac{1}{3}} \|u\|^2.$$

If $1 \le |\beta_1| \le \max(\frac{1}{r_1^4}, r_1^6)$, we only need to check the following cases

$$\begin{aligned}
&r_1 \le 1, && 1 \le |\beta_1| \le \frac{1}{r_1^4} \Longrightarrow \|\mathcal{L}_1 u\| \gtrsim |\beta_1|^{\frac12}\|u\|,\\
&r_1 \ge 1, && 1 \le |\beta_1| \le r_1^4 \Longrightarrow \|\mathcal{L}_1 u\| \gtrsim |\beta_1|^{\frac12}\|u\|,\\
&r_1 \ge 1, && r_1^4 \le |\beta_1| \le r_1^6 \Longrightarrow \|\mathcal{L}_1 u\| \gtrsim |\beta_1|^{\frac13}\|u\|.
\end{aligned}$$

By Lemma 5.6 again, we have

$$|\langle \mathcal{L}_1 u, u\rangle| \gtrsim \big\langle (r^2 + \frac{1}{r^2})u, u\big\rangle + |\beta_1||\langle(\nu_1 - \sigma(r))u, u\rangle|,$$

which along with Lemma A.2 gives our results. □

LEMMA 5.8. – *If* $|\beta_1| \ge \max(\frac{1}{r_1^4}, r_1^6)$*, then we have*

$$\|\mathcal{L}_1 u\| \gtrsim |\beta_1|^{\frac13}\|u\|.$$

Proof. – Let $\delta > 0$ be so that $\delta^3|\beta_1|\min(r_1, r_1^{-3}) = 1$. Thanks to $|\beta_1| \ge \max(\frac{1}{r_1^4}, r_1^6)$, we get

$$|\beta_1|^{-\frac12} \le \min\big(r_1^2, \frac{1}{r_1^3}\big).$$

Thus, we have

$$\delta^3|\beta_1|^{\frac12} \le r_1 \quad \text{for } r_1 \le 1, \quad \delta^3|\beta_1|^{\frac12} \le 1 \quad \text{for } r_1 \ge 1,$$

which in particular give $\delta^2|\beta_1|^{\frac13} \le 1$. Also we have $0 < \delta \le \min(r_1, \frac{1}{r_1})$. Hence, it suffices to show that

$$\|u\| \lesssim \delta^2\|\mathcal{L}_1 u\|. \tag{44}$$

Let us choose $r_- \in (r_1 - \delta, r_1)$ and $r_+ \in (r_1, r_1 + \delta)$ so that

$$|u'(r_-)|^2 + |u'(r_+)|^2 \le \frac{\|u'\|^2}{\delta}. \tag{45}$$

We get by integration by parts that

$$\begin{aligned}
&\mathrm{Re}\big\langle \mathcal{L}_1 u, i\,\mathrm{sgn}(\beta_1)(\chi_{(0,r_-)} - \chi_{(r_+,+\infty)})u\big\rangle\\
&\quad= \mathrm{Re}\langle -\partial_r^2 u + i\beta_1(\sigma - \nu_1)u, i\,\mathrm{sgn}(\beta_1)(\chi_{(0,r_-)} - \chi_{(r_+,+\infty)})u\rangle\\
&\quad= \mathrm{Re}\left(\int_0^{r_-}(-i\,\mathrm{sgn}(\beta_1)|\partial_r u|^2 + |\beta_1|(\sigma - \nu_1)|u|^2)dr + i\,\mathrm{sgn}(\beta_1)(u'\overline{u})(r_-)\right)\\
&\qquad+ \mathrm{Re}\left(\int_{r_+}^{+\infty}(i\,\mathrm{sgn}(\beta_1)|\partial_r u|^2 + |\beta_1|(\nu_1 - \sigma)|u|^2)dr + i\,\mathrm{sgn}(\beta_1)(u'\overline{u})(r_+)\right)\\
&\quad\ge \int_0^{r_-}|\beta_1|(\sigma - \nu_1)|u|^2dr + \int_{r_+}^{+\infty}|\beta_1|(\nu_1 - \sigma)|u|^2dr - |(u'\overline{u})(r_-)| - |(u'\overline{u})(r_+)|.
\end{aligned}$$

Due to $0 < \delta \le \min(r_1, \frac{1}{r_1})$, $0 < r_1 - \delta < r_1 + \delta \le 2r_1$. Then we get by Lemma A.1 that

$$\sigma(r) - \nu_1 \ge \sigma(r_1 - \delta) - \sigma(r_1) \gtrsim \delta|\sigma'(r_1)| \quad 0 < r < r_1 - \delta,$$
$$\nu_1 - \sigma(r) \ge \sigma(r_1) - \sigma(r_1 + \delta) \gtrsim \delta|\sigma'(r_1)| \quad r > r_1 + \delta,$$

from which and (45), we infer that

$$\begin{aligned}
&\mathrm{Re}\langle \mathscr{L}_1 u, i\,\mathrm{sgn}(\beta_1)(\chi_{(0,r_-)} - \chi_{(r_+,+\infty)})u\rangle \\
&\quad\ge \int_0^{r_-} |\beta_1|(\sigma - \nu_1)|u|^2 dr + \int_{r_+}^{+\infty} |\beta_1|(\nu_1 - \sigma)|u|^2 dr - |(u'\overline{u})(r_-)| - |(u'\overline{u})(r_+)| \\
&\quad\ge C^{-1}|\beta_1\delta\sigma'(r_1)|\|u\|^2_{L^2(\mathbb{R}_+\setminus(r_1-\delta,r_1+\delta))} - \frac{2}{\delta^{\frac{1}{2}}}\|u'\|_{L^2}\|u\|_{L^\infty}.
\end{aligned}$$

Thanks to $\sigma'(r) = \frac{2}{r}(e^{-\frac{r^2}{4}} - \frac{1-e^{-\frac{r^2}{4}}}{r^2/4})$, we have $|\sigma'(r)| \sim \frac{1}{r^3}(r \to \infty)$ and $|\sigma'(r)| \sim r(r \to 0)$. Thus, $|\sigma'(r)| \sim \min(r, \frac{1}{r^3})$. Recall that $\delta^3|\beta_1|\min(r_1, r_1^{-3}) = 1$. Then $|\beta_1\delta^3\sigma'(r_1)| \sim 1$. Thus, we obtain

$$\|u\|^2_{L^2(\mathbb{R}_+\setminus(r_1-\delta,r_1+\delta))} \lesssim \delta^2\|u\|\|\mathscr{L}_1 u\|_{L^2} + \delta^{\frac{3}{2}}\|u'\|\|u\|_{L^\infty}.$$

On the other hand, it is obvious that

$$\|u'\|^2 \le \|u\|\|\mathscr{L}_1 u\|_{L^2}, \qquad \|u\|_{L^\infty} \le \|u\|^{\frac{1}{2}}\|u'\|^{\frac{1}{2}}.$$

Consequently, we deduce that

$$\begin{aligned}
\|u\|^2 &= \|u\|^2_{L^2(\mathbb{R}_+\setminus(r_1-\delta,r_1+\delta))} + \|u\|^2_{L^2(r_1-\delta,r_1+\delta)} \\
&\lesssim \delta^2\|u\|\|\mathscr{L}_1 u\| + \delta^{\frac{3}{2}}\|u'\|\|u\|_{L^\infty} + \delta\|u\|^2_{L^\infty} \\
&\lesssim \delta^2\|u\|\|\mathscr{L}_1 u\| + \delta^2\|u'\|^2 + \delta\|u\|^2_{L^\infty} \\
&\le \delta^2\|u\|\|\mathscr{L}_1 u\| + \delta^2\|u\|\|\mathscr{L}_1 u\| + \delta\|u'\|\|u\| \\
&\lesssim \|u\|(\delta^2\|\mathscr{L}_1 u\|) + \|u\|^{\frac{3}{2}}(\delta^2\|\mathscr{L}_1 u\|)^{\frac{1}{2}},
\end{aligned}$$

which implies (44). □

5.4. Sharpness of pseudospectral bound

Finally, let us prove the sharpness of the pseudospectral bound of $\widetilde{\mathscr{H}}_1$. That is, there exist $\lambda \in \mathbb{R}$ and $v \in \{r^{\frac{3}{2}}g(r)\}^\perp \cap D_1$, such that

$$\|\widetilde{\mathscr{H}}_1 v\| \le C|\beta_1|^{\frac{1}{3}}\|v\|. \tag{46}$$

Take $\lambda \in \mathbb{R}$ so that $|\beta_1| = r_1^6 \ge 1$. We take $u(r) = \eta(r_1(r - r_1))$, where $\eta(r) = r^2(r-1)^2$ for $0 < r < 1$, $\eta(r) = 0$ for $r(r-1) \ge 0$. Then we have

$$\|u\| = r_1^{-\frac{1}{2}}\|\eta\|, \quad \|\partial_r^2 u\| = r_1^{\frac{3}{2}}\|\partial_r^2\eta\| \le Cr_1^2\|u\|.$$

By Lemma A.1, we have

$$|\beta_1(\sigma(r) - \sigma(r_1))| \sim |\beta_1\sigma'(r_1)||r - r_1| \le C\frac{|\beta_1|}{r_1^4} \le Cr_1^2, \quad \text{for } |r - r_1| \le \frac{1}{r_1},$$

and we also have that for $0 < r - r_1 < \frac{1}{r_1}$,

$$\big|(\frac{3}{4r^2} + \frac{r^2}{16} - \frac{1}{2} + f)u\big| \leq C r_1^2 |u|.$$

Thus, we can conclude that

$$\|\mathcal{L}_1 u\| \leq C r_1^2 \|u\|,$$

which along with Lemma 5.2 gives

$$\|\widetilde{\mathcal{H}}_1 T^* u\| = \|T\widetilde{\mathcal{H}}_1 T^* u\| = \|\mathcal{L}_1 u\| \leq C|\beta_1|^{\frac{1}{3}}\|u\| = C|\beta_1|^{\frac{1}{3}}\|T^* u\|.$$

This gives (46) by taking $v = T^* u$.

6. Resolvent estimate of $\widetilde{\mathcal{H}}_k$, $|k| \geq 2$

In this section, we will prove the following resolvent estimate for $\widetilde{\mathcal{H}}_k = \widetilde{\mathcal{H}}_{\alpha,k,\lambda}$, $|k| \geq 2$.

THEOREM 6.1. – *Let $|k| \geq 2$. For any $\lambda \in \mathbb{R}$ and $w \in D$, we have*

$$\|\widetilde{\mathcal{H}}_{\alpha,k,\lambda} w\| \gtrsim |\beta_k|^{\frac{1}{3}}\|w\|. \tag{47}$$

In this part, we don't need to use the wave operator. The main reason comes from the following key Lemma 6.2 and Lemma 6.5, where the estimates are better for $|k| \geq 2$.

6.1. Coercive estimates of $\widetilde{A}_k$ and $\widetilde{B}_k$

LEMMA 6.2. – *For any $|k| \geq 1$ and $w \in L^2(\mathbb{R}_+; dr)$, we have*

$$\langle (I - \widetilde{B}_k)w, w\rangle \geq \int_0^{+\infty} (1 - \sigma(r))|w|^2 dr,$$

$$\langle \widetilde{B}_k w, w\rangle \geq (1 - \frac{1}{|k|}) \int_0^{+\infty} \sigma(r)|w|^2 dr.$$

Proof. – Let us first prove that the operator $g\widetilde{\mathcal{K}}_k[g\cdot]$ is nonnegative. For this, we write

$$\begin{aligned}\widetilde{\mathcal{K}}_k[w](r) &= \frac{1}{2|k|}\int_0^{+\infty} \min(\frac{r}{s}, \frac{s}{r})^{|k|}(rs)^{\frac{1}{2}} w(s)ds\\ &= \frac{1}{2|k|}\int_0^r r^{\frac{1}{2}-|k|} s^{\frac{1}{2}+|k|} w(s)ds + \frac{1}{2|k|}\int_r^{+\infty} r^{\frac{1}{2}+|k|} s^{\frac{1}{2}-|k|} w(s)ds.\end{aligned}$$

Then we find that

$$\begin{aligned}(\widetilde{\mathcal{K}}_k[w](r))' &= \frac{\frac{1}{2} - |k|}{2|k|}\int_0^r (\frac{s}{r})^{|k|+\frac{1}{2}} w(s)ds + \frac{\frac{1}{2} + |k|}{2|k|}\int_r^{+\infty} (\frac{r}{s})^{|k|-\frac{1}{2}} w(s)ds,\\ (\widetilde{\mathcal{K}}_k[w](r))'' &= \frac{k^2 - \frac{1}{4}}{r^2}\tilde{\mathcal{K}}_k[w](r) - w(r).\end{aligned}$$

In particular, we find that

$$\big(-\partial_r^2 + \frac{k^2 - \frac{1}{4}}{r^2}\big)\widetilde{\mathcal{K}}_k[w](r) = w(r). \tag{48}$$

Using the following pointwise estimates of $\widetilde{\mathcal{K}}_k[w](r)$

$$|\widetilde{\mathcal{K}}_k[w](r)| \leq \frac{1}{2|k|}\int_0^{+\infty} \min(r,s)|w(s)|ds \leq \frac{1}{2|k|}\min\big(r\|w\|_{L^1}, \|rw\|_{L^1}\big),$$

$$|\partial_r(\widetilde{\mathcal{K}}_k[w](r))| \leq \int_0^{+\infty} \frac{\min(r,s)}{r}|w(s)|ds \leq \min\big(\|w\|_{L^1}, \frac{1}{r}\|rw\|_{L^1}\big),$$

we infer that

$$\widetilde{\mathcal{K}}_k[w](\widetilde{\mathcal{K}}_k[w])'|_{to1.5} r = 0, +\infty = 0.$$

Then we get by using (48) and integration by parts that

$$\begin{aligned}\langle g\widetilde{\mathcal{K}}_k[gw], w\rangle &= \langle \widetilde{\mathcal{K}}_k[gw], gw\rangle = \langle \widetilde{\mathcal{K}}_k[gw], (-\partial_r^2 + \frac{k^2-\frac{1}{4}}{r^2})\widetilde{\mathcal{K}}_k[gw]\rangle \\ &= \|\partial_r(\widetilde{\mathcal{K}}_k[gw])\|^2 + (k^2 - \frac{1}{4})\|\frac{\widetilde{\mathcal{K}}_k[gw]}{r}\|^2 \geq 0.\end{aligned}$$

Next we give a upper bound for $g\widetilde{\mathcal{K}}_k[g\cdot]$.

$$\begin{aligned}\left|\int_0^{+\infty} g\widetilde{\mathcal{K}}_k[gw]\overline{w(r)}dr\right| &\leq \frac{1}{2|k|}\int_0^{+\infty}\int_0^{+\infty} \min(\frac{r}{s},\frac{s}{r})^{|k|}(rs)^{\frac{1}{2}}|g(r)w(s)||g(s)w(r)|dsdr \\ &\leq \frac{1}{4|k|}\int_0^{+\infty}\int_0^{+\infty} \min(\frac{r}{s},\frac{s}{r})(rs)^{\frac{1}{2}}[(\frac{r}{s})^{\frac{3}{2}}g^2(r)|w(s)|^2 + (\frac{s}{r})^{\frac{3}{2}}g^2(s)|w(r)|^2]dsdr \\ &= \frac{1}{|k|}\int_0^{+\infty} \widetilde{\mathcal{K}}_1[r^{\frac{3}{2}}g^2(r)](s)\frac{|w(s)|^2}{s^{\frac{3}{2}}}ds = \frac{1}{|k|}\int_0^{+\infty}\sigma(s)|w(s)|^2ds,\end{aligned}$$

which gives

$$0 \leq g\widetilde{\mathcal{K}}_k[g\cdot] \leq \frac{1}{|k|}\sigma(r).$$

As a consequence, we deduce that

$$\begin{aligned}\langle(1-\widetilde{B}_k)w, w\rangle &= \langle(1-\sigma)w + g\widetilde{\mathcal{K}}_k[gw], w\rangle \geq \langle(1-\sigma)w, w\rangle, \\ \langle\widetilde{B}_k w, w\rangle &= \langle\sigma w, w\rangle - \langle g\tilde{\mathcal{K}}_k[gw], w\rangle \geq (1-\frac{1}{|k|})\langle\sigma w, w\rangle.\end{aligned}$$

The proof is completed. □

The following lemma gives a sharper lower bound of $\widetilde{A}_k$ than Lemma 5.5.

LEMMA 6.3. – *Let $|k| \geq 2$. Then for any $w \in D$, we have*

$$\langle\widetilde{A}_k w, w\rangle \gtrsim \big\langle(\frac{k^2}{r^2} + r^2)w, w\big\rangle.$$

Proof. – For $|k| \geq 2$, we have

$$\begin{aligned}\widetilde{A}_k &\geq \frac{k^2-\frac{1}{4}}{r^2} + \frac{r^2}{16} - \frac{1}{2} \\ &\geq (\frac{2}{r^2} + \frac{r^2}{32} - \frac{1}{2}) + (\frac{\frac{k^2}{2}-\frac{1}{4}}{r^2} + \frac{r^2}{32}) \\ &\geq \frac{\frac{k^2}{2}-\frac{1}{4}}{r^2} + \frac{r^2}{32} \sim \frac{k^2}{r^2} + r^2,\end{aligned}$$

which gives our result. $\square$

6.2. Resolvent estimate for $\nu_k \geq 1$ or $\nu_k \leq 0$

In this subsection, we prove Theorem 6.1 for the case of $\nu_k \geq 1$ or $\nu_k \leq 0$.

First of all, for $\nu_k \geq 1$, we infer from Lemma 6.2 that

$$\begin{aligned}
|\langle \widetilde{\mathcal{H}}_k w, w\rangle| &\sim \langle \widetilde{A}_k w, w\rangle + |\beta_k|\langle(\nu_k - \widetilde{B}_k)w, w\rangle \\
&\geq \langle \widetilde{A}_k w, w\rangle + |\beta_k|\langle(1 - \widetilde{B}_k)w, w\rangle \\
&\gtrsim \big\langle(\frac{1}{r^2} + r^2)w, w\big\rangle + |\beta_k|\langle(1-\sigma(r))w, w\rangle.
\end{aligned}$$

Thanks to $1-\sigma(r) = 1 - \frac{1-e^{-\frac{r^2}{4}}}{r^2/4} \sim r^2 (r \to 0)$ and $\lim\limits_{r\to\infty} 1-\sigma(r) = 1$, we get

$$\begin{aligned}
\int_0^1 \big[\frac{1}{r^2} + |\beta_k|(1-\sigma(r))\big]|w(r)|^2 dr &\gtrsim \int_0^1 (\frac{1}{r^2} + |\beta_k| r^2)|w(r)|^2 dr \\
&\gtrsim \int_0^1 |\beta_k|^{\frac12}|w(r)|^2 dr, \\
\int_1^{+\infty} \big[\frac{1}{r^2} + r^2 + |\beta_k|(1-\sigma(r))\big]|w(r)|^2 dr &\gtrsim \int_1^{+\infty} (1+|\beta_k|)|w(r)|^2 dr \\
&\gtrsim \int_1^{+\infty} |\beta_k|^{\frac12}|w(r)|^2 dr,
\end{aligned}$$

which yield that for $\nu_k \geq 1$,

$$|\langle \widetilde{\mathcal{H}}_k w, w\rangle| \gtrsim \big\langle(\frac{1}{r^2} + r^2)w, w\big\rangle + |\beta_k|\langle(1-\sigma(r))w, w\rangle \gtrsim |\beta_k|^{\frac12}\|w\|^2 \tag{49}$$

For $\nu_k \leq 0$, we infer from Lemma 6.2 that

$$\begin{aligned}
|\langle \widetilde{\mathcal{H}}_k w, w\rangle| &\sim \langle \widetilde{A}_k w, w\rangle + |\beta_k|\langle(\widetilde{B}_k - \nu_k)w, w\rangle \\
&\gtrsim \langle \widetilde{A}_k w, w\rangle + |\beta_k|\langle \widetilde{B}_k w, w\rangle \\
&\gtrsim \big\langle(\frac{1}{r^2} + r^2)w, w\big\rangle + |\beta_k|\langle\sigma(r)w, w\rangle.
\end{aligned}$$

Thanks to $\lim\limits_{r\to 0}\sigma(r) = 1$ and $\sigma(r) \sim \frac{1}{r^2}(r\to\infty)$, we get

$$\begin{aligned}
\int_0^1 \big[\frac{1}{r^2} + r^2 + |\beta_k|\sigma(r)\big]|w(r)|^2 dr \gtrsim \int_0^1 (1+|\beta_k|)|w(r)|^2 dr &\gtrsim \int_0^1 |\beta_k|^{\frac12}|w(r)|^2 dr, \\
\int_1^{+\infty} (r^2 + |\beta_k|\sigma(r))|w(r)|^2 dr &\gtrsim \int_1^{+\infty} (r^2 + \frac{1}{r^2}|\beta_k|)|w(r)|^2 dr \\
&\gtrsim \int_1^{+\infty} |\beta_k|^{\frac12}|w(r)|^2 dr,
\end{aligned}$$

which show that for $\nu_k \leq 0$

$$|\langle \widetilde{\mathcal{H}}_k w, w\rangle| \gtrsim \big\langle(\frac{1}{r^2} + r^2)w, w\big\rangle + |\beta_k|\langle\sigma(r)w, w\rangle \gtrsim |\beta_k|^{\frac12}\|w\|^2. \tag{50}$$

6.3. Resolvent estimate for $0 < \nu_k < 1$

In this subsection, we prove Theorem 6.1 for the case of $0 < \nu_k < 1$.

Let $\nu_k = \sigma(r_k)$ for some $r_k > 0$. We again divide the proof into two cases:

$$|\beta_k| \le \max\big(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6\big) \quad \text{and} \quad |\beta_k| \ge \max\big(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6\big).$$

6.3.1. *Case 1.* $|\beta_k| \le \max\big(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6\big)$.– It suffices to prove the following lemma.

LEMMA 6.4. – *If* $|\beta_k| \le \max\big(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6\big)$, *then we have*

$$\|\widetilde{\mathscr{H}}_k w\| \gtrsim |\beta_k|^{\frac{1}{3}}\|w\|.$$

Proof. – If $|\beta_k| \le |k|^3$, then $\frac{k^2}{r^2} + r^2 \ge |k| \ge |\beta_k|^{\frac{1}{3}}$. Thus,

$$|\langle \widetilde{\mathscr{H}}_k w, w\rangle| \gtrsim \big\langle (\frac{k^2}{r^2} + r^2)w, w\big\rangle \ge |\beta_k|^{\frac{1}{3}}\|w\|^2.$$

If $k^3 \le |\beta_k| \le \max(\frac{|k|^3}{r_k^4}, r_k^6)$, then $r_k \le 1$ or $r_k \ge \sqrt{k}$. Thus, we only need to check the following cases

$$\begin{aligned}
&r_k \le 1, && |k|^3 \le |\beta_k| \le \frac{|k|^3}{r_k^4} \Longrightarrow \|\widetilde{\mathscr{H}}_k w\| \gtrsim |\beta_k|^{\frac{1}{2}}\|w\|,\\
&r_k \ge \sqrt{k}, && |k|^3 \le |\beta_k| \le r_k^4 \Longrightarrow \|\widetilde{\mathscr{H}}_k w\| \gtrsim |\beta_k|^{\frac{1}{2}}\|w\|,\\
&r_k \ge \sqrt{k}, && r_k^4 \le |\beta_k| \le r_k^6 \Longrightarrow \|\widetilde{\mathscr{H}}_k w\| \gtrsim |\beta_k|^{\frac{1}{3}}\|w\|,
\end{aligned}$$

which can be deduced from the following fact

$$|\langle \widetilde{\mathscr{H}}_k w, w\rangle| \gtrsim \big\langle (r^2 + \frac{k^2}{r^2})w, w\big\rangle + |\beta_k| \max(\langle(\nu_k - \sigma(r))w, w\rangle, \langle(\sigma(r)/2 - \nu_k)w, w\rangle, 0)$$

and Lemma A.3. □

6.3.2. *Case 2.* $|\beta_k| \ge \max(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6)$.– Let us introduce the operator

$$\widetilde{\mathscr{K}}_k^{(r_k)}[f](r) = \int_0^{r_k} \widetilde{K}_k^{(r_k)}(r,s) f(s)ds, \tag{51}$$

where for $0 \le r, s \le r_k$,

$$\widetilde{K}_k^{(r_k)}(r,s) = \frac{1}{2|k|}\min(\frac{r}{s}, \frac{s}{r})^{|k|}(rs)^{\frac{1}{2}} - \frac{1}{2|k|}(\frac{rs}{r_k^2})^{|k|}(rs)^{\frac{1}{2}} \ge 0.$$

Let $u(r) = \widetilde{\mathscr{K}}_k^{(r_k)}[w](r)$. Then $u \in H_0^1(0, r_k)$ is the unique solution to

$$\Big(-\partial_r^2 + \frac{k^2 - \frac{1}{4}}{r^2}\Big)u = w \quad \text{in} \quad (0, r_k).$$

LEMMA 6.5. – *It holds that*

$$\mathrm{Re}\int_0^{r_k} g(r)\widetilde{\mathscr{K}}_k^{(r_k)}[gw](r)\overline{w}(r)dr \le \frac{2}{|k|+1}\int_0^{r_k}(\sigma(s) - \nu_k)|w(s)|^2 ds.$$

Proof. – As $(r^3\sigma'(r))' = -r^3 g(r)^2$ and g^2 is decreasing, we have $-r^3\sigma'(r) \geq r^4 g^2(r)/4$, and

$$\begin{aligned}\Big(-\partial_r^2 + \frac{k^2-\frac14}{r^2}\Big)\big(r^{|k|+\frac12}(\sigma(r)-\nu_k)\big) &= r^{|k|+\frac12}(-\partial_r^2\sigma - (2|k|+1)r^{-1}\partial_r\sigma)\\ &= r^{|k|+\frac12}(g^2 - (2|k|-2)r^{-1}\partial_r\sigma)\\ &\geq r^{|k|+\frac12}(g^2 + \frac{|k|-1}{2}g^2)\\ &= r^{|k|+\frac12}\frac{|k|+1}{2}g^2,\end{aligned}$$

which implies that

$$\begin{aligned}r^{|k|+\frac12}(\sigma-\nu_k) &= \widetilde{\mathcal{K}}_k^{(r_k)}\big[r^{|k|+\frac12}(g^2-(2|k|-2)\frac{\partial_r\sigma}{r})\big]\\ &\geq \frac{|k|+1}{2}\widetilde{\mathcal{K}}_k^{(r_k)}\big[r^{|k|+\frac12}g^2\big].\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\mathrm{Re}\int_0^{r_k} g\,\widetilde{\mathcal{K}}_k^{(r_k)}[gw](r)\overline{w}(r)dr\\ &\quad\leq \int_0^{r_k}\int_0^{r_k}\widetilde{K}_k^{(r_k)}(r,s)|g(r)w(s)||g(s)w(r)|dsdr\\ &\quad\leq \frac12\int_0^{r_k}\int_0^{r_k}\widetilde{K}_k^{(r_k)}(r,s)\left(\Big(\frac{r}{s}\Big)^{|k|+\frac12}g^2(r)|w(s)|^2 + \Big(\frac{s}{r}\Big)^{|k|+\frac12}g^2(s)|w(r)|^2\right)dsdr\\ &\quad= \int_0^{r_k}\widetilde{\mathcal{K}}_k^{(r_k)}[r^{|k|+\frac12}g^2](s)\frac{|w(s)|^2}{s^{|k|+\frac12}}ds \leq \frac{2}{|k|+1}\int_0^{r_k}(\sigma(s)-\nu_k)|w(s)|^2 ds.\end{aligned}$$

This completes the proof. □

To proceed, we introduce the following decomposition: let $\psi = \widetilde{\mathcal{K}}_k[gw]$ and decompose

$$\psi(r) = \psi_1(r) + \psi_2(r), \tag{52}$$

where $\psi_2(r)$ is given by

$$\psi_2(r) = \begin{cases}(\frac{r}{r_k})^{|k|+\frac12}\psi(r_k), & 0<r<r_k,\\ (\frac{r}{r_k})^{-|k|+\frac12}\psi(r_k), & r>r_k.\end{cases}$$

Then we find that $\psi_1(r) \in H_0^1(0,r_k)$ and solves

$$\Big(-\partial_r^2 + \frac{k^2-\frac14}{r^2}\Big)\psi_1 = gw, \quad r\in\mathbb{R}_+\backslash\{r_k\}. \tag{53}$$

Thus, $\psi_1(r) = \widetilde{\mathcal{K}}_k^{(r_k)}[gw](r)$ in $(0,r_k)$.

Let $\delta>0$ be such that

$$\delta^3|\beta_k|\min(r_k, r_k^{-3}) = 1.$$

Due to $|\beta_k| \geq \max(\frac{k^3}{r_k^4}, |k|^3, r_k^6)$, we have

$$0 < \delta \leq \min(\frac{r_k}{|k|}, \frac{1}{r_k}). \tag{54}$$

Due to $|\sigma'(r)| \sim \min(r, \frac{1}{r^3})$, we also have

$$|\beta_k \delta^3 \sigma'(r_k)| \sim 1. \tag{55}$$

We denote

$$\begin{aligned}\mathcal{E}(w) = &\frac{\|w'\|\|w\|_{L^\infty}}{|\beta_k \delta^{\frac{3}{2}} \sigma'(r_k)|} + \frac{\|\widetilde{\mathcal{H}}_k w\|\|w\|}{|\beta_k \delta \sigma'(r_k)|} + \frac{\|w'\overline{w}\|_{L^\infty(r_k-\delta, r_k+\delta)}}{|\beta_k \delta \sigma'(r_k)|} + \delta \|w\|_{L^\infty}^2 \\ &+ \frac{|\psi(r_k) J(r_k)|}{\min(1, r_k^2)} + \frac{g(r_k)^2 |\psi(r_k)|^2}{\delta |\sigma'(r_k)|^2} + \frac{|\psi(r_k)|^2}{r_k^5 + 1} = \mathcal{E}_1(w) + \cdots + \mathcal{E}_7(w),\end{aligned}$$

where

$$J(r) = \int_0^r (\frac{s}{r})^{|k|+\frac{1}{2}} g(s) w(s) ds - \int_r^{+\infty} (\frac{r}{s})^{|k|-\frac{1}{2}} g(s) w(s) ds.$$

It is easy to see that

$$\partial_r \psi = -\frac{J}{2} + \frac{\psi}{2r}. \tag{56}$$

LEMMA 6.6. – *It holds that for any $w \in D$,*

$$\|w\|^2 \leq C\, \mathcal{E}(w).$$

Proof. – Recall that

$$\widetilde{\mathcal{H}}_k w = -\partial_r^2 w + \Big(\frac{k^2 - 1/4}{r^2} + \frac{r^2}{16} - \frac{1}{2}\Big) w + i\beta_k((\sigma - \nu_k) w - g\psi).$$

Then we get by integration by parts that

$$\begin{aligned}&\mathrm{Re}\langle \widetilde{\mathcal{H}}_k w, i\,\mathrm{sgn}(\beta_k)(\chi_{(0,r_k)} - \chi_{(r_k,+\infty)}) w\rangle \\ &\quad= \mathrm{Re}\langle -\partial_r^2 w + i\beta_k((\sigma - \nu_k) w - g\psi), i\,\mathrm{sgn}(\beta_k)(\chi_{(0,r_k)} - \chi_{(r_k,+\infty)}) w\rangle \\ &\quad= \mathrm{Re}\left(\int_0^{r_k} (-i\,\mathrm{sgn}(\beta_k)|\partial_r w|^2 + |\beta_k|(\sigma - \nu_k)|w|^2 - |\beta_k| g\psi\overline{w}) dr + i\,\mathrm{sgn}(\beta_k) w'\overline{w}(r_k)\right) \\ &\qquad+ \mathrm{Re}\left(\int_{r_k}^{+\infty} (i\,\mathrm{sgn}(\beta_k)|\partial_r w|^2 + |\beta_k|(\nu_k - \sigma)|w|^2 + |\beta_k| g\psi\overline{w}) dr + i\,\mathrm{sgn}(\beta_k) w'\overline{w}(r_k)\right) \\ &\quad\geq |\beta_k| \int_0^{+\infty} |\sigma - \nu_k||w|^2 dr - |\beta_k| \mathrm{Re}\left(\int_0^{r_k} g\psi\overline{w} dr - \int_{r_k}^{+\infty} g\psi\overline{w} dr\right) - 2|w'\overline{w}(r_k)|.\end{aligned}$$

Using (52), we write

$$\begin{aligned}
&\int_0^{r_k} g\psi\overline{w}dr - \int_{r_k}^{+\infty} g\psi\overline{w}dr\\
&= \int_0^{r_k} g\psi_1\overline{w}dr - \int_{r_k}^{+\infty} g\psi_1\overline{w}dr + \int_0^{r_k} g\psi_2\overline{w}dr - \int_{r_k}^{+\infty} g\psi_2\overline{w}dr\\
&= \int_0^{r_k} g\psi_1\overline{w}dr - \int_{r_k}^{+\infty} \psi_1(-\partial_r^2 + (k^2-1/4)r^{-2})\overline{\psi}_1 dr\\
&\quad + \psi(r_k)\int_0^{r_k}\left(\frac{r}{r_k}\right)^{|k|+\frac12} g\overline{w}dr - \psi(r_k)\int_{r_k}^{+\infty}\left(\frac{r}{r_k}\right)^{-|k|+\frac12} g\overline{w}dr\\
&= \int_0^{r_k} g\psi_1\overline{w}dr - \int_{r_k}^{+\infty}\big(|\partial_r\psi_1|^2 + (k^2-1/4)r^{-2}|\psi_1|^2\big)dr + \psi(r_k)\overline{J}(r_k).
\end{aligned}$$

By (53), we have

$$\int_0^{r_k} g\psi_1\overline{w}dr = \int_0^{r_k}\psi_1(-\partial_r^2 + \frac{k^2-1/4}{r^2})\overline{\psi}_1 dr = \int_0^{r_k}\big(|\partial_r\psi_1|^2 + \frac{k^2-1/4}{r^2}|\psi_1|^2\big)dr.$$

We get by Lemma 6.5 that

$$\int_0^{r_k} g\psi_1\overline{w}dr = \int_0^{r_k} g\widetilde{\mathscr{K}}_k^{(r_k)}[gw](r)\overline{w}dr \le \frac{2}{|k|+1}\int_0^{r_k}(\sigma(s)-\nu_k)|w(s)|^2 ds.$$

Then we conclude that

$$\begin{aligned}
&\operatorname{Re}\langle\widetilde{\mathscr{H}}_k w, i\operatorname{sgn}(\beta_k)(\chi_{(0,r_k)} - \chi_{(r_k,+\infty)})w\rangle\\
&\ge |\beta_k|\left(\frac{|k|+1}{2}\int_0^{r_k} g\psi_1\overline{w}dr - \int_0^{r_k} g\psi_1\overline{w}dr + \int_{r_k}^{+\infty}(|\partial_r\psi_1|^2 + \frac{k^2-1/4}{r^2}|\psi_1|^2)dr\right)\\
&\quad - |\beta_k\psi(r_k)\overline{J(r_k)}| - 2|w'\overline{w}(r_k)|\\
&= |\beta_k|\left(\frac{|k|-1}{2}\int_0^{r_k} g\psi_1\overline{s}dr + \int_{r_k}^{+\infty}(|\partial_r\psi_1|^2 + \frac{k^2-1/4}{r^2}|\psi_1|^2)dr\right)\\
&\quad - |\beta_k\psi(r_k)\overline{J(r_k)}| - 2|w'\overline{w}(r_k)|\\
&= |\beta_k|\left(\frac{|k|-1}{2}\int_0^{r_k}(|\partial_r\psi_1|^2 + \frac{k^2-1/4}{r^2}|\psi_1|^2)dr + \int_{r_k}^{+\infty}(|\partial_r\psi_1|^2 + \frac{k^2-1/4}{r^2}|\psi_1|^2)dr\right)\\
&\quad - |\beta_k\psi(r_k)\overline{J(r_k)}| - 2|w'\overline{w}(r_k)|\\
&\ge |\beta_k|\frac{A_1}{2} - |\beta_k\psi(r_k)\overline{J(r_k)}| - 2|w'\overline{w}(r_k)|,
\end{aligned}$$

where

$$A_1 = \int_0^{+\infty}(|\partial_r\psi_1|^2 + (k^2-1/4)r^{-2}|\psi_1|^2)dr.$$

This shows that

$$A_1 \le 2|\psi(r_k)J(r_k)| + \frac{1}{|\beta_k|}\big(4|w'\overline{w}(r_k)| + 2\|w\|\|\widetilde{\mathscr{H}}_k w\|\big). \tag{57}$$

Similarly, we have

$$\begin{aligned}
&\operatorname{Re}\langle \widetilde{\mathcal{H}}_k w, i\,\mathrm{sgn}(\beta_k)\frac{\chi_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)}}{\sigma-\nu_k}w\rangle\\
&= \operatorname{Re}\Big\langle -\partial_r^2 w + i\beta_k((\sigma-\nu_k)w - g\psi), i\,\mathrm{sgn}(\beta_k)\frac{\chi_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)}}{\sigma-\nu_k}w\Big\rangle\\
&= \operatorname{Re}\int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)}\left(-i\frac{\mathrm{sgn}(\beta_k)|w'|^2}{\sigma-\nu_k} + i\frac{\mathrm{sgn}(\beta_k)w'\overline{w}\sigma'}{(\sigma-\nu_k)^2} + |\beta_k||w|^2 - \frac{|\beta_k|g\psi\overline{w}}{\sigma-\nu_k}\right)dr\\
&\quad + \operatorname{Re}\Big(i\,\mathrm{sgn}(\beta_k)\frac{w'\overline{w}}{\sigma-\nu_k}(r_k-\delta) - i\,\mathrm{sgn}(\beta_k)\frac{w'\overline{w}}{\sigma-\nu_k}(r_k+\delta)\Big)\\
&\geq -\|w'\|\|w\|_{L^\infty}\left\|\frac{\sigma'}{(\sigma-\nu_k)^2}\right\|_{L^2(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))} + |\beta_k|\int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)}|w|^2dr\\
&\quad - \frac{|\beta_k|}{2}\int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)}\Big(|w|^2 + \frac{g^2|\psi|^2}{(\sigma-\nu_k)^2}\Big)dr\\
&\quad - \|w'\overline{w}\|_{L^\infty(r_k-\delta,r_k+\delta)}\Big(\frac{1}{\sigma(r_k-\delta)-\nu_k} + \frac{1}{\nu_k-\sigma(r_k+\delta)}\Big)\\
&= \frac{|\beta_k|}{2}\int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)}\Big(|w|^2 - \frac{g^2|\psi|^2}{(\sigma-\nu_k)^2}\Big)dr - \|w'\|_{L^2}\|w\|_{L^\infty}\left\|\frac{\sigma'}{(\sigma-\nu_k)^2}\right\|_{L^2(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))}\\
&\quad - \|w'\overline{w}\|_{L^\infty(r_k-\delta,r_k+\delta)}\Big(\frac{1}{\sigma(r_k-\delta)-\nu_k} + \frac{1}{\nu_k-\sigma(r_k+\delta)}\Big),
\end{aligned}$$

which gives

$$\begin{aligned}
\|w\|^2_{L^2(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))} \leq& \frac{2}{|\beta_k|}\|\widetilde{\mathcal{H}}_k w\|\|w\|\left\|\frac{1}{\sigma-\nu_k}\right\|_{L^\infty(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))}\\
&+ \int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)}\frac{g^2|\psi|^2}{(\sigma-\nu_k)^2}dr + \frac{2}{|\beta_k|}\|w'\|\|w\|_{L^\infty}\left\|\frac{\sigma'}{(\sigma-\nu_k)^2}\right\|_{L^2(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))}\\
&+ \frac{2}{|\beta_k|}\|w'\overline{w}\|_{L^\infty(r_k-\delta,r_k+\delta)}\Big(\frac{1}{\sigma(r_k-\delta)-\nu_k} + \frac{1}{\nu_k-\sigma(r_k+\delta)}\Big).
\end{aligned}$$

By Lemma A.1, we have

$$\left\|\frac{1}{\sigma-\nu_k}\right\|_{L^\infty(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))} \leq \frac{C}{|\sigma'(r_k)|\delta},$$

and

$$\begin{aligned}
\left\|\frac{\sigma'}{(\sigma-\nu_k)^2}\right\|^2_{L^2(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))} &\leq \left\|\frac{\sigma'}{\sigma-\nu_k}\right\|_{L^\infty(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))}\left\|\frac{\sigma'}{(\sigma-\nu_k)^3}\right\|_{L^1(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))}\\
&\leq \frac{C}{\delta}\left(\frac{1}{(\sigma(r_k-\delta)-\nu_k)^2} + \frac{1}{(\nu_k-\sigma(r_k+\delta))^2}\right)\\
&\leq \frac{C}{\delta(\sigma'(r_k)\delta)^2}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|w\|^2 \le& \|w\|^2_{L^2(\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta))} + \|w\|^2_{L^2((r_k-\delta,r_k+\delta))}\\ \le& \int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)} \frac{g^2|\psi|^2}{(\sigma-\nu_k)^2}dr + \frac{C\|w'\|\|w\|_{L^\infty}}{|\beta_k\delta^{\frac{3}{2}}\sigma'(r_k)|}\\ &+ \frac{C\|w'\overline{w}\|_{L^\infty(r_k-\delta,r_k+\delta)}}{|\beta_k\delta\sigma'(r_k)|} + \frac{C\|\widetilde{\mathcal{H}}_k w\|\|w\|}{|\beta_k\delta\sigma'(r_k)|} + 2\delta\|w\|^2_{L^\infty}\\ \le& \int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)} \frac{g^2|\psi|^2}{(\sigma-\nu_k)^2}dr + C\,\mathcal{E}(w).\end{aligned}$$

It remains to estimate the first term, which is bounded by

$$\int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)} \frac{g^2|\psi|^2}{(\sigma-\nu_k)^2}dr \le 2\int_{\mathbb{R}_+} \frac{g^2|\psi_1|^2}{(\sigma-\nu_k)^2}dr + 2\int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)} \frac{g^2|\psi_2|^2}{(\sigma-\nu_k)^2}dr.$$

Let us first consider the case of $0 < r_k \le 1$. By Lemma A.1, we have

$$\begin{aligned}&|\sigma(r)-\nu_k| \ge C^{-1}|r-r_k||\sigma'(r_k)| \ge C^{-1}|r-r_k|r_k, \quad 0<r<r_k+1,\\ &|\sigma(r)-\nu_k| \ge C^{-1}, \quad r \ge r_k+1.\end{aligned}$$

Due to $\psi_1(r_k)=0$, we get by Hardy's inequality that

$$\int_0^{r_k+1} \frac{g^2|\psi_1|^2}{(\sigma-\nu_k)^2}dr \le C\int_0^{r_k+1} \frac{|\psi_1|^2}{|r-r_k|^2 r_k^2}dr \le \frac{C}{r_k^2}\int_0^{r_k+1} |\partial_r\psi_1|^2 dr \le \frac{CA_1}{r_k^2},$$

and by Lemma A.1,

$$\int_{r_k+1}^{+\infty} \frac{g^2|\psi_1|^2}{(\sigma-\nu_k)^2}dr \le C\int_{r_k+1}^{+\infty} g^2|\psi_1|^2 dr \le C\int_{r_k+1}^{+\infty} \frac{|\psi_1|^2}{r^2}dr \le CA_1.$$

Thanks to $0<\delta<r_k$ and $|\psi_2(r)| \le |\psi(r_k)|$, we have

$$\int_{(0,r_k+1)\setminus(r_k-\delta,r_k+\delta)} \frac{g^2|\psi_2|^2}{(\sigma-\nu_k)^2}dr \le C\int_{(0,r_k+1)\setminus(r_k-\delta,r_k+\delta)} \frac{|\psi(r_k)|^2}{|r-r_k|^2 r_k^2}dr \le \frac{C|\psi(r_k)|^2}{r_k^2\delta},$$

and

$$\int_{r_k+1}^{+\infty} \frac{g^2|\psi_2|^2}{(\sigma-\nu_k)^2}dr \le C|\psi(r_k)|^2\int_{r_k+1}^{+\infty} g^2(r)dr \le C|\psi(r_k)|^2.$$

Therefore, we obtain

$$\int_{\mathbb{R}_+\setminus(r_k-\delta,r_k+\delta)} \frac{g^2|\psi|^2}{(\sigma-\nu_k)^2}dr \le \frac{CA_1}{r_k^2} + \frac{C|\psi(r_k)|^2}{r_k^2\delta} \le C\,\mathcal{E}(w),$$

where we used (57) and the facts that $|\sigma'(r_k)| \sim r_k$, $g(r_k) \ge C^{-1}$ and $|\delta\sigma'(r_k)| \le Cr_k^2$.

Next we consider the case of $r_k \ge 1$. By Lemma A.1, we have

$$\begin{aligned}&|\sigma(r)-\nu_k| \ge C^{-1}|r-r_k||\sigma'(r_k)| \ge C^{-1}|r-r_k|r_k^{-3}, \quad |r-r_k|<1,\\ &|\sigma(r)-\nu_k| \ge C^{-1}(1+r)^{-4}, \quad |r-r_k| \ge 1/r_k,\end{aligned}$$

and $g(r) \le Cg(r_k)$ for $|r-r_k| < 1/r_k$. Thanks to $\psi_1(r_k)=0$ and $g(r_k-1)^2 r_k^6 \le C$, we get by Hardy's inequality that

$$\int_{r_k-1}^{r_k+1} \frac{g^2|\psi_1|^2}{(\sigma-\nu_k)^2}dr \le C\int_{r_k-1}^{r_k+1} \frac{g^2|\psi_1|^2 r_k^6}{|r-r_k|^2}dr \le C\int_{r_k-1}^{r_k+1} |\partial_r\psi_1|^2 dr \le CA_1,$$

and

$$\int_{\mathbb{R}_+\setminus B(r_k,1)} \frac{g^2|\psi_1|^2}{(\sigma-\nu_k)^2}dr \le C\int_{\mathbb{R}_+} g^2(r)(1+r)^8|\psi_1|^2 dr \le C\int_{\mathbb{R}_+}\frac{|\psi_1|^2}{r^2}dr \le CA_1,$$

where we denote $B(a,b)=(a-b,a+b)$. Since $0<\delta<1/r_k$ and $|\psi_2(r)| \le |\psi(r_k)|$, we have

$$\int_{B(r_k,1/r_k)\setminus B(r_k,\delta)} \frac{g^2|\psi_2|^2}{(\sigma-\nu_k)^2}dr \le C\int_{B(r_k,1/r_k)\setminus B(r_k,\delta)} \frac{g^2(r_k)|\psi(r_k)|^2}{|r-r_k|^2\sigma'(r_k)^2}dr \le \frac{Cg^2(r_k)|\psi(r_k)|^2}{\sigma'(r_k)^2\delta},$$

and due to $|\psi_2(r)| \le (r/r_k)^{\frac52}|\psi(r_k)|$, we have

$$\int_{\mathbb{R}_+\setminus B(r_k,1/r_k)} \frac{g^2|\psi_2|^2}{(\sigma-\nu_k)^2}dr \le C\int_{R_+} (r/r_k)^5 g^2(r)|\psi(r_k)|^2(1+r)^8 dr \le \frac{C|\psi(r_k)|^2}{r_k^5}.$$

Therefore, we obtain

$$\int_{\mathbb{R}_+\setminus B(r_k,\delta)} \frac{g^2|\psi|^2}{(\sigma-\nu_k)^2}dr \le CA_1 + \frac{Cg^2(r_k)|\psi(r_k)|^2}{\sigma'(r_k)^2\delta} + \frac{C|\psi(r_k)|^2}{r_k^5} \le C\,\mathcal{E}(w),$$

where we used (57) and the facts that $C^{-1}r_k^{-3} \le |\sigma'(r_k)| \le Cr_k^{-3}$ and $|\delta\sigma'(r_k)| \le C$.

This completes the proof of the lemma. □

Now we are in a position to show that for $|\beta_k| \ge \max\big(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6\big)$,

$$\|\widetilde{\mathcal{H}}_k w\| \gtrsim |\beta_k|^{\frac13}\|w\|. \tag{58}$$

By Lemma 6.6, we have

$$\|w\| \le C\,\mathcal{E}(w) = C\big(\mathcal{E}_1(w)+\cdots+\mathcal{E}_7(w)\big).$$

In the following, we handle each $\mathcal{E}_i(w)$. Using the fact that

$$|J(r)| \le \int_0^r |w(s)|ds + \int_r^{+\infty}\Big(\frac{r}{s}\Big)^{\frac32}|w(s)|ds \le Cr\|w\|_{L^\infty}, \tag{59}$$

$$|J(r)| \le \int_0^{+\infty}\Big(\frac{s}{r}\Big)^{\frac52}|gw|(s)ds \le Cr^{-\frac52}\min(\|w\|_{L^\infty},\|w\|), \tag{60}$$

we deduce that

(61)

$$\mathcal{E}_5(w) \le \frac{C}{r_k}|\psi(r_k)|\|w\|_{L^\infty} \le \mathcal{E}_4(w) + C\frac{1}{\delta}\frac{|\psi(r_k)|^2}{r_k^2} \le \mathcal{E}_4(w) + C\,\mathcal{E}_6(w), \quad 0<r_k\le 1,$$

(62)

$$\mathcal{E}_5(w) \le C\frac{1}{r_k^{5/2}}|\psi(r_k)|\|w\| \le C\,\mathcal{E}_7(w)^{\frac12}\|w\|, \quad r_k \ge 1.$$

Using the fact that

$$|\psi(r)| \le r\int_0^r |w(s)|ds + \int_r^{+\infty}\Big(\frac{r}{s}\Big)^2 (rs)^{\frac12}|w(s)|ds \le Cr^2\|w\|_{L^\infty}, \tag{63}$$

$$|\psi(r)| \le \int_0^{+\infty}\Big(\frac{s}{r}\Big)^2 (rs)^{\frac12}|gw|(s)ds \le Cr^{-\frac32}\min(\|w\|_{L^\infty}, \|w\|), \tag{64}$$

we deduce that

$$\mathcal{E}_7(w) \le Cr_k^{-8}\|w\|^2, \tag{65}$$

As $\delta < 1,\ |\sigma'(r_k)| < C$, we also have

$$\mathcal{E}_7(w) \le C\,\mathcal{E}_6(w)g(r_k)^{-2}. \tag{66}$$

We introduce

$$\mathcal{J}(w) = \delta\|w\|_{L^\infty}^2 + \delta^2\|w'\|^2 + \delta^2\|w\|\|\widetilde{\mathcal{H}}_k w\| + \delta^4\|\widetilde{\mathcal{H}}_k w\|^2. \tag{67}$$

It is easy to see that

$$\mathcal{E}_4(w) \le \mathcal{J}(w), \tag{68}$$

and by (55), we have

$$\mathcal{E}_1(w) + \mathcal{E}_2(w) \le C\,\mathcal{J}(w). \tag{69}$$

To proceed, we need the following L^∞ estimate of w' and ψ.

Lemma 6.7. – *It holds that*

$$\delta^3\|w'\|_{L^\infty(B(r_k,\delta))}^2 + \frac{1}{(\sigma'(r_k))^2\delta}\|g\psi\|_{L^\infty(B(r_k,\delta))}^2 \le C\,\mathcal{J}(w).$$

Proof. – Let

$$u = \widetilde{\mathcal{H}}_k w, \quad u_1 = g\psi, \quad u_2 = \Big(\frac{k^2-1/4}{r^2} + \frac{r^2}{16} - \frac12\Big)w + i\beta_k(\sigma-\nu_k)w.$$

Then we have

$$-w'' + u_2 - i\beta_k u_1 = u. \tag{70}$$

Due to $0 < \delta \le \min(\frac{r_k}{|k|}, \frac{1}{r_k})$ and Lemma A.1, we have

$$\begin{aligned}\|u_2\|_{L^\infty(B(r_k,\delta))} &\le \Big\|\frac{k^2-1/4}{r^2} + \frac{r^2}{16} - \frac12\Big\|_{L^\infty(B(r_k,\delta))}\|w\|_{L^\infty} + \|\beta_k(\sigma-\nu_k)\|_{L^\infty(B(r_k,\delta))}\|w\|_{L^\infty}\\ &\le C\Big(\frac{k^2}{r_k^2} + r_k^2 + |\beta_k\delta\sigma'(r_k)|\Big)\|w\|_{L^\infty}\\ &\le C(\delta^{-2} + |\beta_k\delta\sigma'(r_k)|)\|w\|_{L^\infty} \le C\delta^{-2}\|w\|_{L^\infty}.\end{aligned} \tag{71}$$

By (63), (59) and (56), we get

$$|\psi/r| + |\partial_r\psi| \le Cr\|w\|_{L^\infty},$$

which gives

$$\begin{aligned}|\partial_r u_1| &\le g|\partial_r\psi| + |\partial_r g||\psi| \le g(|\partial_r\psi| + r|\psi|)\\ &\le Cg(r)(r+r^3)\|w\|_{L^\infty} \le C|\sigma'(r)|\|w\|_{L^\infty}.\end{aligned}$$

In particular, for $r, s \in B(r_k, \delta)$,

$$|u_1(r) - u_1(s)| \le C\delta|\sigma'(r_k)|\|w\|_{L^\infty}. \tag{72}$$

Choose $r_* \in (r_k - \delta, r_k)$ such that

$$|w'(r_*)|^2 + |w'(r_* + \delta)|^2 \le \frac{1}{\delta}\|w'\|^2,$$

which along with (70) gives

$$\left|\int_{r_*}^{r_*+\delta} (u_2 - i\beta_k u_1 - u)dr\right| = |w'(r_* + \delta) - w'(r_*)| \le 2\delta^{-\frac{1}{2}}\|w'\|,$$

from which and (71), we infer that

$$\begin{aligned}\left|\int_{r_*}^{r_*+\delta} u_1 dr\right| &\le |\beta_k|^{-1}(\|u_2\|_{L^1(r_*,r_*+\delta)} + \|u\|_{L^1(r_*,r_*+\delta)} + 2\delta^{-\frac{1}{2}}\|w'\|)\\ &\le |\beta_k|^{-1}(\delta\|u_2\|_{L^\infty(r_*,r_*+\delta)} + \delta^{\frac{1}{2}}\|u\| + 2\delta^{-\frac{1}{2}}\|w'\|)\\ &\le |\beta_k|^{-1}(C\delta^{-1}\|w\|_{L^\infty} + \delta^{\frac{1}{2}}\|\widetilde{\mathscr{H}}_k w\| + 2\delta^{-\frac{1}{2}}\|w'\|)\\ &\le C|\beta_k|^{-1}\delta^{-\frac{3}{2}}\mathscr{J}(w)^{\frac{1}{2}}.\end{aligned}$$

For $s \in B(r_k, \delta)$, we get by (72) that

$$\left|\delta u_1(s) - \int_{r_*}^{r_*+\delta} u_1 dr\right| \le \int_{r_*}^{r_*+\delta} |u_1(s) - u_1(r)|dr \le C\delta^2|\sigma'(r_k)|\|w\|_{L^\infty},$$

which gives

$$|u_1(s)| \le C(\delta^{\frac{1}{2}}|\sigma'(r_k)| + |\beta_k|^{-1}\delta^{-\frac{5}{2}})A^{\frac{1}{2}} \le C\delta^{\frac{1}{2}}|\sigma'(r_k)|\mathscr{J}(w)^{\frac{1}{2}}, \tag{73}$$

that is,

$$\|g\psi\|_{L^\infty(B(r_k,\delta))} = \|u_1\|_{L^\infty(B(r_k,\delta))} \le C\delta^{\frac{1}{2}}|\sigma'(r_k)|\mathscr{J}(w)^{\frac{1}{2}}.$$

Using (70), (71) and (73), we infer that

$$\begin{aligned}\|w'\|_{L^\infty(B(r_k,\delta))} &\le |w'(r_*)| + \|w''\|_{L^1(B(r_k,\delta))}\\ &\le \delta^{-\frac{1}{2}}\|w'\| + \|u_2\|_{L^1(B(r_k,\delta))} + |\beta_k|\|u_1\|_{L^1(B(r_k,\delta))} + \|u\|_{L^1(B(r_k,\delta))}\\ &\le \delta^{-\frac{1}{2}}\|w'\| + 2\delta\|u_2\|_{L^\infty(B(r_k,\delta))} + 2|\beta_k|\delta\|u_1\|_{L^\infty(B(r_k,\delta))} + 2\delta^{\frac{1}{2}}\|u\|\\ &\le \delta^{-\frac{1}{2}}\|w'\| + C\delta^{-1}\|w\|_{L^\infty} + C|\beta_k|\delta^{\frac{3}{2}}|\sigma'(r_k)|\mathscr{J}(w)^{\frac{1}{2}} + 2\delta^{\frac{1}{2}}\|u\|\\ &\le C\delta^{-\frac{3}{2}}\mathscr{J}(w)^{\frac{1}{2}}.\end{aligned}$$

This completes the proof of the lemma. □

Now we infer from Lemma 6.7 that

$$\mathscr{E}_6(w) \le C\,\mathscr{J}(w), \tag{74}$$

and by (55),

$$\mathcal{E}_3(w) \le \frac{1}{|\beta_k \delta \sigma'(r_k)|}\|w'\|_{L^\infty(B(r_k,\delta))}\|w\|_{L^\infty}$$

$$\le C\delta^2\|w'\|_{L^\infty(B(r_k,\delta))}\|w\|_{L^\infty} \le C\,\mathcal{J}(w). \tag{75}$$

If $0 < r_k \le 1$, we deduce from (69), (75), (68), (61), (74), (66) that

$$\|w\|^2 \le C\,\mathcal{J}(w).$$

If $r_k \ge 1$, we similarly have

$$\|w\|^2 \le C\big(\mathcal{J}(w) + \mathcal{E}_7(w)^{\frac{1}{2}}\|w\| + \mathcal{E}_7(w)\big) \le C_0\big(\mathcal{J}(w) + \mathcal{E}_7(w)\big).$$

Now if $\|w\|^2 < 2C_0\,\mathcal{E}_7(w)$, we get by (65) that

$$\|w\|^2 < 2Cr_k^{-8}\|w\|_{L^2}^2,$$

which implies that $r_k \le C$, thus, $g(r_k)^{-2} \le C$. Hence,

$$\mathcal{E}_7(w) \le C\,\mathcal{E}_6(w) \le C\,\mathcal{J}(w) \Longrightarrow \|w\|^2 \le C\,\mathcal{J}(w).$$

While, if $\|w\|^2 \ge 2C_0\,\mathcal{E}_7(w)$, we have

$$\|w\|^2 \le 2C\,\mathcal{J}(w).$$

Thanks to $\|w\|_{L^\infty}^2 \le \|w'\|\|w\|$ and $\|w'\|^2 \le \|w\|\|\widetilde{\mathcal{H}}_k w\|$, we have

$$\begin{aligned}\mathcal{J}(w) &\le \delta\|w'\|\|w\| + 2\delta^2\|w\|\|\widetilde{\mathcal{H}}_k w\| + \delta^4\|\widetilde{\mathcal{H}}_k w\|^2\\ &\le \|w\|^{\frac{3}{2}}\big(\delta^2\|\widetilde{\mathcal{H}}_k w\|\big)^{\frac{1}{2}} + 2\|w\|_{L^2}\big(\delta^2\|\widetilde{\mathcal{H}}_k w\|\big) + \big(\delta^2\|\widetilde{\mathcal{H}}_k w\|\big)^2,\end{aligned}$$

which along with $\|w\|^2 \le C\,\mathcal{J}(w)$ implies that

$$\|w\| \le C\delta^2\|\widetilde{\mathcal{H}}_k w\|.$$

As $|\beta_k| \ge \max\big(\frac{|k|^3}{r_k^4}, |k|^3, r_k^6\big) \ge \max\big(r_k^{-1}, r_k^3\big)^2$, $1 = \delta^3|\beta_k|\min(r_k, r_k^{-3}) \ge \delta^3|\beta_k||\beta_k|^{-\frac{1}{2}}$, we have $\delta^2 \le |\beta_k|^{-\frac{1}{3}}$ and that

$$\|\widetilde{\mathcal{H}}_k w\| \gtrsim |\beta_k|^{\frac{1}{3}}\|w\|.$$

7. Spectral lower bound

Recall that

$$\widetilde{\mathcal{H}}_{\alpha,k,0} = -\partial_r^2 + \frac{k^2 - \frac{1}{4}}{r^2} + \frac{r^2}{16} - \frac{1}{2} + i\beta_k\sigma(r) - i\beta_k g\,\widetilde{\mathcal{K}}_k[g\cdot].$$

We know that for $|k| = 1$, $\widetilde{\mathcal{H}}_{\alpha,k,0}$ in $\{r^{\frac{3}{2}}g(r)\}^\perp \cap L^2(\mathbb{R}_+, dr)$ is isometric with $T\widetilde{\mathcal{H}}_{\alpha,k,0}T^{-1} = -\partial_r^2 + \frac{3}{4r^2} + \frac{r^2}{16} - \frac{1}{2} + f(r) + i\beta_k\sigma(r)$ in $L^2(\mathbb{R}_+, dr)$. Hence, we just consider $\widetilde{\mathcal{H}}_{\alpha,k}$ in the form

$$\widetilde{\mathcal{H}}_{\alpha,k} = \begin{cases} -\partial_r^2 + \dfrac{3}{4r^2} + \dfrac{r^2}{16} - \dfrac{1}{2} + f(r) + i\beta_k\sigma(r), & |k| = 1,\\ -\partial_r^2 + \dfrac{k^2 - \frac{1}{4}}{r^2} + \dfrac{r^2}{16} - \dfrac{1}{2} + i\beta_k\sigma(r) - i\beta_k g\,\tilde{\mathcal{K}}_k[g\cdot], & |k| \ge 2.\end{cases}$$

Notice that

$$\sigma(L-\alpha\Lambda|_{(\ker\Lambda)^\perp}) = \bigcup_{k\in\mathbb{Z}\setminus\{0\}} \sigma(\widetilde{\mathscr{H}}_{\alpha,k}).$$

Then we define

$$\Sigma(\alpha,k) = \inf \operatorname{Re}\sigma(\widetilde{\mathscr{H}}_{\alpha,k}), \quad \Sigma(\alpha) = \inf_{k\in\mathbb{Z}\setminus\{0\}} \Sigma(\alpha,k).$$

Our main result is the following spectral lower bound.

THEOREM 7.1. – *For any $|k|\geq 1$, we have*

$$\Sigma(\alpha,k)\geq C^{-1}|\beta_k|^{\frac12}, \quad \Sigma(\alpha)\geq C^{-1}|\alpha|^{\frac12}.$$

Motivated by [10], we will use the complex deformation method.

7.1. Complex deformation

We introduce the group of dilations

$$(U_\theta w)(r) = e^{\theta/2} w(e^\theta r),$$

which are unitary operators for $\theta\in\mathbb{R}$. We consider

$$\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k} = U_\theta \widetilde{\mathscr{H}}_{\alpha,k} U_\theta^{-1}. \tag{76}$$

Then we have

$$\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k} = \begin{cases} -e^{-2\theta}\partial_r^2 + \left(\dfrac{3e^{-2\theta}}{4r^2} + \dfrac{r^2e^{2\theta}}{16} - \dfrac12 + f(re^\theta)\right) + i\beta_k\sigma(re^\theta), & |k|=1,\\ -e^{-2\theta}\partial_r^2 + \left(\dfrac{k^2-1/4}{4r^2e^{2\theta}} + \dfrac{r^2e^{2\theta}}{16} - \dfrac12\right) + i\beta_k(\sigma(re^\theta) - e^{2\theta}g(re^\theta)\widetilde{\mathscr{K}}_k[g(re^\theta)\cdot]), & |k|\geq 2.\end{cases}$$

Now we consider the analytic continuation of $\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k}$. For this, we first consider the analytic continuation of the functions f,σ,g. Let

$$F_0(z) = e^z - z - 1, \qquad F_1(z) = (1-e^{-z})/z,$$

$$F_2(z) = e^{-z/2}, \qquad F_3(z) = \left(\frac{2z^2}{F_0(z)} - 3 + 2z\right)\frac{z}{F_0(z)}.$$

Then $F_0,\ F_1,\ F_2$ are holomorphic in $\mathbb{C}$ (0 is a removable singularity of F_1) and F_3 is meromorphic in $\mathbb{C}$, and we have

$$f(r) = F_3(r^2/4), \quad \sigma(r) = F_1(r^2/4), \quad g(r) = F_2(r^2/4).$$

The poles of F_3 are the zeros of F_0. If $F_0(z)=0,\ z=x+iy,\ x,y\in\mathbb{R},\ x>0$, then

$$e^{2x} = |e^z| = |1+z|^2 = (1+x)^2+y^2,$$
$$y^2 = e^{2x}-(1+x)^2 > 1+2x+(2x)^2/2-(1+x)^2 = x^2 \Longrightarrow |y|>x,$$

hence, $F_3(z)$ is holomorphic in a neighborhood of Γ, which is defined as

$$\Gamma = \{x+iy | x>0, -x\leq y\leq x\} = \{re^{i\theta} | r>0, -\pi/4\leq\theta\leq\pi/4\}.$$

Let $F_4(z) = F_3(z) - \frac{2}{z}$. As $\lim_{z\to 0} zF_3(z) = 2$, $F_4(z)$ is holomorphic in a neighborhood of $\Gamma \cup \{0\}$. We have

$$|F_0(z)| \geq |e^z| - |z| - 1 \geq e^{|z|/2} - |z| - 1, \quad z \in \Gamma,$$

and

$$\lim_{z\to\infty, z\in\Gamma} \frac{z^2}{F_0(z)} = 0, \quad \lim_{z\to\infty, z\in\Gamma} F_3(z) = 0, \quad \lim_{z\to\infty, z\in\Gamma} F_4(z) = 0,$$

thus $|F_4(z)| \leq C$ in Γ. We also have

$$F_2(z) \leq C(1+|z|)^{-1}.$$

Now we rewrite $\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k} w$ as follows, for $|k| = 1$,

$$\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k} w = -e^{-2\theta}\partial_r^2 w + \left(\frac{35e^{-2\theta}}{4r^2} + \frac{r^2e^{2\theta}}{16} - \frac{1}{2} + F_4(\frac{r^2e^{2\theta}}{4})\right) w + i\beta_k F_1(\frac{r^2e^{2\theta}}{4})w,$$

and for $|k| \geq 2$,

$$\begin{aligned}\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k} w = &-e^{-2\theta}\partial_r^2 w + \left(\frac{k^2-1/4}{r^2e^{2\theta}} + \frac{r^2e^{2\theta}}{16} - \frac{1}{2}\right) w + i\beta_k (F_1(\frac{r^2e^{2\theta}}{4})w \\ &- e^{2\theta}F_2(\frac{r^2e^{2\theta}}{4})\widetilde{\mathscr{K}}_k[F_2(\frac{r^2e^{2\theta}}{4})w]).\end{aligned}$$

Thanks to the properties of $F_i(z)(i = 0, 1, \ldots, 4)$ which are shown above, $\{\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k}\}$ are defined as an analytic family of type (A)(see [17]) in the strip $\Gamma_1 = \{\theta \in \mathbb{C} \big| |\mathrm{Im}_{,,}| < \frac{1}{8}\}$ with common domain $D = \{w \in H^2(\mathbb{R}_+),\ w/r^2,\ r^2w \in L^2(\mathbb{R}_+)\}$. In particular, the spectrum of $\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k}$ is always discrete and depends holomorphically on θ. Since the eigenvalues of $\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k}$ are constant for $\theta \in \mathbb{R}$, they are also constant for $\theta \in \Gamma_1$.

Now we have

$$\Sigma(\alpha, k) = \inf \mathrm{Re}\sigma(\widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k}) \geq \inf_{w\in D, \|w\|=1} \mathrm{Re}\langle \widetilde{\mathscr{H}}^{(\theta)}_{\alpha,k} w, w\rangle. \tag{77}$$

7.2. Proof of Theorem 7.1

We need the following lemma.

Lemma 7.2. – *For $r > 0, 0 < \theta < \frac{\pi}{4}$, we have*

$$-\mathrm{Im}F_1(re^{i\theta}) \geq C^{-1} \sin\theta \min\big(r, \frac{1}{r}\big).$$

Proof. – Thanks to

$$F_1(re^{i\theta}) = \frac{1 - e^{-re^{i\theta}}}{re^{i\theta}} = \frac{e^{-i\theta} - e^{-re^{i\theta} - i\theta}}{r},$$

we have

$$-\mathrm{Im}F_1(re^{i\theta}) = \frac{F_5(r,\theta)}{r},$$

where

$$F_5(r,\theta) = -\mathrm{Im}(e^{-i\theta} - e^{-re^{i\theta}-i\theta})$$
$$= \sin\theta - e^{-r\cos\theta}\sin(r\sin\theta+\theta).$$

Using the inequality

$$|\sin(r\sin\theta+\theta)| \le |\sin(r\sin\theta)\cos\theta| + |\cos(r\sin\theta)\sin\theta|$$
$$\le r\sin\theta\cos\theta + \sin\theta,$$

we get

$$F_5(r,\theta) \ge \sin\theta(1 - e^{-r\cos\theta}(1+r\cos\theta)). \tag{78}$$

This shows that

$$F_5(r,\theta) \ge C^{-1}\min((r\cos\theta)^2, 1)\sin\theta \ge C^{-1}\min(r^2,1)\sin\theta,$$

thus,

$$-\mathrm{ImF}_1(re^{i\theta}) = \frac{F_5(r,\theta)}{r} \ge C^{-1}\min(r, \frac{1}{r})\sin\theta.$$

This completes the proof. □

Now we are in a position to prove Theorem 7.1.

Let us first consider the case of $|k| = 1$. It follows from Lemma 5.5 that

$$\mathrm{Re}\langle \widetilde{\mathscr{H}}_{\alpha,k} w, w\rangle \ge \frac{\|w\|^2}{2},$$

which gives

$$\Sigma(\alpha,k) \ge 1/2. \tag{79}$$

For $\theta \in (-\pi/8, \pi/8),\ \mathrm{sgn}\theta = \mathrm{sgn}\beta_k$, we have

$$\mathrm{Im}F_1(\frac{r^2e^{2i\theta}}{4}) = \mathrm{sgn}\theta\,\mathrm{Im}F_1(\frac{r^2e^{2i|\theta|}}{4}),$$

from which and Lemma 7.2, we infer that

$$\begin{aligned}\mathrm{Re}\langle \widetilde{\mathscr{H}}^{(i\theta)}_{\alpha,k} w, w\rangle &= \int_{\mathbb{R}_+} \mathrm{Re}\left(\frac{35e^{-2i\theta}}{4r^2} + \frac{r^2e^{2i\theta}}{16} - \frac{1}{2} + F_4(\frac{r^2e^{2i\theta}}{4}) + i\beta_k F_1(\frac{r^2e^{2i\theta}}{4})\right)|w|^2dr \\ &\quad + \cos 2\theta\|\partial_r w\|^2 \\ &\ge \int_{\mathbb{R}_+}\left(\frac{35\cos(2\theta)}{4r^2} + \frac{r^2\cos(2\theta)}{16} - \frac{1}{2} - C - |\beta_k|\mathrm{Im}F_1(\frac{r^2e^{2i|\theta|}}{4})\right)|w|^2dr \\ &\ge \int_{\mathbb{R}_+}\left(C^{-1}(\frac{1}{r^2}+r^2) - C + |\beta_k|C^{-1}\sin|\theta|\min(r^2,\frac{1}{r^2})\right)|w|^2dr \\ &\ge \int_{\mathbb{R}_+}\left(C^{-1}|\beta_k\sin\theta|^{\frac12} - C\right)|w|^2dr = \left(C^{-1}|\beta_k\sin\theta|^{\frac12} - C\right)\|w\|^2,\end{aligned}$$

which shows that

$$\Sigma(\alpha,k) \ge C^{-1}|\beta_k\sin\theta|^{\frac12} - C \ge C^{-1}|\beta_k|^{\frac12} - C,$$

if we take $\theta = (\mathrm{sgn}\beta_k)\frac{\pi}{12}$. Then by (79), we get

$$\Sigma(\alpha, k) \geq \max(C^{-1}|\beta_k|^{\frac{1}{2}} - C, 1/2) \geq C^{-1}|\beta_k|^{\frac{1}{2}}.$$

Next we consider the case of $|k| \geq 2$. We still assume $\theta \in (-\frac{\pi}{8}, \frac{\pi}{8})$, $\mathrm{sgn}\theta = \mathrm{sgn}\beta_k$. Then we have

$$\begin{aligned}\mathrm{Re}\langle \widetilde{\mathscr{H}}^{(i\theta)}_{\alpha,k} w, w\rangle &= \cos 2\theta \|\partial_r w\|^2_{L^2} \\ &+ \int_{\mathbb{R}_+} \mathrm{Re}\left(\frac{k^2 - 1/4}{r^2 e^{-2i\theta}} + \frac{r^2 e^{2i\theta}}{16} - \frac{1}{2} + i\beta_k F_1(\frac{r^2 e^{2i\theta}}{4})\right)|w|^2 dr \\ &+ \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_k(r,s)\mathrm{Re}(-i\beta_k e^{2i\theta} F_2(\frac{r^2 e^{2i\theta}}{4}) F_2(s^2 e^{2i\theta}/4))\overline{w(s)} w(r) dr ds,\end{aligned}$$

here $K_k(r,s) = \frac{1}{2|k|}\min\left(\frac{r}{s}, \frac{s}{r}\right)^{|k|}(rs)^{\frac{1}{2}}$. Notice that

$$\begin{aligned}&\mathrm{Re}\Big(-i\beta_k e^{2i\theta} F_2(\frac{r^2 e^{2i\theta}}{4}) F_2(\frac{s^2 e^{2i\theta}}{4})\Big) = \mathrm{Re}\big(-i\beta_k e^{2i\theta} e^{-(r^2+s^2)\frac{e^{2i\theta}}{8}}\big) \\ &= \beta_k e^{-(r^2+s^2)\frac{\cos(2\theta)}{8}} \sin(2\theta - \frac{(r^2+s^2)\sin(2\theta)}{8}) \\ &= \frac{\beta_k}{\sin(2\theta)} e^{-(r^2+s^2)\frac{\cos(2\theta)}{8}} \Big(\sin(2\theta - \frac{r^2\sin(2\theta)}{8})\sin(2\theta - \frac{s^2\sin(2\theta)}{8}) \\ &\quad - \sin(\frac{r^2\sin(2\theta)}{8})\sin(\frac{s^2\sin(2\theta)}{8})\Big) \\ &= \frac{|\beta_k|}{|\sin(2\theta)|}(g_2(r)g_2(s) - g_3(r)g_3(s)),\end{aligned}$$

where

$$g_2(r) = e^{-r^2\frac{\cos(2\theta)}{8}} \sin\Big(2\theta - \frac{r^2\sin(2\theta)}{8}\Big), \quad g_3(r) = e^{-r^2\frac{\cos(2\theta)}{8}}\sin\Big(\frac{r^2\sin(2\theta)}{8}\Big),$$

and here we used the fact that

$$\sin(a-b-c)\sin a = \sin(a-b)\sin(a-c) - \sin b \sin c.$$

Thus, we obtain

$$\begin{aligned}\mathrm{Re}\langle \widetilde{\mathscr{H}}^{(i\theta)}_{\alpha,k} w, w\rangle &= \cos 2\theta \|\partial_r w\|^2_{L^2} \\ &+ \int_{\mathbb{R}_+}\left(\frac{k^2-1/4}{r^2}\cos(2\theta) + \frac{r^2}{16}\cos(2\theta) - \frac{1}{2} - |\beta_k|\mathrm{Im}\nu_k F_1(\frac{r^2 e^{2i|\theta|}}{4})\right)|w|^2 dr \\ &+ \frac{|\beta_k|}{|\sin(2\theta)|}\big(\langle \widetilde{\mathscr{K}}_k[g_2 w], g_2 w\rangle - \langle \widetilde{\mathscr{K}}_k[g_3 w], g_3 w\rangle\big).\end{aligned}$$

By the proof of Lemma 6.2, we know that

$$\langle \widetilde{\mathscr{K}}_k[g_2 w], g_2 w\rangle \geq 0.$$

Due to $0 < K_k(r,s) \le K_2(r,s)$, we have

$$\begin{aligned}
\langle \widetilde{\mathcal{K}}_k[g_3 w], g_3 w\rangle &\le \int_0^{+\infty}\int_0^{+\infty} K_k(r,s)|g_3 w(s)||g_3 w(r)|dsdr\\
&\le \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} K_2(r,s)\left(\left(\frac{r}{s}\right)^{\frac12} g_3(r)^2|w(s)|^2 + \left(\frac{s}{r}\right)^{\frac12} g_3(s)^2|w(r)|^2\right) dsdr\\
&= \int_0^{+\infty} \widetilde{\mathcal{K}}_2[r^{\frac12} g_3^2](s)\frac{|w(s)|^2}{s^{\frac12}}ds.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\mathrm{Re}\langle \widetilde{\mathcal{H}}_{\alpha,k}^{(i\theta)} w, w\rangle \ge &\int_{\mathbb{R}_+}\left(\frac{k^2-1/4}{r^2}\cos(2\theta) + \frac{r^2}{16}\cos(2\theta) - \frac{1}{2}\right)|w|^2 dr\\
&+ |\beta_k|\int_{\mathbb{R}_+}\left(-\mathrm{Im}F_1(\frac{r^2 e^{2i|\theta|}}{4}) - \frac{\widetilde{\mathcal{K}}_2[r^{\frac12} g_3^2]}{|\sin(2\theta)|r^{\frac12}}\right)|w|^2 dr.
\end{aligned}$$

Due to $0 \le g_3^2(r) \le e^{-r^2\frac{\cos(2\theta)}{4}}|\frac{r^2\sin(2\theta)}{8}|^2$, we have

$$\begin{aligned}
\frac{\widetilde{\mathcal{K}}_2[r^{\frac12} g_3^2]}{|\sin(2\theta)|r^{\frac12}} &\le \frac{|\sin(2\theta)|}{64 r^{\frac12}}\widetilde{\mathcal{K}}_2[r^{\frac92} e^{-r^2\frac{\cos(2\theta)}{4}}]\\
&= \frac{|\sin(2\theta)|}{64 r^{\frac12}}\frac{1}{4}\left(\int_0^r \left(\frac{s}{r}\right)^2 (rs)^{\frac12} s^{\frac92} e^{-s^2\frac{\cos(2\theta)}{4}} ds + \int_r^{+\infty}\left(\frac{r}{s}\right)^2 (rs)^{\frac12} s^{\frac92} e^{-s^2\frac{\cos(2\theta)}{4}} ds\right)\\
&= \frac{|\sin(2\theta)|}{256 r^2}\left(\int_0^r s^7 e^{-s^2\cos(2\theta)/4} ds + r^4\int_r^{+\infty} s^3 e^{-s^2\cos(2\theta)/4} ds\right)\\
&= \frac{|\sin(2\theta)|}{2r^2|\cos(2\theta)|^4}\left(\int_0^a \rho^3 e^{-\rho} d\rho + r^4\left(\frac{\cos(2\theta)}{4}\right)^2\int_a^{+\infty}\rho e^{-\rho} d\rho\right)\\
&= \frac{|\sin(2\theta)|}{2r^2|\cos(2\theta)|^4}\Big(6 - (6 + 6a + 3a^2 + a^3)e^{-a} + a^2(1+a)e^{-a}\Big)\\
&= \frac{|\sin(2\theta)|\left(3 - (3 + 3a + a^2)e^{-a}\right)}{r^2|\cos(2\theta)|^4},
\end{aligned}$$

here $a = \frac{r^2\cos(2\theta)}{4}$ and we used the change of variable $\rho = \frac{s^2\cos(2\theta)}{4}$. On the other hand, thanks to $-\mathrm{Im}F_1(\frac{r^2 e^{2i|\theta|}}{4}) = \frac{F_5(\frac{r^2}{4}, 2|\theta|)}{r^2/4}$, we get by (78) that

$$\begin{aligned}
-\mathrm{Im}F_1(\frac{r^2 e^{2i|\theta|}}{4}) &\ge |\sin(2\theta)|\frac{1 - e^{-r^2\frac{\cos 2\theta}{4}}(1 + r^2\frac{\cos 2\theta}{4})}{r^2/4}\\
&= 4|\sin(2\theta)|\frac{1 - e^{-a}(1+a)}{r^2}.
\end{aligned}$$

Then we have

$$\frac{\widetilde{\mathcal{K}}_2[r^{\frac12} g_3^2]}{|\sin(2\theta)|r^{\frac12}} \le \frac{-3\mathrm{Im}F_1(\frac{r^2 e^{2i|\theta|}}{4})}{4|\cos(2\theta)|^4}.$$

Now we take $\theta = (\mathrm{sgn}\beta_k)\frac{\pi}{24}$, then we have $|\cos(2\theta)|^4 > 3/4$, and

$$\frac{k^2 - 1/4}{r^2}\cos(2\theta) + \frac{r^2}{16}\cos(2\theta) - \frac{1}{2} \ge C^{-1}\big(\frac{1}{r^2} + r^2\big).$$

Then we conclude that

$$\begin{aligned}
\mathrm{Re}\langle \widetilde{\mathscr{H}}^{(i\theta)}_{\alpha,k} w, w\rangle \ge & C^{-1}\int_{\mathbb{R}_+}\left(\frac{1}{r^2} + r^2\right)|w|^2 dr \\
& + |\beta_k|\int_{\mathbb{R}_+}\left(-\mathrm{Im}F_1\big(\frac{r^2 e^{2i|\theta|}}{4}\big) + \frac{3\mathrm{Im}F_1\big(\frac{r^2 e^{2i|\theta|}}{4}\big)}{4|\cos(2\theta)|^4}\right)|w|^2 dr \\
\ge & C^{-1}\int_{\mathbb{R}_+}\left(\frac{1}{r^2} + r^2 - |\beta_k|\mathrm{Im}F_1\big(\frac{r^2 e^{2i|\theta|}}{4}\big)\right)|w|^2 dr \\
\ge & C^{-1}\int_{\mathbb{R}_+}\left(\frac{1}{r^2} + r^2 + |\beta_k||\sin(2\theta)|\min(r^2, \frac{1}{r^2})\right)|w|^2 dr \\
\ge & C^{-1}\int_{\mathbb{R}_+}|\beta_k|^{\frac{1}{2}}|w|^2 dr = C^{-1}|\beta_k|^{\frac{1}{2}}\|w\|^2,
\end{aligned}$$

which shows that for $|k| \ge 2$,

$$\Sigma(\alpha, k) \ge C^{-1}|\beta_k|^{\frac{1}{2}}.$$

Appendix

In this appendix, let us present some properties of the function $\sigma(r) = \frac{1-e^{-r^2/4}}{r^2/4}$.

Lemma A.1. – *It holds that*

1. *for any* $r_0 > 0$,

$$\begin{aligned}
&|\sigma'(r)| \sim |\sigma'(r_0)|, && \frac{r_0}{2} \le r \le 2r_0, \\
&|\sigma(r) - \sigma(r_0)| \gtrsim |r - r_0||\sigma'(r_0)|, && 0 < r \le 2r_0;
\end{aligned}$$

2. *for* $0 < r_0 < 1$ *and* $r_0/2 < r \le 2r_0 + 1$,

$$|\sigma'(r)| \gtrsim |\sigma'(r_0)|, \quad |\sigma(r) - \sigma(r_0)| \gtrsim |r - r_0||\sigma'(r_0)|;$$

3. *for* $r_0 \ge 1$ *and* $|r - r_0| \ge \frac{1}{r_0}$,

$$|\sigma(r) - \sigma(r_0)| \gtrsim \frac{1}{(1+r)^4}.$$

Here $a \sim b$ *means* $ca \le a \le c^{-1}b$ *and* $a \gtrsim b$ *means* $a \ge cb$, *where* c *and* C *are constants independent of* r_0.

Proof. – Let us prove the first property. Thanks to $\sigma'(r) = \frac{2}{r}\big(e^{-\frac{r^2}{4}} - \frac{1-e^{-\frac{r^2}{4}}}{r^2/4}\big)$, we have

$$|\sigma'(r)| \sim \min\big(r, \frac{1}{r^3}\big), \tag{80}$$

which shows that if $r_0 \geq 1$, i.e $r \geq \frac{1}{2}$, then

$$|\sigma'(r)| \sim \frac{1}{r^3} \sim \frac{1}{r_0^3} \sim |\sigma'(r_0)|,$$

and if $r_0 \leq 1$, i.e $r \leq 2$, then

$$|\sigma'(r)| \sim r \sim r_0 \sim |\sigma'(r_0)|.$$

Thus, $|\sigma'(r)| \sim |\sigma'(r_0)|$ for $\frac{r_0}{2} \leq r \leq 2r_0$.

If $\frac{r_0}{2} \leq r \leq 2r_0$, $|\sigma'(\theta r + (1-\theta)r_0)| \sim |\sigma'(r_0)|(0 \leq \theta \leq 1)$. Thus, for some $\theta \in (0,1)$,

$$|\sigma(r) - \sigma(r_0)| = |r - r_0||\sigma'(\theta r + (1-\theta)r_0)| \sim |r - r_0||\sigma'(r_0)|.$$

While, if $0 < r \leq \frac{r_0}{2}$, we get by $\sigma'(r) < 0$ that

$$|\sigma(r) - \sigma(r_0)| \geq \sigma(\frac{r_0}{2}) - \sigma(r_0) \sim \frac{r_0}{2}|\sigma'(r_0)| \sim |r - r_0||\sigma'(r_0)|.$$

The second property could be proved similarly.

Now we prove the third property. If $r_0 \geq 1$ and $r \geq r_0 + \frac{1}{r_0}$, we get by (80) that

$$|\sigma(r) - \sigma(r_0)| \geq \sigma(r_0) - \sigma(r_0 + \frac{1}{r_0}) \gtrsim \frac{1}{r_0}|\sigma'(r_0)| \sim \frac{1}{r_0^4} \gtrsim \frac{1}{(1+r)^4},$$

and if $r_0 \geq 1$, $0 < r \leq r_0 - 1$, then

$$|\sigma(r) - \sigma(r_0)| \geq \sigma(r) - \sigma(r+1) \gtrsim |\sigma'(r+1)| \gtrsim \frac{1}{(1+r)^4},$$

and if $r_0 \geq 1$, $r_0 - 1 < r \leq r_0 - \frac{1}{r_0}$, then

$$|\sigma(r) - \sigma(r_0)| \geq \sigma(r_0 - \frac{1}{r_0}) - \sigma(r_0) \gtrsim \frac{1}{r_0}|\sigma'(r_0)| \sim \frac{1}{r_0^4} \gtrsim \frac{1}{(1+r)^4}.$$

This shows the third property of $\sigma(r)$. $\square$

LEMMA A.2. – *Let* $0 < \nu_1 < 1$ *and* $\sigma(r_1) = \nu_1$. *There exist constants* $c_i \sim 1$ $(i = 1, \ldots, 4)$*, such that for any* $r > 0$*, we have*

1. *if* $r_1 \leq 1$ *and* $1 \leq |\beta_1| \leq \frac{1}{r_1^4}$*, then*

$$c_1 \frac{1}{r^2} + c_2|\beta_1|(\nu_1 - \sigma(r)) \geq |\beta_1|^{\frac{1}{2}};$$

2. *if* $r_1 \geq 1$ *and* $1 \leq |\beta_1| \leq r_1^4$*, then*

$$c_3(1 + r^2) + c_4|\beta_1|(\sigma(r) - \nu_1) \geq |\beta_1|^{\frac{1}{2}};$$

3. *if* $r_1 \geq 1$ *and* $r_1^4 \leq |\beta_1| \leq r_1^6$*, then*

$$c_3(1 + r^2) + c_4 r_1^4(\sigma(r) - \nu_1) \geq |\beta_1|^{\frac{1}{3}}.$$

Proof. – We consider the first case. Let $F(r) = c_1\frac{1}{r^2}+c_2|\beta_1|(\sigma(r_1)-\sigma(r))$, $r_0 = |\beta_1|^{-\frac{1}{4}}$. Then we have $r_1 \le r_0 \le 1$ and

$$F'(r) = -c_2|\beta_1|\sigma'(r) - \frac{2c_1}{r^3}.$$

If we choose $2c_1 = -c_2|\beta_1|\sigma'(r_0)r_0^3$, due to $-\sigma'(r) \sim \min(r, \frac{1}{r^3})$, we have $c_1 \sim c_2$ and $F'(r_0) = 0$. As $-(r^3\sigma'(r))' = r^3 g^2(r) > 0$, we conclude that

$$F'(r) < 0 \text{ for } 0 < r < r_0 \quad \text{and} \quad F'(r) > 0 \text{ for } r > r_0,$$

which implies that

$$\min_{r>0} F(r) = F\big(r_0\big) \ge \frac{c_1}{r_0^2} = c_1|\beta_1|^{\frac{1}{2}}.$$

That is, for $c_1 = 1$ we have,

$$F(r) \ge |\beta_1|^{\frac{1}{2}}.$$

Next we consider the second case. Let $G(r) = c_3(1 + r^2) + c_4|\beta_1|(\sigma(r) - \sigma(r_1))$. Then

$$G(r) \ge c_3(1 + r^2) + \frac{c_4|\beta_1|}{C(1 + r^2)} - \frac{Cc_4|\beta_1|}{r_1^2} \ge C^{-1}(c_3c_4|\beta_1|)^{\frac{1}{2}} - \frac{Cc_4|\beta_1|}{|\beta_1|^{\frac{1}{2}}}.$$

We can choose constants $c_3, c_4 > 0$ such that $C^{-1}(c_3c_4)^{\frac{1}{2}} - Cc_4 = 1$. Then

$$G(r) \ge |\beta_1|^{\frac{1}{2}}.$$

Finally, we prove the third case. Let $H(r) = c_3(1 + r^2) + c_4r_1^4(\sigma(r) - \nu_1)$. Then we have

$$H(r) \ge c_3(1 + r^2) + \frac{c_4r_1^4}{C(1 + r^2)} - \frac{Cc_4r_1^4}{r_1^2} \ge C^{-1}(c_3c_4)^{\frac{1}{2}}r_1^2 - Cc_4r_1^2 = r_1^2 \ge |\beta_1|^{\frac{1}{3}}.$$

The proof is finished. □

Similar to Lemma A.2, we have

LEMMA A.3. – *Let $0 < \nu_k < 1$ and $\sigma(r_k) = \nu_k$. There exist constants $c_i \sim 1$ $(i = 1, \dots, 4)$ such that for any $r > 0$, we have*

1. *if $r_k \le 1$ and $|k|^3 \le |\beta_k| \le \frac{|k|^3}{r_k^4}$, then*

$$c_1\frac{k^2}{r^2} + c_2\min(|\beta_k|, \frac{|k|^2}{r_k^4})(\nu_k - \sigma(r)) \ge |\beta_k|^{\frac{1}{2}};$$

2. *if $r_k \ge \sqrt{k}$ and $|k|^3 \le |\beta_k| \le r_k^4$, then*

$$c_3(1 + r^2) + c_4|\beta_k|(\sigma(r)/2 - \nu_k) \ge |\beta_k|^{\frac{1}{2}};$$

3. *if $r_k \ge \sqrt{k}$ and $r_k^4 \le |\beta_k| \le r_k^6$, then*

$$c_3(1 + r^2) + c_4r_k^4(\sigma(r)/2 - \nu_k) \ge |\beta_k|^{\frac{1}{3}}.$$

Proof. – We consider the first case. Due to $1 \le \min(\frac{|\beta_k|}{|k|^2}, \frac{1}{r_k^4}) \le \frac{1}{r_k^4}$, by Lemma A.2, we have

$$c_1\frac{1}{r^2} + c_2\min(\frac{|\beta_k|}{|k|^2}, \frac{1}{r_k^4})(\nu_k - \sigma(r)) \ge \min(\frac{|\beta_k|}{|k|^2}, \frac{1}{r_k^4})^{\frac{1}{2}} = \min(\frac{|\beta_k|^{\frac{1}{2}}}{|k|}, \frac{1}{r_k^2}),$$

which implies that

$$c_1\frac{k^2}{r^2} + c_2\min(|\beta_k|, \frac{|k|^2}{r_k^4})(\nu_k - \sigma(r)) \ge |k|^2\min(\frac{|\beta_k|^{\frac{1}{2}}}{|k|}, \frac{1}{r_k^2}) \ge \min(|\beta_k|^{\frac{1}{2}}, \frac{|k|^{\frac{3}{2}}}{r_k^2}) \ge |\beta_k|^{\frac{1}{2}}.$$

Next we consider the second case. Let $G(r) = c_3(1+r^2) + c_4|\beta_k|(\sigma(r)/2 - \sigma(r_k))$. Then

$$G(r) \ge c_3(1+r^2) + \frac{c_4|\beta_k|}{C(1+r^2)} - \frac{Cc_4|\beta_1|}{r_k^2} \ge C^{-1}(c_3c_4|\beta_k|)^{\frac{1}{2}} - \frac{Cc_4|\beta_k|}{|\beta_k|^{\frac{1}{2}}}.$$

We can choose constants $c_3, c_4 > 0$ such that $C^{-1}(c_3c_4)^{\frac{1}{2}} - Cc_4 = 1$. Then

$$G(r) \ge |\beta_k|^{\frac{1}{2}}.$$

Finally, we prove the third case. Let $H(r) = c_3(1+r^2) + c_4r_k^4(\sigma(r)/2 - \nu_k)$. Then we have

$$H(r) \ge c_3(1+r^2) + \frac{c_4r_k^4}{C(1+r^2)} - \frac{Cc_4r_1k^4}{r_k^2} \ge C^{-1}(c_3c_4)^{\frac{1}{2}}r_k^2 - Cc_4r_k^2 = r_k^2 \ge |\beta_k|^{\frac{1}{3}}.$$

The proof is finished. □

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PSEUDO-SPLIT FIBERS AND ARITHMETIC SURJECTIVITY

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Abstract. – Let $f : X \to Y$ be a dominant morphism of smooth, proper and geometrically integral varieties over a number field k, with geometrically integral generic fiber. We give a necessary and sufficient geometric criterion for the induced map $X(k_v) \to Y(k_v)$ to be surjective for almost all places v of k. This generalizes a result of Denef which had previously been conjectured by Colliot-Thélène, and can be seen as an optimal geometric version of the celebrated Ax-Kochen theorem.

Résumé. – Soit $f : X \to Y$ un morphisme dominant de variétés lisses, propres et géométriquement intègres définies sur un corps de nombres k, dont la fibre générique est géométriquement intègre. Nous donnons un critère géometrique, à la fois nécessaire et suffisant, pour que l'application induite $X(k_v) \to Y(k_v)$ soit surjective pour presque toute place v de k. Ceci généralise un résultat de Denef précédemment conjecturé par Colliot-Thélène. Notre résultat peut être vu comme une version géométrique optimale du célèbre théorème de Ax-Kochen.

1. Introduction

1.1. – A famous theorem of Ax-Kochen [6] states that any homogeneous polynomial over $\mathbf{Q}_p$ of degree d in at least d^2+1 variables has a non-trivial zero, provided that p avoids a certain finite exceptional set of primes depending only on d. This was originally proved using model theory. Denef recently found purely algebro-geometric proofs [12, 13]. In [13], he did so by proving a more general conjecture of Colliot-Thélène [8, §3, Conjecture].

The essential notion (first introduced by the second author in [32, Definition 0.1]) appearing in this conjecture is that of a *split scheme*:

Definition 1.1. – Let k be a perfect field. A scheme X of finite type over k is called *split* if X contains an irreducible component of multiplicity 1 which is geometrically irreducible.

Here the *multiplicity* of an irreducible component Z of X is the length of the local ring of X at the generic point of Z. In particular, it has multiplicity 1 if and only if it is generically reduced. Denef's result [13, Theorem 1.2] is the following.

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THEOREM 1.2 (Denef). – *Let $f : X \to Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over a number field k, with geometrically integral generic fiber. Assume that for every modification $f' : X' \to Y'$ of f with X' and Y' smooth such that the generic fibers of f and f' are isomorphic, the fiber $(f')^{-1}(D)$ is a split $\kappa(D)$-variety for every $D \in (Y')^{(1)}$.*

Then $Y(k_v) = f(X(k_v))$ for all but finitely many places v of k.

Here k_v denotes the completion of k at the place v, $(Y')^{(1)}$ denotes the set of points of codimension 1 in Y', and $\kappa(D)$ is the residue field of D. A *modification* of f is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\alpha_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\alpha_Y"] & Y, \end{tikzcd} \tag{1.1}$$

where $f' : X' \to Y'$ is a dominant morphism of proper and geometrically integral varieties over k, and $\alpha_X : X' \to X$ and $\alpha_Y : Y' \to Y$ are birational morphisms.

One obtains the Ax-Kochen theorem by applying Theorem 1.2 to the universal family of all hypersurfaces of degree d in $\mathbf{P}^n$ with $n \geq d^2$; that the hypotheses of the theorem are satisfied in this case was shown by Colliot-Thélène (see [8, Remarque 4]).

1.2. – In this paper we strengthen Denef's result, by determining conditions which are both *necessary and sufficient* for the map $f : X(k_v) \to Y(k_v)$ to be surjective for almost all places v. Our result uses the following weakening of Definition 1.1 (in §2.2 we also give a more general definition over arbitrary ground fields).

DEFINITION 1.3. – Let k be a perfect field with algebraic closure $\bar{k}$. A scheme X of finite type over k is called *pseudo-split* if every element of $\mathrm{Gal}(\bar{k}/k)$ fixes some irreducible component of $X \times_k \bar{k}$ of multiplicity 1.

It is clear that pseudo-splitness is weaker than splitness, the latter meaning that a *single* irreducible component of $X \times_k \bar{k}$ of multiplicity 1 is fixed by *all* of $\mathrm{Gal}(\bar{k}/k)$. With this terminology, we can state our generalization of Denef's result as follows:

THEOREM 1.4. – *Let k be a number field. Let $f : X \to Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over k with geometrically integral generic fiber. Then $Y(k_v) = f(X(k_v))$ for all but finitely many places v of k if and only if for every modification $f' : X' \to Y'$ of f, with X' and Y' smooth, and for every point $D \in (Y')^{(1)}$, the fiber $(f')^{-1}(D)$ is a pseudo-split $\kappa(D)$-variety.*

In the notation introduced by the first and third named authors in their recent work [25, §3], the morphisms $f : X \to Y$ satisfying the conclusion of the theorem are exactly the morphisms such that $\Delta(f') = 0$ for every modification f' of f.

1.3. – Theorem 1.4 will be deduced from finer results. With $f : X \to Y$ as in Theorem 1.2, Colliot-Thélène asked in [9, §13.1] how the geometry of f relates to the surjectivity of the map $X(k_v) \to Y(k_v)$, for a possibly infinite collection of places v. He called this phenomenon "surjectivité arithmétique" (note that this is different from the notion of arithmetic surjectivity studied in [16]). We develop general criteria which allow one to decide whether, for an *individual* (but large) place v, the map $X(k_v) \to Y(k_v)$ is surjective. They involve certain invariants which we call "s-invariants," defined in §3—local versions of the δ-invariants introduced in [25, §3]; their definition is given in terms of the geometry of f and does not involve model theory.

The following result is proved in §6 using tools from logarithmic geometry, in particular, a logarithmic version of Hensel's lemma and "weak toroidalisation". It should be viewed as the main theorem of the paper and is a geometric criterion, in the style of Colliot-Thélène's conjecture, for surjectivity of the map $X(k_v) \to Y(k_v)$.

THEOREM 1.5. – *Let k be a number field. Let $f : X \to Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over k, with geometrically integral generic fiber. Then there exist a modification $f' : X' \to Y'$ of f with X' and Y' smooth, and a finite set of places S of k such that for all $v \notin S$ the following are equivalent:*

(1) *the map $X(k_v) \to Y(k_v)$ is surjective;*

(2) *for every codimension 1 point $D' \in (Y')^{(1)}$, we have $s_{f',D'}(v) = 1$.*

The invariants $s_{f',D'}(v)$ appearing in the statement will be defined in §3. They are defined in terms of the Galois action on the irreducible components of the fiber of f' over D'. One benefit of our approach is that it yields a single model for f which can be used to test arithmetic surjectivity using a finite list of criteria.

A simple consequence of Theorem 1.5 is the following:

THEOREM 1.6. – *Let $f : X \to Y$ be a dominant morphism of smooth, proper and geometrically integral varieties over a number field k, with geometrically integral generic fiber. The set of places v such that $Y(k_v) = f(X(k_v))$ is Frobenian.*

Here we use the term "Frobenian" in the sense of Serre [31, §3.3] (see §3.1). Frobenian sets of places have a density, but being Frobenian is much stronger than just having a density; for example, an infinite Frobenian set has positive density. It is also possible to prove Theorem 1.6 using model-theoretic results and techniques such as quantifier elimination [5, 28]; our method avoids these and is completely algebro-geometric. However, we know of no model-theoretic proof of the finer Theorems 1.4 and 1.5. (From a model-theoretic perspective, one may view Theorem 1.5 as an explicit instance of quantifier elimination).

1.4. – Some of the ingredients of our proof are already present in the work of Denef [12, 13], e.g., the use of the weak toroidalisation theorem [4, 3]. We need more ingredients from logarithmic geometry, cf. §5—essentially a few basic properties of log smooth morphisms and log blow-ups. The choice of a log smooth model for the morphism makes some of its arithmetic properties more transparent, and can be seen as a convenient way to come up with a Galois stratification, in the sense of Fried and Sacerdote [14]. On the other hand, we also use work of Serre [31] on Frobenian functions, expanding upon what was done in [25].

Let us finally give an overview of the structure of our paper. In §2 we introduce the class of *pseudo-split varieties* and discuss their elementary properties; this section also includes some examples involving torsors under coflasque tori. In §3, we introduce the "s-invariants". These allow us to prove in §4 that our geometric conditions are *necessary* for arithmetic surjectivity. To prove that this criterion is also *sufficient*, we introduce the necessary logarithmic tools in §5. We finish the proof of our main result in §6.

1.5. Notation

A variety over a field k is a separated k-scheme of finite type.

Let k be a number field. We denote by $\Omega_{k,\mathrm{f}}$ the set of finite places of k. Given a place v of k, we write k_v for the completion of k at v. If v is non-archimedean, then we denote by $\mathcal{O}_v$ the ring of integers of k_v, by $\mathbf{F}_v$ its residue field, and by $\mathrm{N}(v) = \#\mathbf{F}_v$ its norm.

For a variety X over a number field k, a *model* of X is a scheme $\mathscr{X}$ of finite type over $\mathcal{O}_k$ together with a choice of isomorphism $X \cong \mathscr{X} \times_{\mathcal{O}_k} k$. For a morphism $f : X \to Y$ of k-varieties, a *model* of f is a morphism $f : \mathscr{X} \to \mathscr{Y}$ of finite type over $\mathcal{O}_k$, again denoted by f, of models of X and Y, such that the induced map on generic fibers is identified with the original morphism $X \to Y$ via the isomorphisms $\mathscr{X} \times_{\mathcal{O}_k} k \cong X$ and $\mathscr{Y} \times_{\mathcal{O}_k} k \cong Y$.

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2. Pseudo-split varieties

In this section, we collect some observations on the characterisation and properties of pseudo-split algebras and pseudo-split varieties.

2.1. Pseudo-split algebras

Let k be a field, with algebraic closure $\bar{k}$, and let A be a finite étale k-algebra; then $A = \prod_{i=1}^n k_i$ for some finite separable field extensions k_i/k. Write $d_i = [k_i : k]$. Let K_i/k be the Galois closure of k_i/k in $\bar{k}$. Let K be the compositum of $K_1, \dots, K_n$, i.e., the smallest subfield of $\bar{k}$ containing these fields.

Write $G = \mathrm{Gal}(K/k)$, $G_i = \mathrm{Gal}(K_i/k)$ and $H_i = \mathrm{Gal}(K_i/k_i)$. The index of H_i in G_i is equal to d_i. By the normal basis theorem the G-module $A \otimes_k K$ is identified with $\bigoplus_{i=1}^n K[G_i/H_i]$, where G acts on G_i/H_i through the natural homomorphism $G \to G_i$. We have $(\mathrm{Spec}\, A)(\bar{k}) = (\mathrm{Spec}\, A)(K) = \coprod_{i=1}^n G_i/H_i$.

Definition 2.1. – The following conditions are equivalent:

(1) $A = k \oplus A'$ for some k-algebra A',

(2) the natural morphism $\operatorname{Spec} A \to \operatorname{Spec} k$ has a section,

(3) at least one point of $(\operatorname{Spec} A)(\bar{k})$ is fixed by G.

If these conditions are satisfied, we say that A is *split*. For example, a separable polynomial $p(X) \in k[X]$ has a root in k if and only if $k[X]/(p(X))$ is a split k-algebra.

Definition 2.2. – The following conditions are equivalent:

(1) $A \otimes_k F$ is split for each field F such that $k \subseteq F \subseteq K$ and K/F is cyclic,

(2) each element of G fixes at least one point of $(\operatorname{Spec} A)(\bar{k})$.

If these conditions are satisfied, we say that A is *pseudo-split*.

Condition (2) in the above definition can be rephrased by saying that G is the union of the stabilizers of points of $\coprod_{i=1}^n G_i/H_i$. Let $\widetilde{H}_i \subset G$ be the inverse image of H_i under the surjective homomorphism $G \to G_i$. Then $G_i/H_i = G/\widetilde{H}_i$ with its natural G-action. Thus condition (2) is equivalent to the equality

$$G = \bigcup g\widetilde{H}_i g^{-1}, \tag{2.1}$$

where the union is taken over all $g \in G$ and $i = 1, \ldots, n$.

Remark 2.3. – Pseudo-split algebras naturally arise in the study of the Hasse principle for finite schemes over a number field (see e.g., [20, Lemma 2.2] and [33, Proposition 1]).

Remark 2.4. – By Jordan's theorem [30, Theorem 4], any transitive subgroup of a permutation group contains a permutation without fixed points. Hence if A is a pseudo-split k-algebra such that $\operatorname{Spec} A$ is connected, then $A = k$.

Remark 2.5. – One immediately sees that if a k-algebra A is pseudo-split, then for any field extension L/k the L-algebra $A \otimes_k L$ is pseudo-split.

Remark 2.6. – If G is cyclic, then any pseudo-split k-algebra is in fact split.

The following gives a description of pseudo-split algebras over number fields. The proof is an exercise using the Chebotarev density theorem; we omit it as the result will not be used in the sequel (it is also a special case of Proposition 2.11).

Proposition 2.7. – *Let k be a number field. Then a finite étale k-algebra is pseudo-split if and only if it is split over almost all completions of this field. In particular, a separable polynomial $p(x) \in k[x]$ has a root in almost all completions of k if and only if $k[x]/(p(x))$ is a pseudo-split k-algebra.*

See Lemma 3.11 for a variant of this for finitely generated fields over $\mathbf{Q}$. Returning to a general field k one can classify all pseudo-split non-split k-algebras as follows.

PROPOSITION 2.8. – *Let K/k be a Galois extension. Let $E_1, \dots, E_n$ be subgroups of $G = \mathrm{Gal}(K/k)$ such that G is the union of $gE_i g^{-1}$ for all $g \in G$ and $i = 1, \dots, n$. Then $\bigoplus_{i=1}^n K^{E_i}$ is a pseudo-split k-algebra, and all pseudo-split k-algebras A such that $A \otimes_k K$ is isomorphic to $K^{\dim A}$ are obtained in this way. Under this bijection, the non-split k-algebras are those for which $E_1, \dots, E_n$ are proper subgroups of G.*

Proof. – Write $A = \bigoplus_{i=1}^n K^{E_i}$. Then $(\mathrm{Spec}\, A)(\overline{k})$ is the set $\coprod_{i=1}^n G/E_i$. The subgroups $gE_i g^{-1} \subset G$ are precisely the G-stabilizers of points of this set. This shows that A is pseudo-split. Conversely, each pseudo-split k-algebra A can be written in our previous notation as the direct sum of $k_i = K^{\widetilde{H}_i}$, where the subgroups $\widetilde{H}_i$ satisfy (2.1). It is clear that A is split if and only if $E_i = G$ for some i. □

Any non-cyclic Galois extension K/k gives rise to at least one pseudo-split non-split k-algebra: take $E_1, \dots, E_n$ to be all cyclic subgroups of $\mathrm{Gal}(K/k)$.

EXAMPLE 2.9. – Let K/k be a Galois extension such that $G = \mathrm{Gal}(K/k) \cong D_n$ is the dihedral group of degree n, where n is odd. Then $D_n = \mathbf{Z}/n \rtimes \mathbf{Z}/2$, and D_n is the union of $E_1 = \mathbf{Z}/n$ and the conjugates of $E_2 = \mathbf{Z}/2$. Hence we obtain a pseudo-split non-split algebra $A = k_1 \oplus k_2$, where $[k_1 : k] = 2$ and $[k_2 : k] = n$. For $n = 3$, we obtain a 5-dimensional pseudo-split non-split k-algebra; this is the smallest possible dimension of such an algebra. If $k = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt{-3}, \sqrt[3]{2})$, then $(X^2 + 3)(X^3 - 2)$ is solvable in all completions of $\mathbf{Q}$ except $\mathbf{Q}_2$ and $\mathbf{Q}_3$.

2.2. Pseudo-split varieties

Let k be a field (not necessarily perfect). For a k-variety X, we denote by X_{sm} the maximal open subscheme of X which is smooth over k. We let $X_{\mathrm{sm},1}, \dots, X_{\mathrm{sm},n}$ be the irreducible components of X_{sm}. Let k_i be the algebraic closure of k in the function field $k(X_{\mathrm{sm},i})$, for $i = 1, \dots, n$. Consider the finite k-algebra

$$A_X = \bigoplus_{i=1}^n k_i. \tag{2.2}$$

We call $\mathrm{Spec}\, A_X$ the *scheme of irreducible components of geometric multiplicity* 1 of X. The map $X_{\mathrm{sm}} \to \mathrm{Spec}\, k$ factors as $X_{\mathrm{sm}} \to \mathrm{Spec}\, A_X \to \mathrm{Spec}\, k$, where $X_{\mathrm{sm}} \to \mathrm{Spec}\, A_X$ has geometrically integral fibers. Moreover, $\mathrm{Spec}\, A_X$ is smooth over k by [1, Lemma 34.11.5, Tag 05B5], thus $\mathrm{Spec}\, A_X$ is finite and étale over k.

DEFINITION 2.10. – We say that X is *split* if the finite étale k-algebra A_X is split in the sense of Definition 2.1. Similarly, we say that X is *pseudo-split* if the finite étale k-algebra A_X is pseudo-split in the sense of Definition 2.2.

If $X_{\mathrm{sm}} = \emptyset$, then (2.2) is the empty direct sum, hence A_X is the zero ring and $\mathrm{Spec}\, A_X$ is the empty scheme. In this case it follows easily from the definitions that X is both non-split and non-pseudo-split.

The definitions in Definition 2.10 are easily checked to be equivalent to Definitions 1.1 and 1.3 over a perfect field. In this case, in the notation of [25, §3.2], being pseudo-split is equivalent to having $\delta(X) = 1$. In the case where k is a number field, we obtain the following characterisation.

PROPOSITION 2.11. – *Let k be a number field. A k-variety is pseudo-split if and only if it has a smooth k_v-point for almost all completions k_v of k.*

Proof. – This follows immediately from [25, Lemma 3.9]. □

Another immediate consequence of the results in [25, §3.2] is the following.

PROPOSITION 2.12. – *Let R be a discrete valuation ring with a perfect residue field k. Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be regular schemes which are proper over R, and whose generic fibers are birational. Then the special fiber of $\mathscr{X}_1$ is pseudo-split if and only if the special fiber of $\mathscr{X}_2$ is pseudo-split.*

Proof. – This is a special case of [25, Lemma 3.11]. □

2.3. Pseudo-split algebras and coflasque tori

We make some observations in the style of Colliot-Thélène's paper [10]. Given any finite étale k-algebra $A = \bigoplus_{i=1}^n k_i$, one defines the associated multinorm k-torus $\mathrm{R}^1_{A/k}\,\mathbf{G}_\mathrm{m}$ by the exact sequence

$$1 \to \mathrm{R}^1_{A/k}\,\mathbf{G}_\mathrm{m} \to \prod_{i=1}^n \mathrm{R}_{k_i/k}\,\mathbf{G}_m \to \mathbf{G}_m \to 1, \tag{2.3}$$

where the third map is induced by the norms from k_i to k. In [10], Colliot-Thélène studied the special case $A = k(\sqrt{a}) \oplus k(\sqrt{b}) \oplus k(\sqrt{ab})$, where $a, b \in k^*$ are such that none of a, b and ab is a square and $\mathrm{char}(k) \neq 2$; this algebra is clearly pseudo-split.

PROPOSITION 2.13. – *If A is a pseudo-split k-algebra, then $\mathrm{R}^1_{A/k}\,\mathbf{G}_\mathrm{m}$ is a coflasque k-torus. If k is a local field, then $\mathrm{H}^1(k, \mathrm{R}^1_{A/k}\,\mathbf{G}_\mathrm{m}) = 0$.*

We refer to [11, §1] for the definition of (and some background on) coflasque tori.

Proof. – Let M be the module of characters of $\mathrm{R}^1_{A/k}\,\mathbf{G}_\mathrm{m}$ and let G be as in §2.1. To prove that $\mathrm{R}^1_{A/k}\,\mathbf{G}_\mathrm{m}$ is coflasque we need to show that $\mathrm{H}^1(H, M) = 0$ for each subgroup $H \subset G$. By Remark 2.5, we can assume without loss of generality that $H = G$.

Consider the exact sequence of Galois-modules dual to (2.3)

$$0 \to \mathbf{Z} \to \bigoplus_{i=1}^n \mathbf{Z}[G/\widetilde{H}_i] \to M \to 0,$$

where we use the notation of (2.1). The Galois action factors through the action of G, and $\mathrm{H}^1(G, \mathbf{Z}[G/\widetilde{H}_i]) = 0$. To prove that $\mathrm{H}^1(G, M) = 0$, we must therefore show that any element of the group $\mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z}) = \mathrm{H}^2(G, \mathbf{Z})$ which vanishes when restricted to each $\widetilde{H}_i$, is zero. This follows from (2.1), proving the first statement. The second statement is a general property of coflasque tori, as over the local field k the finite abelian groups $\mathrm{H}^1(k, \mathrm{R}^1_{A/k}\,\mathbf{G}_\mathrm{m})$ and $\mathrm{H}^1(k, M) = \mathrm{H}^1(G, M)$ are dual to each other by Tate-Nakayama duality [26, Corollary I.2.4, Thm. I.2.13]. □

Let k be a number field and let $A = \bigoplus_{i=1}^{n} k_i$ be a pseudo-split k-algebra such that the extensions k_i/k satisfy $\gcd([k_1 : k], \dots, [k_n : k]) \neq 1$. The family of torsors for $\mathbf{R}^1_{A/k}\,\mathbf{G}_\mathrm{m}$

$$\prod_{i=1}^{n} N_{k_i/k}(x_i) = t \neq 0 \tag{2.4}$$

can be compactified to a smooth, proper, geometrically integral variety with a morphism $\pi : X \to \mathbf{P}^1_k$ extending the projection to the coordinate t.

The map $X(k_v) \to \mathbf{P}^1(k_v)$ is surjective for *all* places v of k. For smooth fibers this follows from Proposition 2.13. However since π is proper, the image $\pi(X(k_v))$ is closed, hence $\pi(X(k_v)) = \mathbf{P}^1(k_v)$ as claimed. The singular fibers are pseudo-split, but non-split. That they are pseudo-split is clear; that they are non-split follows from [25, Lemma 5.4]. In particular, this surjectivity is not implied by Denef's result (Theorem 1.2), but is implied by our Theorem 1.4—at least Theorem 1.4 gives surjectivity for all but finitely many v. It was this family of examples which in fact originally motivated Definition 1.3.

3. Splitting densities

Let k be a number field. In this section we will introduce the "s-invariants"; these are certain explicit Frobenian functions which, for a morphism of k-varieties $f : X \to Y$, measure the "density" of the split fibers of $X_{k_v} \to Y_{k_v}$ as v varies.

3.1. Frobenian functions

We first recall some of the theory of Frobenian functions, following Serre's treatment in [31, §3.3]. Recall that a function $\varphi : \Gamma \to \mathbf{C}$ on a group Γ is called a *class function* if it is constant on each conjugacy class. We denote by $\Omega_{k,\mathrm{f}}$ the set of finite places of k.

Definition 3.1. – A *Frobenian function* is a map $s : \Omega_{k,\mathrm{f}} \to \mathbf{C}$ satisfying the following properties. There exist a finite Galois extension K/k with Galois group Γ, a finite set of places $S \subset \Omega_{k,\mathrm{f}}$ and a class function $\varphi : \Gamma \to \mathbf{C}$ such that:

(1) K/k is unramified outside of S;

(2) $s(v) = \varphi(\mathrm{Frob}_v)$ for all $v \notin S$.

A subset of $\Omega_{k,\mathrm{f}}$ is called *Frobenian* if its indicator function is Frobenian.

Given $v \in \Omega_{k,\mathrm{f}}$ and a place $w \in \Omega_{K,\mathrm{f}}$ above v, we denote by $\mathrm{Frob}_{w/v} \in \Gamma$ the associated Frobenius element. In Definition 3.1, we adopt a common abuse of notation (see [31, §3.2.1]), and denote by $\mathrm{Frob}_v \in \Gamma$ the choice of such an element for some w. Note that $\varphi(\mathrm{Frob}_v)$ is well-defined as φ is a class function.

Example 3.2. – Let E/k be a finite extension of number fields. Then the set of all prime ideals of $\mathcal{O}_k$ which split completely in E is Frobenian: in Definition 3.1, one takes K to be the Galois closure of E/k with Galois group Γ, S the set of primes which ramify in K/k and $\varphi : \Gamma \to \mathbf{C}$ the indicator of the identity element of Γ.

For any function $s : \Omega_{k,\mathfrak{f}} \to \mathbf{C}$, we define its *density* to be

$$\mathrm{dens}(s) = \lim_{B\to\infty} \frac{\sum_{v\in\Omega_{k,\mathfrak{f}},\, \mathrm{N}(v)\leq B} s(v)}{B/\log B},$$

if the limit exists. The density of a subset of $\Omega_{k,\mathfrak{f}}$ is defined to be the density of its indicator function. If s is Frobenian with associated class function $\varphi : \Gamma \to \mathbf{C}$, then we define its *mean* to be

$$m(s) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\varphi(\gamma)$$

(this does not depend on the choice of φ). A simple application of the Chebotarev density theorem (see [31, §3.3.3.5]) shows that in this case $\mathrm{dens}(s)$ exists and

$$m(s) = \mathrm{dens}(s). \tag{3.1}$$

In particular, a Frobenian set has positive density if and only if it is infinite.

3.2. s-invariants

We now define our s-invariants and study their properties.

3.2.1. *Set-up*. – Let k be a number field, let K be a finitely generated field extension of k and let I be a non-empty finite étale K-scheme. We associate to this situation some group theoretic data as follows.

The K-algebra $K(I)$ is finite étale over K. Let L be a finite Galois extension of K such that $K(I)\otimes_K L \cong L^d$ for some $d\in\mathbf{N}$. The Galois group $G = \mathrm{Gal}(L/K)$ acts on $I(L)$ in a natural way. We let k_L (resp. k_K) be the algebraic closure of k in L (resp. K).

If k_L/k is not Galois then we change L as follows: let M be a Galois closure of k_L/k and let $L_M := L\otimes_{k_L} M$. Note that L_M is a field as k_L is algebraically closed in L. Moreover L_M is clearly still Galois over K and the algebraic closure of k_L in L_M is M. In conclusion, replacing L by L_M if necessary, we may assume that k_L/k is Galois.

Let N be the normal subgroup of G which acts trivially on k_L. Define $\Gamma = \mathrm{Gal}(k_L/k_K)$ and $\Lambda = \mathrm{Gal}(k_L/k)$. Note that $G/N = \Gamma \subset \Lambda$. We summarize this set-up with the following commutative diagram of field extensions and Galois groups.

$$\begin{tikzcd} k \arrow[r, dash] \arrow[d, equal] & K \arrow[r, dash, "G"] \arrow[d, dash] & L \arrow[d, dash] \\ k \arrow[r, dash] \arrow[rr, dash, bend right, "\Lambda"'] & k_K \arrow[r, dash, "\Gamma=G/N"] & k_L. \end{tikzcd} \tag{3.2}$$

Definition 3.3. – With the above set-up, let $v\in\Omega_{k,\mathfrak{f}}$.

— If v ramifies in k_L or there is no place w of k_K of degree 1 over v, then $s_I(v) := 1$.

— Otherwise, we set

$$s_I(v) := \frac{\sum_{\substack{w\in\Omega_{k_K}\\ \mathrm{N}\,w=\mathrm{N}\,v\\ w|v}} \frac{1}{|N|}\#\left\{g\in G : \begin{array}{l} g \bmod N = \mathrm{Frob}_w \text{ and}\\ g \text{ acts with a fixed point on } I(L)\end{array}\right\}}{\#\{w\in\Omega_{k_K} : w\mid v, \mathrm{N}\,w=\mathrm{N}\,v\}}. \tag{3.3}$$

Note that as

$$\#\left\{g \in G : \begin{array}{l} g \bmod N = \mathrm{Frob}_w \text{ and} \\ g \text{ acts with a fixed point on } I(L) \end{array}\right\} \leq |N|,$$

we see that Definition 3.3 yields a well-defined function $s_I : \Omega_{k,\mathrm{f}} \to [0,1] \cap \mathbf{Q}$.

3.2.2. *Basic properties*. – Let us first give a purely group-theoretic formula for the invariant which we introduced in Definition 3.3. For a subset $Z \subset H$ of a group H, we denote by $C_H(Z)$ the smallest subset of H that is stable under conjugacy and contains Z, and by $\mathrm{Cl}_H(Z)$ the set of conjugacy classes in $C_H(Z)$.

LEMMA 3.4. – *If v ramifies in k_L or $C_\Lambda(\mathrm{Frob}_v) \cap \Gamma = \emptyset$, then $s_I(v) = 1$. Otherwise*

$$s_I(v) = \frac{\sum_C \frac{1}{|C| \cdot |N|} \#\left\{g \in G : \begin{array}{l} g \bmod N \in C \text{ and} \\ g \text{ acts with a fixed point on } I(L) \end{array}\right\}}{\#\,\mathrm{Cl}_\Gamma(C_\Lambda(\mathrm{Frob}_v) \cap \Gamma)}, \tag{3.4}$$

where the sum is over $C \in \mathrm{Cl}_\Gamma(C_\Lambda(\mathrm{Frob}_v) \cap \Gamma)$.

Proof. – We claim that

$$C_\Lambda(\mathrm{Frob}_v) \cap \Gamma = \bigsqcup_{\substack{w \in \Omega_{k_K} \\ \mathrm{N}\,w = \mathrm{N}\,v \\ w \mid v}} C_\Gamma(\mathrm{Frob}_w). \tag{3.5}$$

Indeed, let u be a finite place of k_L over v and let w be its restriction to k_K. Then

$$\mathrm{Frob}_{u/w} = \mathrm{Frob}_{u/v}^{[\mathbf{F}_w:\mathbf{F}_v]} \in \Gamma.$$

It follows that if $[\mathbf{F}_w : \mathbf{F}_v] = 1$, then $\mathrm{Frob}_{u/w} \in C_\Lambda(\mathrm{Frob}_v)$, so the left hand side of (3.5) contains the right hand side. Conversely, if $\mathrm{Frob}_{u/v} \in \Gamma$, then $\mathrm{Frob}_{u/v}$ leaves k_K invariant, hence fixes w. Therefore $\mathrm{N}\,w = \mathrm{N}\,v$ and $\mathrm{Frob}_{u/w} = \mathrm{Frob}_{u/v}$, whence (3.5).

Using (3.5) we find that

$$\#\{w \in \Omega_{k_K} : w \mid v, \mathrm{N}\,w = \mathrm{N}\,v\} = \#\,\mathrm{Cl}_\Gamma(C_\Lambda(\mathrm{Frob}_v) \cap \Gamma). \tag{3.6}$$

This shows that $s_I(v) = 1$ if $C_\Lambda(\mathrm{Frob}_v) \cap \Gamma = \emptyset$ or if v ramifies in k_L, by definition. So assume that we are not in these cases. By (3.6) we see that the denominators in (3.3) and (3.4) agree. As for the numerators, using (3.5) we obtain

$$\sum_{\substack{w \in \Omega_{k_K} \\ \mathrm{N}\,w = \mathrm{N}\,v \\ w \mid v}} \frac{1}{|N|} \#\left\{g \in G : \begin{array}{l} g \bmod N = \mathrm{Frob}_w \text{ and} \\ g \text{ acts with a fixed point on } I(L) \end{array}\right\}$$

$$= \sum_{C \in \mathrm{Cl}_\Gamma(C_\Lambda(\mathrm{Frob}_v) \cap \Gamma)} \sum_{\gamma \in C} \frac{1}{|C| \cdot |N|} \#\left\{g \in G : \begin{array}{l} g \bmod N = \gamma \text{ and} \\ g \text{ acts with a fixed point on } I(L) \end{array}\right\}$$

$$= \sum_{C \in \mathrm{Cl}_\Gamma(C_\Lambda(\mathrm{Frob}_v) \cap \Gamma)} \frac{1}{|C| \cdot |N|} \#\left\{g \in G : \begin{array}{l} g \bmod N \in C \text{ and} \\ g \text{ acts with a fixed point on } I(L) \end{array}\right\}$$

as required. □

COROLLARY 3.5. – *The function*

$$s_I : \Omega_{k,\mathrm{f}} \to [0,1] \cap \mathbf{Q}, \quad v \mapsto s_I(v),$$

is Frobenian.

Proof. – This follows from Lemma 3.4, which shows that $s_I(v)$ only depends on the conjugacy class of $\mathrm{Frob}_v \in \Lambda$ for all v which are unramified in k_L. □

The invariant $s_I(v)$ simplifies in special cases.

EXAMPLE 3.6. – Let v be unramified in k_L.

1. Assume that $k = k_K$, i.e., K is geometrically irreducible. Then

$$s_I(v) = \frac{\#\left\{g \in G : \begin{array}{l} g \bmod N = \mathrm{Frob}_v \text{ and} \\ g \text{ acts with a fixed point on } I(L) \end{array}\right\}}{|N|}.$$

2. Assume that $N = 0$, e.g., K is a number field. If there is no place of k_K of degree 1 over v then $s_I(v) = 1$. Otherwise

$$s_I(v) = \frac{\#\left\{w \in \Omega_{k_K} : \begin{array}{l} w \mid v, \mathrm{N}\,w = \mathrm{N}\,v, \text{ and} \\ \mathrm{Frob}_w \text{ acts with a fixed point on } I(L) \end{array}\right\}}{\#\{w \in \Omega_{k_K} : w \mid v, \mathrm{N}\,w = \mathrm{N}\,v\}}.$$

In special cases, one can relate the s-invariants to the δ-invariants from [25].

LEMMA 3.7. – *Assume that* $k = k_K$. *Then*

$$\mathrm{dens}(s_I) = \frac{\#\{g \in G : g \text{ acts with a fixed point on } I(L)\}}{\#G}.$$

In particular $\mathrm{dens}(s_I) = \delta(I)$ *in the notation of* [25, §3.2].

Proof. – Since $k = k_K$, we have $\Lambda = \Gamma = G/N$. Therefore Example 3.6 and the Chebotarev density Theorem (3.1) imply that the density in question equals

$$\begin{aligned} &\frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} \frac{\#\{g \in G : g \bmod N = \gamma \text{ and } g \text{ acts with a fixed point on } I(L)\}}{|N|} \\ &= \frac{1}{|\Gamma| \cdot |N|} \sum_{\substack{g \in G \\ g \text{ acts with a fixed point on } I(L)}} \#\{\gamma \in \Gamma : g \bmod N = \gamma\} \\ &= \frac{\#\{g \in G : g \text{ acts with a fixed point on } I(L)\}}{\#G}, \end{aligned}$$

as required. □

EXAMPLE 3.8. – In general, one can have $\mathrm{dens}(s_I) \neq \delta(I)$ when $k \neq k_K$. Indeed, let $a, b \in k^*$ be such that $a, b, ab \notin k^{*2}$. Take $K = k(\sqrt{b})$ and $I = \mathrm{Spec}\, k(\sqrt{a}, \sqrt{b})$ over K. In this case we have $G = \mathbf{Z}/2\mathbf{Z}$, $\Gamma = G = \mathbf{Z}/2\mathbf{Z}$, $\Lambda = (\mathbf{Z}/2\mathbf{Z})^2$ and $N = 0$. Using Example 3.6, one easily checks that $3/4 = \mathrm{dens}(s_I) \neq \delta(I) = 1/2$.

3.2.3. *Determining when* $s_I(v) = 1$. – Of particular interest to us will be the set of places with $s_I(v) = 1$. Here we have the following criterion.

LEMMA 3.9. – *Let* v *be a place of* k *which is unramified in* k_L. *Then* $s_I(v) < 1$ *if and only if there exists some* $g \in G$ *such that*

(1) g *does not act with a fixed point on* $I(L)$,

and such that g *satisfies one of the following equivalent conditions.*

(2) *There is a place* w *of* k_K *of degree* 1 *over* v *such that* $g \bmod N = \mathrm{Frob}_w$.

(3) $g \bmod N \in C_\Lambda(\mathrm{Frob}_v) \cap \Gamma$.

Proof. – This follows immediately from Definition 3.3 and Lemma 3.4 □

LEMMA 3.10. – *The set*

$$\{v \in \Omega_{k,\mathrm{f}} : s_I(v) = 1\} \tag{3.7}$$

is Frobenian; its density is equal to

$$\frac{1}{|\Lambda|}\#\left\{\lambda \in \Lambda : \begin{array}{l} \textit{each } g \in G \textit{ with } g \bmod N \in C_\Lambda(\lambda) \cap \Gamma \\ \textit{acts with a fixed point on } I(L) \end{array}\right\}.$$

Proof. – That (3.7) is Frobenian follows immediately from Corollary 3.5. The density is easily calculated using the Chebotarev density Theorem (3.1) and Lemma 3.9. □

We now relate the s-invariants to the notion of pseudo-splitness introduced in §2.

LEMMA 3.11. – *With notation as in §*3.2.1*, the finite étale* K*-scheme* I *is pseudo-split over* K *if and only if* $s_I(v) = 1$ *for all but finitely many places* v *of* k.

Proof. – It follows easily from Lemma 3.10 that the set (3.7) has density 1 if and only if every element of G acts with a fixed point of $I(L)$, i.e., if and only if I is pseudo-split. As (3.7) is Frobenian, this completes the proof. □

3.3. s-invariants in families

We now consider s-invariants in families and give an application to splitting densities over finite fields. We begin with some formalities concerning irreducible components in families.

3.3.1. *Irreducible components*. – For a morphism of schemes $X \to Y$ we denote by $\mathrm{Irr}_{X/Y}$ the *functor of open irreducible components* of X over Y, defined by Romagny in [29, Définition 2.1.1]. We call $\mathrm{Irr}^1_{X/Y} := \mathrm{Irr}_{X_{\mathrm{sm}}/Y}$ the *subfunctor of open irreducible components of* X *over* Y *of geometric multiplicity* 1, where X_{sm} denotes the maximal open subscheme of X which is smooth over Y. This parametrises those components of the fibers of f which are geometrically generically reduced.

LEMMA 3.12. – *Let* $X \to Y$ *be a morphism of schemes of finite presentation, with* Y *irreducible. Then there exists a dense open subset* $U \subset Y$ *such that the restrictions of* $\mathrm{Irr}_{X/Y}$ *and* $\mathrm{Irr}^1_{X/Y}$ *to* U *are representable by finite étale* U*-schemes.*

Proof. – It clearly suffices to prove the result for $I := \mathrm{Irr}_{X/Y}$. By [29, Lemmes 2.1.2 et 2.1.3], there exists a dense open $U \subset Y$ such that $I|_U$ is representable by a quasi-compact algebraic space étale over U. Moreover by [29, Proposition 2.1.4], the generic fiber of I is a finite affine scheme. In particular the generic fiber is separated. As being separated is a constructible property (for schemes this is [18, Proposition 9.6.1]; the case of algebraic spaces follows from similar arguments to those given in [29, §A]), on shrinking U we may assume that $I|_U \to U$ is separated. Thus I is a scheme by Knutson's criterion [23, Corollary II.6.17]. Finally, as the generic fiber of $I|_U \to U$ is finite, we may again shrink U further to assume that $I|_U \to U$ is finite étale, as required. □

3.3.2. *Definition*. – We can now define s-invariants in families. Let $f : X \to Y$ be a morphism of schemes of finite type over a number field k. Let $y \in Y$ be an arbitrary point. The residue field $\kappa(y)$ is a finitely generated extension of k. By Lemma 3.12, the functor $\mathrm{Irr}^1_{f^{-1}(y)/\kappa(y)}$ of irreducible components of multiplicity 1 of the fiber $f^{-1}(y)$ is representable by a finite étale $\kappa(y)$-scheme. If this scheme happens to be empty, define $s_{f,y}(v) = 0$ for all $v \in \Omega_{k,\mathrm{f}}$. Otherwise, we are in the set-up of §3.2.1, with $K = \kappa(y)$ and $I = \mathrm{Irr}^1_{f^{-1}(y)/\kappa(y)}$. Hence, given $v \in \Omega_{k,\mathrm{f}}$, we define

$$s_{f,y}(v) := s_{\mathrm{Irr}^1_{f^{-1}(y)/\kappa(y)}}(v) \tag{3.8}$$

using the notation of Definition 3.3.

3.3.3. *Splitting densities over finite fields*. – Our next proposition is the main result of this section, and shows how s-invariants arise "in nature".

PROPOSITION 3.13. – *Let $f : X \to Y$ be a morphism of schemes of finite type over a number field k, with Y integral. Let $f : \mathscr{X} \to \mathscr{Y}$ be a model of $X \to Y$ over $\mathcal{O}_k$. Let $n = \dim Y$ and let η be the generic point of Y. Then*

$$\#\{y \in \mathscr{Y}(\mathbf{F}_v) : f^{-1}(y) \textit{ is split}\} = s_{f,\eta}(v)\#\mathscr{Y}(\mathbf{F}_v) + O((\mathrm{N}\,v)^{n-1/2}), \quad \textit{as } \mathrm{N}\,v \to \infty,$$

where the implied constant depends on f and the choice of the model.

Proof. – The key analytic ingredient for this result is [31, Prop. 9.15], which is a version of the Chebotarev density theorem for arithmetic schemes.

Choose a finite set S of finite places of k, which we will enlarge throughout the proof. Let $I := \mathrm{Irr}^1_{X/Y}$ and $\mathscr{J} := \mathrm{Irr}^1_{\mathscr{X}/\mathscr{Y}}$. First, by Lemma 3.12, the restriction of $\mathscr{J}$ to an open dense subset of $\mathscr{Y}$ is representable by a finite étale cover. However, by the Lang-Weil estimates [24], strict closed subsets contribute to the error term only. We may therefore replace $\mathscr{Y}$ by a dense open subset, if necessary, to assume that $\pi : \mathscr{J} \to \mathscr{Y}$ is finite étale. Similarly, we may assume that $\mathscr{Y}$ is normal and that $\mathscr{Y}_{\mathbf{F}_v}$ is normal for all $v \notin S$.

Our next step is to obtain a version of the diagram (3.2) over $\mathcal{O}_{k,S}$. Let k_Y denote the algebraic closure of k inside $\kappa(Y)$. As Y is normal and integral, the ring $\mathcal{O}_Y(Y)$ is an integrally closed domain. In particular $k_Y \subset \mathcal{O}_Y(Y)$, hence the structure morphism $Y \to \operatorname{Spec} k$ factors through a morphism $Y \to \operatorname{Spec} k_Y$. As in §3.2.1 we choose a common Galois closure for the connected components of I over Y (for the existence of the Galois closure of a connected finite étale cover, see [34, Prop. 5.3.9]). This yields a finite étale Galois morphism

$\mathscr{Z} \to \mathscr{Y}$ with $\mathscr{Z}$ integral such that, enlarging S if necessary and choosing $\mathscr{Z}$ appropriately as in §3.2.1, we obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{k,S} \arrow[d, equal] & \mathscr{Y}_{\mathcal{O}_{k,S}} \arrow[l] \arrow[d] & \mathscr{Z}_{\mathcal{O}_{k,S}} \arrow[l, "G"'] \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{k,S} & \operatorname{Spec} \mathcal{O}_{k_Y,S} \arrow[l] & \operatorname{Spec} \mathcal{O}_{k_L,S}, \arrow[l, "\Gamma=G/N"'] \arrow[ll, bend left, "\Lambda"] \end{tikzcd} \tag{3.9}$$

which recovers the diagram (3.2) on the level of function fields. Here $L = \kappa(\mathscr{Z})$ and k_L is the algebraic closure of k in L. We denote by $\mathscr{Y}_{\mathcal{O}_{k,S}}$ and $\mathscr{Z}_{\mathcal{O}_{k,S}}$ the base change to $\mathcal{O}_{k,S}$, and abuse notation by denoting $\mathcal{O}_{k_Y,S}$ and $\mathcal{O}_{k_L,S}$ the localisations of $\mathcal{O}_{k_Y}$ and $\mathcal{O}_{k_L}$ respectively at those places which lie above places of S. We also choose S sufficiently large so that each morphism in the bottom row is finite étale.

Let v be a finite place of k not in S. We now give an asymptotic formula for $\#\mathscr{Y}(\mathbf{F}_v)$. Enlarging the set S if necessary, we may assume that all fibers of $\mathscr{Y}_{\mathcal{O}_{k,S}} \to \operatorname{Spec} \mathcal{O}_{k_Y,S}$ are geometrically integral. Therefore the irreducible components of $\mathscr{Y}_{\mathbf{F}_v}$ are in bijection with those places w of k_Y which divide v. We denote the corresponding component by $\mathscr{Y}_w$; this is geometrically irreducible over $\mathbf{F}_v$ if and only if $\mathrm{N}\, w = \mathrm{N}\, v$. The $\mathscr{Y}_w$ are disjoint as $\mathscr{Y}_{\mathbf{F}_v}$ is normal [1, Tag 033M] and, again by normality, any non-geometrically-irreducible component has no $\mathbf{F}_v$-point. Thus if there is no w with $\mathrm{N}\, w = \mathrm{N}\, v$, then the proposition trivially holds. So assume that there is a place w of k_Y with $\mathrm{N}\, w = \mathrm{N}\, v$. The Lang-Weil estimates now yield

$$\#\mathscr{Y}(\mathbf{F}_v) = \sum_{\substack{w \in \Omega_{k_Y} \\ \mathrm{N}\, w = \mathrm{N}\, v \\ w \mid v}} \#\mathscr{Y}_w(\mathbf{F}_w) = (\mathrm{N}\, v)^n \sum_{\substack{w \in \Omega_{k_Y} \\ \mathrm{N}\, w = \mathrm{N}\, v \\ w \mid v}} 1 + O((\mathrm{N}\, v)^{n-1/2}). \tag{3.10}$$

Next let w be a place of k_Y with $w \mid v$. For $y \in \mathscr{Y}_w(\mathbf{F}_w)$, we let $\mathrm{Frob}_y \in G$ denote the choice of some Frobenius element of y (well-defined up to conjugacy; see [31, §9.3.1]). Note that $f^{-1}(y)$ is split if and only if Frob_y acts with a fixed point on $I(L)$. We therefore let $F : G \to \{0, 1\}$ be the indicator function of those elements of G which act with a fixed point on $I(L)$, which is a class function on G. We obtain

$$\#\{y \in \mathscr{Y}_w(\mathbf{F}_w) : f^{-1}(y) \text{ is split}\} = \sum_{y \in \mathscr{Y}_w(\mathbf{F}_w)} F(\mathrm{Frob}_y).$$

We are now in the set-up of [31, Prop. 9.15]. Hence we may apply loc. cit. to obtain

$$\#\{y \in \mathscr{Y}_w(\mathbf{F}_w) : f^{-1}(y) \text{ is split}\} = F^N(\mathrm{Frob}_w)(\mathrm{N}\, w)^n + O((\mathrm{N}\, w)^{n-1/2}),$$

where for $\gamma \in \Gamma$ we define

$$F^N(\gamma) = \frac{1}{|N|} \sum_{g \bmod N = \gamma} F(g)$$

(cf. [31, §5.1.4]). As in the proof of (3.10) we obtain

$$\#\{y \in \mathcal{Y}(\mathbf{F}_v) : f^{-1}(y) \text{ is split}\}$$
$$= \frac{(\mathrm{N}\, v)^n}{|N|} \sum_{\substack{w \in \Omega_{k_Y} \\ \mathrm{N}\, w = \mathrm{N}\, v \\ w | v}} \# \left\{ g \in G : \begin{array}{l} g \bmod N = \mathrm{Frob}_w \text{ and} \\ g \text{ acts with a fixed point on } I(L) \end{array} \right\} + O((\mathrm{N}\, v)^{n-1/2}).$$

Combining this with (3.10) and recalling Definition 3.3 completes the proof. □

Corollary 3.14. – *Let $f : X \to Y$ be a morphism of schemes of finite type over a number field k, with Y integral. Let $f : \mathcal{X} \to \mathcal{Y}$ be a model of $X \to Y$ over $\mathcal{O}_k$.*

There exists a finite set of places S of k such that for all $v \notin S$ with $s_{f,\eta}(v) < 1$, there exists $y \in \mathcal{Y}(\mathbf{F}_v)$ such that $f^{-1}(y)$ is non-split.

Proof. – As $s_{f,\eta}(v) < 1$, Definition 3.3 implies that there is a place w of k_Y of degree 1 over v; hence Y_{k_v} is split. The Lang-Weil estimates [24] show that $\mathcal{Y}(\mathbf{F}_v) \neq \emptyset$ for all $v \notin S$, for a suitably large set of places S. Enlarging S if necessary, Proposition 3.13 implies that there exists $y \in \mathcal{Y}(\mathbf{F}_v)$ for which $f^{-1}(y)$ is non-split, as required. □

Corollary 3.15. – *Let $f : X \to Y$ be a morphism of schemes of finite type over a number field k, with Y normal and integral. Assume that $\mathrm{Irr}^1_{X/Y}$ is representable by a finite étale scheme over Y. Let $f : \mathcal{X} \to \mathcal{Y}$ be a model of $X \to Y$ over $\mathcal{O}_k$.*

There exists a finite set of places S of k such that for all $v \notin S$, the fiber over every *point $y \in \mathcal{Y}(\mathbf{F}_v)$ is split if and only if $s_{f,\eta}(v) = 1$.*

Proof. – By Lemma 3.12 there is a dense open subset $\mathcal{U} \subset \mathcal{Y}$ such that $\mathrm{Irr}^1_{\mathcal{X}_{\mathcal{U}}/\mathcal{U}}$ is representable by a finite étale scheme over $\mathcal{U}$. The restriction $\mathrm{Irr}^1_{X/Y}$ of the functor $\mathrm{Irr}^1_{\mathcal{X}/\mathcal{Y}}$ to Y is representable by a finite étale scheme over Y, by assumption. As $\mathrm{Irr}^1_{\mathcal{X}/\mathcal{Y}}$ is an étale sheaf [29, Lemme 2.1.2] we may glue these representations together. Thus we may assume that the generic fiber of $\mathcal{U} \to \operatorname{Spec} \mathcal{O}_k$ is Y. Spreading out we see that there is a finite set of places S of k such that $\mathrm{Irr}^1_{\mathcal{X}_{\mathcal{O}_{k,S}}/\mathcal{Y}_{\mathcal{O}_{k,S}}}$ is representable by a finite étale scheme.

To continue, we use some of the techniques from the proof of Proposition 3.13 and keep the notation of that proof. We will use the diagram (3.9); this is valid on enlarging S, because $\mathrm{Irr}^1_{X/Y} \to Y$ is finite étale and Y is normal.

If $s_{f,\eta}(v) < 1$ then, enlarging S, the result follows from Corollary 3.14. So let $v \notin S$, assume that $s_{f,\eta}(v) = 1$ and let $y \in \mathcal{Y}(\mathbf{F}_v)$. Let w be the place of k_Y lying below y in (3.9); this has degree 1 over v as $y \in \mathcal{Y}(\mathbf{F}_v)$. Let $l \in \mathcal{L}$ be a closed point lying above y and let u be the place of k_L lying below l in (3.9). Let $\mathbf{F}_w, \mathbf{F}_u, \mathbf{F}_l$ and $\mathbf{F}_y$ be the respective residue fields. From (3.9) we obtain the tower of extensions of finite fields

$$\mathbf{F}_y = \mathbf{F}_w \subset \mathbf{F}_u \subset \mathbf{F}_l.$$

The functoriality of Frobenius elements in extensions of finite fields implies that

$$\mathrm{Frob}_{l/w} \bmod N = \mathrm{Frob}_{u/w}, \tag{3.11}$$

where $\mathrm{Frob}_{l/w} \in \mathrm{Gal}(\mathbf{F}_l/\mathbf{F}_w) = \mathrm{Gal}(\mathbf{F}_l/\mathbf{F}_y) \subset G$ and $\mathrm{Frob}_{u/w} \in \mathrm{Gal}(\mathbf{F}_u/\mathbf{F}_w) \subset \Gamma$ denote the associated Frobenius elements. However, as $s_{f,\eta}(v) = 1$, Lemma 3.9 and (3.11) imply that $\mathrm{Frob}_{l/w} = \mathrm{Frob}_{l/y} \in G$ acts with a fixed point on $I(L)$. Thus $f^{-1}(y)$ is split. □

4. Non-surjectivity

Using the material developed in §3, we now prove one implication of Theorem 1.4, namely that our geometric conditions are *necessary* for arithmetic surjectivity. We require the following criterion for non-existence of an $\mathcal{O}_v$-point in a fiber.

Proposition 4.1. – *Let k be a number field. Let $f : X \to Y$ be a dominant morphism of smooth and geometrically integral k-varieties and let $f : \mathcal{X} \to \mathcal{Y}$ be a model over $\mathcal{O}_k$. Let $\mathcal{T}$ be a reduced divisor in $\mathcal{Y}$ such that the restriction of $\mathcal{X} \to \mathcal{Y}$ to $\mathcal{Y} \setminus \mathcal{T}$ is smooth.*

There exist a finite set $S \subseteq \Omega_{k,\mathrm{f}}$ and a closed subset $\mathcal{Z} \subset \mathcal{T}_{\mathcal{O}_{k,S}}$ containing the singular locus of $\mathcal{T}_{\mathcal{O}_{k,S}}$, of codimension 2 in $\mathcal{Y}_{\mathcal{O}_{k,S}}$, such that for all finite places $v \notin S$ the following holds:

> *Let $\mathcal{P} \in \mathcal{Y}(\mathcal{O}_v)$ be such that the image of $\mathcal{P} : \mathrm{Spec}\, \mathcal{O}_v \to \mathcal{Y}$ meets $\mathcal{T}_{\mathcal{O}_{k,S}}$ transversally outside of $\mathcal{Z}$ and such that the fiber above $\mathcal{P} \bmod v \in \mathcal{T}(\mathbf{F}_v)$ is non-split. Then $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{P})(\mathcal{O}_v) = \emptyset$.*

Proof. – For rational points this is proved in [25, Thm. 2.8]. The adaptation to integral points is straightforward. □

Here is the main result of this section.

Theorem 4.2. – *Let k be a number field. Let $f : X \to Y$ be a dominant morphism of smooth and geometrically integral k-varieties and let $f : \mathcal{X} \to \mathcal{Y}$ be a model over $\mathcal{O}_k$. Let $D \in Y^{(1)}$. Then there exists a finite set S of finite places of k such that for all finite places $v \notin S$, the following holds: if $s_{f,D}(v) < 1$ then $\mathcal{X}(\mathcal{O}_v) \to \mathcal{Y}(\mathcal{O}_v)$ is not surjective.*

Proof. – Let $\mathcal{D}$ be the closure of D inside $\mathcal{Y}$. Enlarge S so that Proposition 4.1 may be applied, and let $\mathcal{T}$ and $\mathcal{Z}$ be as in the statement of that proposition. Enlarging S further if necessary, we may assume that $\mathcal{Y}_{\mathcal{O}_{k,S}}$ is smooth over $\mathcal{O}_{k,S}$.

Note that $s_{f,D}(v)$ only depends on $f^{-1}(D)$, i.e., on the generic fiber of $\mathcal{D}$. In particular, enlarging S further if necessary, we may apply Corollary 3.14 to the restriction of f to $\mathcal{D} \setminus \mathcal{Z}$. Thus if $v \notin S$ and $s_{f,D}(v) < 1$, then there exists a point $y \in \mathcal{D}(\mathbf{F}_v) \setminus \mathcal{Z}(\mathbf{F}_v)$ such that $f^{-1}(y)$ is non-split.

Therefore, by Proposition 4.1, to prove the result it suffices to show that we may lift y to a point $\mathcal{P} \in \mathcal{Y}(\mathcal{O}_v)$ which meets $\mathcal{D}$ transversally at y. This is well-known; let us give a short geometric proof of this statement. Let $\psi : \mathcal{Y}' \to \mathcal{Y}$ be the blow-up of $\mathcal{Y}$ at y. The exceptional divisor $E = \mathbf{P}(T_{\mathcal{Y},y})$ is the projectivisation of the tangent space to $\mathcal{Y}$ at y (since $\mathcal{Y}$ is smooth at y). The linear subvariety $\mathbf{P}(T_{\mathcal{D},y})$ of E is strictly smaller (since $\mathcal{D}$ is smooth at y). Choose any $y' \in E(\mathbf{F}_v)$ which does not lie in this linear subvariety. By Hensel's lemma, y' can be lifted to an $\mathcal{O}_v$-point $\mathcal{P}' \in \mathcal{Y}'(\mathcal{O}_v)$. The image $\mathcal{P} = \psi(\mathcal{P}') \in \mathcal{Y}(\mathcal{O}_v)$ now satisfies the requirements. □

The following result (see [13, Observation 2.2]) follows from Greenberg's theorem [17].

LEMMA 4.3. – *Let R be an excellent henselian discrete valuation ring with fraction field K. Let $f : X \to Y$ be a dominant morphism of integral schemes which are separated and of finite type over R, with $X \times_R K$ and $Y \times_R K$ smooth over K. Let $f' : X' \to Y'$ be a modification of f, i.e., a commutative diagram as in* (1.1)*; here $\alpha_X : X' \to X$ and $\alpha_Y : Y' \to Y$ are proper birational morphisms over R, and $f' : X' \to Y'$ is a dominant morphism of integral separated schemes over of finite type over R, with smooth generic fiber. Then $X(R) \to Y(R)$ is surjective if and only if $X'(R) \to Y'(R)$ is surjective.*

We now prove one implication of Theorem 1.4.

COROLLARY 4.4. – *Let $f : X \to Y$ be a dominant morphism of smooth, geometrically integral varieties over a number field k. Assume that there exists a modification $f' : X' \to Y'$ of f, with X' and Y' smooth, and a point $D \in (Y')^{(1)}$ such that the fiber $(f')^{-1}(D)$ is not pseudo-split over $\kappa(D)$. Let $f : \mathscr{X} \to \mathscr{Y}$ be a model of $X \to Y$ over $\mathscr{O}_k$. Then there exists a set of places v of positive density such that the map $\mathscr{X}(\mathscr{O}_v) \to \mathscr{Y}(\mathscr{O}_v)$ is not surjective.*

Proof. – Ignoring finitely many places, by Lemma 4.3 we may assume that $X = X'$, $Y = Y'$, $f = f'$. By Lemmas 3.10 and 3.11, there exists a set of places v of positive density such that $s_{f,D}(v) < 1$. The result then follows from Theorem 4.2. □

5. Logarithmic geometry

5.1. Preliminaries

The results and proofs in this section are written in the language of logarithmic geometry. Although it would be possible to use the older language of toroidal embeddings, the logarithmic framework is a convenient and flexible language, which makes arguments more conceptual and transparent. A good reference for basic terminology on log schemes is Kato's foundational paper [21]. Let us briefly recall the notions which are most essential for us. We work exclusively with *Zariski log schemes*:

DEFINITION 5.1. – A *Zariski log scheme* is a pair $(X, \mathscr{M}_X)$, where X is a scheme and $\mathscr{M}_X$ is a sheaf of monoids on X (for the Zariski topology), equipped with a homomorphism of sheaves of monoids $\alpha_X : \mathscr{M}_X \to (\mathscr{O}_X, \cdot)$ inducing an isomorphism

$$\alpha_X^{-1}(\mathscr{O}_X^*) \cong \mathscr{O}_X^*.$$

The fundamental example for the purpose of this paper is the situation where the *log structure* $\alpha_X : \mathscr{M}_X \to \mathscr{O}_X$ on X is *divisorial*, i.e., induced by a divisor on X:

EXAMPLE 5.2. – Let X be any scheme and let D be a divisor on X. Denote the corresponding open immersion by $j : X \setminus D \hookrightarrow X$. Then the monoid

$$\mathscr{M}_X = j_* \mathscr{O}_{X \setminus D}^* \cap \mathscr{O}_X$$

(together with the natural inclusion $\alpha_X : \mathscr{M}_X \hookrightarrow \mathscr{O}_X$) defines a log structure on X, the *divisorial log structure induced by D*. This yields a Zariski log scheme $(X, \mathscr{M}_X)$, which we will sometimes denote by (X, D) depending on the context.

Zariski log schemes form a category, with morphisms defined in the obvious way, cf. [21, §1.1]. We will almost always work with a full subcategory: the category of *fine log schemes* [21, Definition 2.9], or the category of *fs ("fine and saturated") log schemes*.

5.1.1. *Log regular schemes and their fans*. – Given an arbitrary monoid P, we write P^* for the subgroup of invertible elements, $P^\sharp = P/P^*$ for the associated sharp monoid and P^{gp} for the group envelope of P. (A monoid is *sharp* if 0 is the only invertible element.)

DEFINITION 5.3. – A *log regular scheme* is a Zariski log scheme which is log regular.

Recall that a Zariski log scheme X is said to be log regular if it is fs and the following property is satisfied for every $x \in X$: if $I(x, \mathcal{M}_X)$ is the ideal of $\mathcal{O}_{X,x}$ generated by $\mathcal{M}_{X,x} \setminus \mathcal{O}^*_{X,x}$, then $\mathcal{O}_{X,x}/I(x, \mathcal{M}_X)$ is a regular local ring and

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}/I(x, \mathcal{M}_X) + \mathrm{rk}_{\mathbf{Z}}\,(\mathcal{M}^\sharp_{X,x})^{\mathrm{gp}}.$$

Log regular schemes are normal and Cohen-Macaulay by [22, Theorem 4.4]. Moreover, they admit the following description by [22, Theorem 11.6].

REMARK 5.4. – Let $(X, \mathcal{M}_X)$ be a log regular scheme. Let U be the largest open subset of X on which the log structure $\mathcal{M}_X$ is trivial, i.e., coincides with the sheaf $\mathcal{O}^*_X$ of invertible functions. Let $j : U \hookrightarrow X$ be the corresponding open immersion. Then $X \setminus U$ is a divisor and $\mathcal{M}_X = j_*\mathcal{O}^*_U \cap \mathcal{O}_X$, as in Example 5.2.

To a log regular scheme $(X, \mathcal{M}_X)$ one can associate a useful combinatorial object, its *fan*. This notion has been introduced by Kato [22, §9.1, §9.3]:

DEFINITION 5.5. – A *fan* is a locally monoidal space $(F, \mathcal{M}_F)$ which admits an open covering by affine monoidal spaces $\operatorname{Spec} P$, where P is an fs monoid. A fan is called *smooth* if it has an open covering by monoidal spaces of the form $\operatorname{Spec} \mathbf{N}^r$; see [2, Definition 4.11].

An affine fan $\operatorname{Spec} P$ is sometimes called a *Kato cone*, cf. [2, §4]. The definition of the sheaf of monoids $\mathcal{M}_{\operatorname{Spec} P}$ is motivated by the definition of an affine scheme. The stalks of $\mathcal{M}_{\operatorname{Spec} P}$ are sharp, hence the stalks of $\mathcal{M}_F$ are sharp too.

To an arbitrary log regular scheme, Kato associates a fan as follows:

DEFINITION 5.6. – Let $(X, \mathcal{M}_X)$ be a log regular scheme. Consider the monoidal space $F(X)$ with underlying set $\{x \in X : I(x, \mathcal{M}_X) = \mathfrak{m}_x\}$; here $I(x, \mathcal{M}_X)$ is defined as above, and $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}_{X,x}$. The topology on $F(X)$ is the subspace topology induced by the Zariski topology on X. The sheaf of monoids on $F(X)$ is the restriction to $F(X)$ of the sheaf $\mathcal{M}^\sharp_X$ of (sharp) monoids on X.

Kato proves that $F(X)$ is indeed a fan [22, Theorem 10.1]. If X is quasi-compact, then the underlying set of $F(X)$ is finite. If X is a regular scheme and if the log structure on X is the divisorial log structure associated to a divisor on X with strict normal crossings, then $F(X)$ is smooth (see [2, §4] and the references given there).

There exists a continuous, open morphism of monoidal spaces $\pi : (X, \mathcal{M}^\sharp_X) \to F(X)$ [22, §10.2]. This allows one to stratify the scheme X into locally closed subsets:

Definition 5.7. – Given a log regular scheme $(X, \mathcal{M}_X)$ and given $x \in F(X)$, denote $U(x) = \pi^{-1}(x)$, with $\pi : (X, \mathcal{M}_X) \to F(X)$ as above. When equipped with its reduced subscheme structure, this is a locally closed subscheme of X. This yields the so-called *logarithmic stratification* $(U(x))_{x \in F(X)}$ of X into locally closed subsets.

With notation as above, we denote by $\overline{U}(x)$ the Zariski closure of $U(x)$. Since π is continuous and open, $\overline{U}(x)$ is the inverse image under π of the closure of x in $F(X)$. Any stratum $U(x)$ is regular and irreducible, but its closure $\overline{U}(x)$ may very well be singular.

The points of a Kato fan F are in a natural bijection with the Kato subcones of $(F, \mathcal{M}_F)$: every subcone has a unique closed point and can be recovered as the smallest open affine neighborhood of this point, see [2, Lemma 4.6].

Let $F(\mathbf{N}) = \mathrm{Hom}(\mathrm{Spec}\, \mathbf{N}, F)$. For a smooth fan F we define the *height* of a morphism $g : \mathrm{Spec}\, \mathbf{N} \to F$ as follows. The morphism g sends the closed point $\mathbf{N}_{>0}$ of $\mathrm{Spec}\, \mathbf{N}$ to the closed point of a unique Kato subcone $\mathrm{Spec}\, \mathbf{N}^r$, so g factors through $\mathrm{Spec}\, \mathbf{N} \to \mathrm{Spec}\, \mathbf{N}^r$. The dual map is a morphism of monoids $\mathbf{N}^r \to \mathbf{N}$; we define the height $h_F(g) \in \mathbf{N}$ to be the image of the sum of the canonical generators of $\mathbf{N}^r$. It is clear that for any $m \in \mathbf{N}$ the set $F(\mathbf{N})_{\leq m} = \{P \in F(\mathbf{N}) : h_F(P) \leq m\}$ is finite.

5.1.2. *Log smooth morphisms and log blow-ups.* – Log smooth morphisms can be defined in many different ways. We recall the definition which is usually called *Kato's criterion*, stated and proven for étale log structures in [21, Theorem 3.5]. The appropriate version for Zariski log structures seems to be folklore and can be found in [15, Corollary 12.3.37].

Theorem 5.8. – *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of Zariski fs log schemes. Then f is log smooth (resp. log étale) if and only if the following condition is satisfied. Given any point $x \in X$, an affine open neighborhood $V = \mathrm{Spec}\, B$ of $f(x)$ in Y, and a chart $Q \to B$ for the log structure around y, there exist*

- *an* étale *affine neighborhood $g : U = \mathrm{Spec}\, A \to X$ of x,*
- *a chart $P \to A$ for the log structure $g^* \mathcal{M}_X$ on U, and*
- *a homomorphism $\varphi : Q \to P$ yielding a chart for $(U, g^* \mathcal{M}_X) \to (V, \mathcal{M}_V)$,*

such that both of the following conditions are satisfied:

- $\ker \varphi^{\mathrm{gp}}$ *and the torsion of* $\mathrm{coker}\, \varphi^{\mathrm{gp}}$ *(resp.* $\ker \varphi^{\mathrm{gp}}$ *and* $\mathrm{coker}\, \varphi^{\mathrm{gp}}$*) are finite abelian groups, the orders of which are invertible on U;*
- *the induced morphism $U \to V \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[P]$ is classically smooth.*

In fact, an equivalent version of the criterion says that $U \to V \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[P]$ can even be taken to be étale instead of smooth (see [21, Theorem 3.5]).

An important basic fact is that any fs log scheme which is log smooth over a log regular scheme is again a log regular scheme (this is [22, Theorem 8.2]).

An essential class of log smooth and log étale morphisms is the class of log blow-ups. They are constructed in terms of subdivisions. A *subdivision* of a fan F as in Definition 5.5 is a morphism of fans $\varphi : F' \to F$ satisfying the properties in [22, Definition 9.6]. The following key example will be used in the proof of Theorem 1.4.

EXAMPLE 5.9. – The *barycentric subdivision* $B(F)$ of a fan F is defined as follows. The *barycenter* of a Kato cone Spec P is a canonical morphism Spec $\mathbf{N} \to$ Spec P which, under the canonical isomorphism $\mathrm{Hom}(\mathrm{Spec}\,\mathbf{N}, \mathrm{Spec}\,P) = \mathrm{Hom}(P, \mathbf{N})$, corresponds to the map sending the generator of each 1-dimensional face of P to 1. One constructs $B(F)$ by performing the so-called star subdivision of each cone of F along its barycenter, in decreasing order of dimension. See [2, Example 4.10 (ii)] for details.

To a subdivision $\varphi : F' \to F(X)$ of the fan of a log regular scheme $(X, \mathscr{M}_X)$ one associates a log regular scheme $(X', \mathscr{M}'_X)$ with $F(X') = F'$ and a morphism, called a *log blow-up*,

$$\mathrm{Bl}_\varphi : (X', \mathscr{M}'_X) \to (X, \mathscr{M}_X),$$

such that the induced map $F(X') \to F(X)$ is exactly φ. The morphism Bl_φ is log étale and birational [22, Proposition 10.3]. If φ is proper [22, Definition 9.7], then the natural map $\varphi_* : F'(\mathbf{N}) \to F(X)(\mathbf{N})$ is bijective; in this case Bl_φ is a proper morphism [22, Proposition 9.11].

Log blow-ups can be used to resolve singularities of log regular schemes, as explained by Kato in [22, §10.4]. They are stable under base change in the category of fs log schemes [27, Corollary 4.8] and under composition [27, Corollary 4.11].

LEMMA 5.10. – *Let F be a smooth quasi-compact Kato fan. Let m be a positive integer. Then there exists a smooth, proper subdivision $\varphi : F' \to F$ such that for every point $f \in F(\mathbf{N})_{\leq m}$ the induced point $\varphi_*^{-1} f \in F'(\mathbf{N})$ lies in $F'(\mathbf{N})_{\leq 1}$.*

Proof. – It is enough to iterate the barycentric subdivision $m - 1$ times. □

5.1.3. *Weak toroidalisation.* – We now briefly recall the *weak toroidalisation theorem* of Abramovich-Karu in the language of log geometry; this will be a major tool for us in §6, and figured already prominently in [12] and [13].

THEOREM 5.11 (Abramovich-Karu, Denef). – *Let $f : X \to Y$ be a dominant morphism of integral varieties over a field k of characteristic zero.*

Then there exist a dominant morphism $f' : X' \to Y'$ of smooth integral k-varieties, proper birational morphisms $\alpha_X : X' \to X$ and $\alpha_Y : Y' \to Y$, and strict normal crossings divisors $D' \subset X'$ and $E' \subset Y'$ such that

(1) *the diagram*

$$\begin{tikzcd} X' \arrow[r, "\alpha_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\alpha_Y"] & Y \end{tikzcd}$$

commutes;

(2) *f' induces a log smooth morphism of log regular schemes $(X', D') \to (Y', E')$;*

(3) *$(f')^{-1}(Y' \setminus E') = X' \setminus D'$.*

This result was first proved over $\mathbf{C}$ in [4] and subsequently over arbitrary fields of characteristic zero in [3]. For an alternative treatment by Gabber and Illusie-Temkin in a more general setting, see [19, §3.8].

We work with Zariski log schemes instead of étale log schemes; Theorem 5.11 is true in both settings, but the statement for Zariski log schemes is slightly stronger. This corresponds to the fact that in [3, Theorem 1.1], the toroidal embeddings can be taken to be *strict*. The following example illustrates the difference between both settings:

EXAMPLE 5.12. – Let k be a field of characteristic zero such that -1 is not a square in k. Consider the conic bundle $X \subset \mathbf{P}^2_k \times \mathbf{A}^1_k$ given by $x^2 + y^2 = tz^2$. Let $\pi : X \to \mathbf{A}^1_k$ be the natural projection to the coordinate t. Here the total space X is smooth, but the fiber $\pi^{-1}(0)$ is irreducible and singular. Hence, when equipped with the log structure induced by $\pi^{-1}(0)$, the log scheme $(X, \pi^{-1}(0))$ is *not* log regular in our sense, i.e., as a Zariski log scheme (it is however log regular as an *étale* log scheme).

However, the normalization of $\pi^{-1}(0)$ is smooth. If $\psi : X' \to X$ is the blow-up of X at the point $x = y = t = 0, z = 1$, then $(X', (\pi\circ\psi)^{-1}(0))$ is log smooth, and $(X', (\pi\circ\psi)^{-1}(0)) \to (\mathbf{A}^1_k, 0)$ satisfies the requirements of Theorem 5.11.

5.2. Logarithmic Hensel's lemma

In [12, §3.2], Denef proved a logarithmic version of Hensel's lemma. We will present a reformulation of this result, with a different proof, written down by Cao in his unpublished MSc thesis [7].

PROPOSITION 5.13. – *Let $f : (X, \mathscr{M}_X) \to (Y, \mathscr{M}_Y)$ be a log smooth morphism of fs log schemes. Let R be a complete discrete valuation ring. Let $S = \operatorname{Spec} R$ and let $j : s \hookrightarrow S$ be the inclusion of the closed point. Given a commutative diagram*

$$\begin{tikzcd} s^\dagger \arrow[r, "u"] \arrow[d, "j"'] & (X, \mathscr{M}_X) \arrow[d, "f"] \\ S^\dagger \arrow[r, "t"] \arrow[ur, dotted, "g"] & (Y, \mathscr{M}_Y) \end{tikzcd}$$

of fs log schemes, there is a morphism $g : S^\dagger \to (X, \mathscr{M}_X)$ of fs log schemes such that $gj = u$ and $fg = t$.

Here $S^\dagger$ denotes the scheme S equipped with the divisorial log structure induced by the closed point s. If π is a uniformiser of R, a chart for the log structure is given by the map $\mathbf{N} \to R$ which sends 1 to π. Similarly, $s^\dagger$ denotes the log point, i.e., $\operatorname{Spec} k$ equipped with the pullback of the log structure on $S^\dagger$ to s (with a chart $\mathbf{N} \to k$ sending 1 to 0).

Proof. – Let $\mathfrak{m} = (\pi)$ be the maximal ideal of R. Write $R_n = R/\mathfrak{m}^n$ and $S_n = \operatorname{Spec} R_n$. In particular, $s = S_1$. We have strict closed immersions

$$i_n : S_n^\dagger \to S^\dagger, \quad i_{n-1,n} : S_{n-1}^\dagger \to S_n^\dagger$$

for all $n \geq 1$. The definition of a log smooth morphism in terms of infinitesimal liftings [21, §3.2, §3.3] implies that for any $n \geq 1$ one can find a morphism

$$g_n : S_n^\dagger \to (X, \mathscr{M}_X),$$

such that $g_1 = u$ and such that for any $n \geq 1$ both triangles in the diagram

$$\begin{tikzcd} S^\dagger_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "i_{n-1,n}"'] & (X, \mathcal{M}_X) \arrow[d, "f"] \\ S^\dagger_n \arrow[r, "t \circ i_n"] \arrow[ur, dotted, "g_n"] & (Y, \mathcal{M}_Y) \end{tikzcd}$$

commute. Since the ring $R = \varinjlim R_n$ is complete, the morphisms g_n induce a well-defined morphism of schemes $g : S \to X$. It is then easy to see that g actually defines a morphism of *log* schemes $g : S^\dagger \to (X, \mathcal{M}_X)$ satisfying all requirements. □

5.3. Log smooth morphisms and irreducible components

The goal of this section is to prove a basic result (probably well-known to experts) concerning the variation of the (geometric) irreducible components of the fibers of a proper, log smooth morphism of log regular schemes.

If $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a log smooth morphism of log regular schemes, then f induces a morphism of the associated Kato fans $F(f) : F(X) \to F(Y)$. The morphisms f and $F(f)$ are compatible with the characteristic morphisms $\pi_X : (X, \mathcal{M}^\sharp_X) \to F(X)$ and $\pi_Y : (X, \mathcal{M}^\sharp_Y) \to F(Y)$. Hence, given $x \in F(X)$ and $y = F(f)(x) \in F(Y)$, we get induced morphisms $U(x) \to U(y)$ and $\overline{U}(x) \to \overline{U}(y)$.

We recall the localisation procedure in [22, §7]. If $(X, \mathcal{M}_X)$ is a log regular scheme, then the "boundary" $\partial U(x) = \overline{U}(x) \setminus U(x)$ of the closed subscheme $\overline{U}(x)$ is a divisor, inducing a log structure (cf. Example 5.2). Kato proves that the resulting log scheme $(\overline{U}(x), \partial U(x))$ is again log regular; the locus of triviality of this log structure is $U(x)$.

The following lemma establishes a "relative" version of this localisation procedure; we are not aware of a published account of this basic statement.

LEMMA 5.14. – *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a log smooth morphism of log regular schemes. Let $x \in F(X)$ and let $y = F(f)(x) \in F(Y)$. Then*

$$(\overline{U}(x), \partial U(x)) \to (\overline{U}(y), \partial U(y)) \tag{5.1}$$

is again a log smooth morphism of log regular schemes.

Proof. – The log regularity is taken care of by [22, Proposition 7.2]; what we really need to prove is log smoothness. To do so, we will use Kato's criterion (Theorem 5.8).

Choose an affine open $W = \operatorname{Spec} B$ of Y containing y and a compatible affine scheme $V = \operatorname{Spec} A$, equipped with an étale map $g : V \to X$ such that $g(V)$ contains x, on which there exists a chart for $(V, g^* \mathcal{M}_X) \to (W, \mathcal{M}_W)$ given by homomorphisms of fs monoids

$$P \to A, \ Q \to B \ \text{ and } \varphi : Q \to P$$

compatible with the homomorphism of rings $B \to A$. We can assume that the monoid Q is toric, i.e., that Q^{gp} is torsion free [22, Lemma 1.6]. Kato's criterion says that we can then choose P such that the following conditions are satisfied:

(1) the induced morphism $V \to W \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$ is classically smooth, and

(2) the induced homomorphism of abelian groups $\varphi^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, and the torsion part of the cokernel has order invertible on V.

Localizing further if needed, we may assume that x has a unique preimage v under g. Now v corresponds to an ideal $\mathfrak{p} \in \operatorname{Spec} P$, and y corresponds to $\mathfrak{q} = \varphi^{-1}(\mathfrak{p}) \in \operatorname{Spec} Q$. Moreover, $g^{-1}(\overline{U}(x)) = \operatorname{Spec} A/(\mathfrak{p})$, with log structure given by $P \setminus \mathfrak{p} \to A/(\mathfrak{p})$ induced by the map $P \to A$. Here $(\mathfrak{p})$ is the ideal of A generated by the elements of $\mathfrak{p}$ via $P \to A$. Similarly, $\overline{U}(y) \cap W = \operatorname{Spec} B/(\mathfrak{q})$, with log structure given by $Q \setminus \mathfrak{q} \to B/(\mathfrak{q})$. We have an induced morphism $\widetilde{\varphi} : Q \setminus \mathfrak{q} \to P \setminus \mathfrak{p}$ giving a chart for the morphism

$$\big(g^{-1}(\overline{U}(x)), g^{-1}(\partial\overline{U}(x))\big) \to (\overline{U}(y) \cap W, \partial\overline{U}(y) \cap W).$$

It suffices to check that this chart satisfies Kato's criterion, i.e., that

(1′) the induced morphism

$$\operatorname{Spec} A/(\mathfrak{p}) \to \operatorname{Spec} B/(\mathfrak{q}) \times_{\operatorname{Spec} \mathbf{Z}[Q\setminus\mathfrak{q}]} \operatorname{Spec} \mathbf{Z}[P \setminus \mathfrak{p}]$$

is classically smooth;

(2′) the induced homomorphism of abelian groups $\widetilde{\varphi}^{\mathrm{gp}} : (Q \setminus \mathfrak{q})^{\mathrm{gp}} \to (P \setminus \mathfrak{p})^{\mathrm{gp}}$ is injective, and the torsion part of the cokernel has order invertible on $g^{-1}(\overline{U}(x))$.

Now (1′) follows immediately from the fact that smooth morphisms are stable under base change, since $V \to W \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$ is already smooth.

Concerning (2′), the injectivity of $\widetilde{\varphi}^{\mathrm{gp}}$ is trivial. Consider the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (Q \setminus \mathfrak{q})^{\mathrm{gp}} \arrow[r] \arrow[d, hook] & (P \setminus \mathfrak{p})^{\mathrm{gp}} \arrow[r] \arrow[d, hook] & \operatorname{coker} \widetilde{\varphi}^{\mathrm{gp}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q^{\mathrm{gp}} \arrow[r] & P^{\mathrm{gp}} \arrow[r] & \operatorname{coker} \varphi^{\mathrm{gp}} \arrow[r] & 0.
\end{tikzcd}$$

The fact that the order of the torsion part of the cokernel of $\widetilde{\varphi}^{\mathrm{gp}}$ is invertible on $g^{-1}(\overline{U}(x))$ follows from the same statement for the cokernel of φ^{gp}, together with the snake lemma and the observation that the cokernel of $(Q\setminus\mathfrak{q})^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is torsion free—indeed, the fs monoid Q is toric, and $Q \setminus \mathfrak{q}$ is one of its faces. This finishes the proof. $\square$

The following lemma concerns the fibers of log smooth families over a base with trivial log structure. In this setting, the functor $\mathrm{Irr}_{X/Y}$ from §3.3.1 becomes rather simple.

Lemma 5.15. – *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{O}_Y^*)$ be a proper, log smooth morphism of log regular schemes, where Y is given the trivial log structure. Then $f : X \to Y$ is flat, and $\mathrm{Irr}_{X/Y}$ is represented by a finite étale scheme over Y.*

Proof. – Since the log structure on Y is trivial, f is integral [21, Corollary 4.4(ii)]. Log smooth, integral morphisms are flat [21, Corollary 4.5], hence f is flat.

The fibers of f are log regular, being log smooth over a field with trivial log structure [22, Theorem 8.2], and hence normal. Moreover, as log smooth morphisms are preserved by base change, the fibers are geometrically normal, hence geometrically reduced.

Let $X \to \mathrm{St}_{X/Y} \to Y$ be the Stein factorisation of $X \to Y$. As f is flat and the fibers are geometrically reduced, it follows from [1, Tag 0BUN] that $\mathrm{St}_{X/Y}$ is finite étale over Y. The result now follows from the fact that $\mathrm{Irr}_{X/Y}$ and $\mathrm{St}_{X/Y}$ are isomorphic under the assumptions of the lemma. Indeed, the fibers of f are normal, so the connected components of the geometric fibers are precisely the irreducible components. □

Another important property of proper, log smooth morphisms is that the multiplicities of the fibers are "constant along strata". Such a result is well-known (and easy to check) for toric morphisms, and readily extends to the log smooth setting.

PROPOSITION 5.16. – *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a proper, log smooth morphism of log regular schemes. Let $y' \in Y$, and let $y \in F(Y)$ be the unique point with the property that $y' \in U(y)$. Let x' be one of the generic points of the fiber $f^{-1}(y')$, and let $x \in F(X)$ be the unique point such that $x' \in U(x)$.*

Then $f(x) = y$ and the multiplicity of x in $f^{-1}(y)$ is equal to the multiplicity of x' in $f^{-1}(y')$, that is, the lengths of the Artinian local rings $\mathcal{O}_{f^{-1}(y),x}$ and $\mathcal{O}_{f^{-1}(y'),x'}$ are equal.

Proof. – Choose a chart for f around x'. This involves choosing

— a Zariski open $j : V \hookrightarrow Y$ such that $y' \in V$ together with a chart $c_V : V \to \operatorname{Spec} \mathbf{Z}[Q]$ for $j^* \mathcal{M}_Y$,

— an étale morphism $h : U \to X$ such that $x' \in h(U)$ and $(f \circ h)(U) \subset V$ together with a chart $c_U : U \to \operatorname{Spec} \mathbf{Z}[P]$ for $h^* \mathcal{M}_X$,

— a homomorphism $\varphi : Q \to P$ compatible with $f \circ h : U \to V$, c_U and c_V,

such that the induced map

$$\psi : U \to V \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$$

is étale. Let

$$\pi_1 : V \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P] \to V$$
$$\pi_2 : V \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} \mathbf{Z}[P]$$

be the natural projections. Choose a point $z' \in h^{-1}(x')$.

Since h and ψ are étale, the lengths of the Artinian local rings

$$\mathcal{O}_{f^{-1}(y'),x'}, \ \mathcal{O}_{(f \circ h)^{-1}(y'),z'} \ \text{and} \ \mathcal{O}_{\pi_1^{-1}(y'),\psi(z')}$$

are equal. Since the projection π_1 is obtained by base change from the toric morphism $\operatorname{Spec} \mathbf{Z}[\varphi] : \operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} \mathbf{Z}[Q]$, the length of the local ring $\mathcal{O}_{\pi_1^{-1}(y'),\psi(z')}$ only depends on the toric strata of $\operatorname{Spec} \mathbf{Z}[P]$ and $\operatorname{Spec} \mathbf{Z}[Q]$ which contain $\pi_2(\psi(z')) = c_U(z')$ and $c_V(y') = (\operatorname{Spec} \mathbf{Z}[\varphi])(c_U(z'))$, respectively, i.e., on the prime ideals of P and Q which correspond to x and to y, respectively. This gives the desired result. □

We will need the following elementary lemma.

LEMMA 5.17. – *Let $f : X \to Y$ be a morphism of schemes of finite presentation. Let $Z \subset X$ be a closed subset such that $Z|_{f^{-1}(y)}$ contains no irreducible component of $f^{-1}(y)$, for all points $y \in Y$. Then the natural morphism of functors*

$$\operatorname{Irr}_{(X \setminus Z)/Y} \to \operatorname{Irr}_{X/Y}$$

is an isomorphism.

Proof. – Our assumptions imply that removing Z does not change the open irreducible components of the fibers; the result follows. (See also [29, Corollary 2.6.2].) □

From the previous results we will now deduce the final statement of this section.

PROPOSITION 5.18. – *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a proper, log smooth morphism of log regular schemes. For each point $y \in F(Y)$, the functors*

$$\mathrm{Irr}_{f^{-1}(U(y))/U(y)} \quad \text{and} \quad \mathrm{Irr}^1_{f^{-1}(U(y))/U(y)}$$

are representable by finite étale schemes over $U(y)$.

Proof. – The set $f^{-1}(U(y))$ is the disjoint union of locally closed subsets $U(x)$, where $x \in F(X)$ is such that $f(x) = y$. For each such x consider the induced morphism

$$(\overline{U}(x), \partial U(x)) \to (\overline{U}(y), \partial U(y)).$$

By Lemma 5.14, this is again a proper and log smooth morphism of log regular schemes. Let f_x be the restriction of this morphism to $U(y)$ equipped with the trivial log structure. The scheme $\overline{U}(x) \times_{\overline{U}(y)} U(y)$ has a natural log structure, obtained from the divisorial log structure on $(\overline{U}(x), \partial U(x))$ by restriction, making

$$f_x : \overline{U}(x) \times_{\overline{U}(y)} U(y) \to U(y)$$

a proper and log smooth morphism of log regular schemes. The log structure on $U(y)$ is trivial, so by Lemma 5.15 the morphism f_x is flat. The fibers of a proper flat morphism of irreducible varieties have the same pure dimension; therefore all fibers of f_x have pure dimension $\dim(U(x)) - \dim(U(y))$.

The stratum $U(x)$ is a dense open subset of $\overline{U}(x) \times_{\overline{U}(y)} U(y)$, whose complement is the union of closed subsets $\overline{U}(x') \subset \overline{U}(x)$, where $x' \in F(X)$ is a specialization of x, $f(x') = y$ and $x' \neq x$. In particular, $\dim(U(x')) < \dim(U(x))$. Thus for any $t \in U(y)$ the intersection $f_x^{-1}(t) \cap U(x) \subset f_x^{-1}(t)$ is the complement of the union of the closed subsets $f_x^{-1}(t) \cap \overline{U}(x') = f_{x'}^{-1}(t)$, with x' as above. We have seen that $f_{x'}^{-1}(t)$ has pure dimension

$$\dim(U(x')) - \dim(U(y)) < \dim(U(x)) - \dim(U(y)),$$

hence $f_x^{-1}(t) \cap U(x)$ is a dense open subset of $f_x^{-1}(t)$. It now follows from Lemma 5.17 that there is a natural isomorphism of functors

$$\mathrm{Irr}_{U(x)/U(y)} = \mathrm{Irr}_{\overline{U}(x) \times_{\overline{U}(y)} U(y)/U(y)}\,. \tag{5.2}$$

Now let $x_1, \ldots, x_n \in F(X)$ be the minimal elements of $f^{-1}(y)$ with respect to the partial ordering given by the topology on $F(X)$. Then $f^{-1}(U(y))$ is the union of the closed subsets $\overline{U}(x_i) \times_{\overline{U}(y)} U(y)$, where $i = 1, \ldots, n$. Hence the fiber $f^{-1}(t)$ is the union of the closed subsets $f_{x_i}^{-1}(t)$. For each $i = 1, \ldots, n$, the intersection $f_{x_i}^{-1}(t) \cap U(x_i)$ is open and dense in $f_{x_i}^{-1}(t)$, hence $\bigcup_{i=1}^n (f_{x_i}^{-1}(t) \cap U(x_i))$ is open and dense in $f^{-1}(t)$. Therefore Lemma 5.17 yields a natural isomorphism of functors

$$\mathrm{Irr}_{f^{-1}(U(y))/U(y)} = \mathrm{Irr}_{\left(\bigcup_{i=1}^n U(x_i)\right)/U(y)} = \coprod_{i=1}^{n} \mathrm{Irr}_{U(x_i)/U(y)}$$

as the $U(x_i)$ are pairwise disjoint.

By (5.2) the result for the functor $\mathrm{Irr}_{f^{-1}(U(y))/U(y)}$ now follows from Lemma 5.15. Next, $\mathrm{Irr}^1_{f^{-1}(y)/y} = \mathrm{Irr}_{f^{-1}(y)_{\mathrm{sm}}/y}$ represents the irreducible components of $f^{-1}(y)$ of geometric multiplicity 1. By Proposition 5.16, the functor $\mathrm{Irr}^1_{f^{-1}(U(y))/U(y)} = \mathrm{Irr}_{f^{-1}(U(y))_{\mathrm{sm}}/U(y)}$ represents the Zariski closure of its generic fiber, which is represented by $\mathrm{Irr}^1_{f^{-1}(y)/y}$. This is the union of some of the irreducible components of the finite étale $U(y)$-scheme represented by $\mathrm{Irr}_{f^{-1}(U(y))/U(y)}$, and hence is also finite and étale over $U(y)$. □

6. Surjectivity

In this section we prove Theorem 1.5, and explain how to use this and other results from the paper to prove Theorems 1.4 and 1.6.

6.1. Notation and hypotheses

Let $f : X \to Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over a number field k with geometrically integral generic fiber. By Lemma 4.3 and Theorem 5.11 we can assume that f gives rise to a log smooth morphism $f : (X, D) \to (Y, E)$ of log regular schemes.

Let $U = X \setminus D$ and $V = Y \setminus E$. By condition (3) of Theorem 5.11 we have $f^{-1}(V) = U$. Thus $f : U \to V$ is a smooth proper morphism with geometrically integral generic fiber. By [18, Corollaire 15.5.4] all fibers of $f : U \to V$ are geometrically connected. Since connected noetherian normal schemes are integral [1, Lemma 27.7.6, Tag 033H], all fibers of $f : U \to V$ are geometrically integral.

Recall that we are working with Zariski log structures; in particular, D and E are strict normal crossing divisors, so their irreducible components are smooth. Let $F(f) : F_X \to F_Y$ be the attached morphism of smooth Kato fans. We have the height $h_Y : F_Y(\mathbf{N}) \to \mathbf{N}$ introduced at the end of §5.1.1.

Let E_i be an irreducible component of E and let D_j be an irreducible component of D. Since X and Y are smooth, each of the local rings $\mathcal{O}_{E_i,Y}$ and $\mathcal{O}_{D_j,X}$ is a discrete valuation ring. Let $\mathrm{val}_{E_i} : \mathcal{O}_{E_i,Y} \to \mathbf{N}$ and $\mathrm{val}_{D_j} : \mathcal{O}_{D_j,X} \to \mathbf{N}$ be the respective discrete valuations. There is an $m_{ij} \in \mathbf{N}$ such that the restriction of val_{D_j} to $\mathcal{O}_{E_i,Y}$ is $m_{ij}\,\mathrm{val}_{E_i}$.

Let S be a finite set of primes of k, such that $f : X \to Y$ extends to a morphism of smooth and proper $\mathcal{O}_S$-schemes $f : \mathcal{X} \to \mathcal{Y}$. For each irreducible component $D' \subset D$ let k' be the algebraic closure of k in the field of functions $\kappa(D')$. We assume that S contains all the primes of k ramified in any of the fields k'.

Let v be a prime of k which is not in S. We denote by $\mathrm{val}_v : \mathcal{O}_v \setminus \{0\} \to \mathbf{Z}$ the valuation at v and by π_v a generator of the maximal ideal of $\mathcal{O}_v$, so that $\mathbf{F}_v = \mathcal{O}_v/(\pi_v)$. We write $\mathcal{Y}_v = \mathcal{Y} \times_{\mathrm{Spec}\, O_S} \mathrm{Spec}\, O_v$ and $\mathcal{E}_v = \mathcal{E} \times_{\mathrm{Spec}\, O_S} \mathrm{Spec}\, O_v$.

Let $\mathcal{D} \subset \mathcal{X}$ and $\mathcal{E} \subset \mathcal{Y}$ be the Zariski closures of D and E in $\mathcal{X}$ and $\mathcal{Y}$, respectively. By adding finitely many primes to S we can assume that $\mathcal{E}$ and $\mathcal{D}$ are strict normal crossing divisors, so each of their irreducible components is smooth over $\mathrm{Spec}\, \mathcal{O}_S$. We can also assume that $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{Y}, \mathcal{E})$ are log regular schemes with Kato fans F_X and F_Y, respectively, and that there is a log smooth morphism $f : (\mathcal{X}, \mathcal{D}) \to (\mathcal{Y}, \mathcal{E})$ inducing the given morphism $(X, D) \to (Y, E)$ on the generic fibers.

By the valuative criterion of properness any point $P \in Y(k_v)$ extends to a local section $\mathscr{P} : \operatorname{Spec} \mathcal{O}_v \to \mathscr{Y}$ of the structure morphism $\mathscr{Y} \to \operatorname{Spec} \mathcal{O}_S$. We write $P \bmod \pi_v$ for $\mathscr{P}(\operatorname{Spec} \mathbf{F}_v)$, which is an $\mathbf{F}_v$-point of the closed fiber $\mathscr{Y} \times_{\operatorname{Spec} O_S} \operatorname{Spec} \mathbf{F}_v$.

For any point $P \in V(k_v)$ the local section $\mathscr{P} : \operatorname{Spec} \mathcal{O}_v \to \mathscr{Y}$ gives rise to a morphism of log schemes $\mathscr{P} : (\operatorname{Spec} \mathcal{O}_v)^\dagger \to (\mathscr{Y}, \mathscr{E})$, where $(\operatorname{Spec} \mathcal{O}_v)^\dagger$ is equipped with the natural (divisorial) log structure. The associated morphism of Kato fans is $F(\mathscr{P}) : \operatorname{Spec} \mathbf{N} \to F_Y$. It sends the closed point $\mathbf{N}_{>0}$ of $\operatorname{Spec} \mathbf{N}$ to $F(P \bmod \pi_v)$.

6.2. A surjectivity criterion

The following intermediate result is an adaptation of Denef's "surjectivity criterion" [13, 4.2] in the language of log geometry.

PROPOSITION 6.1. – *There exists an integer $m \geq 1$ such that whenever $v \notin S$, we have $f(X(k_v)) = Y(k_v)$ if and only if $f(X(k_v))$ contains all $P \in V(k_v)$ with $h_Y(F(\mathscr{P})) \leq m$.*

Proof. – For each $s \in F_X$ let F_X^s be the open Kato subcone of F_X defined by s. Similarly, for each $t \in F_Y$ let F_Y^t be the open Kato subcone of F_Y defined by t. The morphism $F(f) : F_X \to F_Y$ induces a map $F(f)_* : F_X(\mathbf{N}) \to F_Y(\mathbf{N})$. For each $t \in F_Y$ define

$$m_t = \min\{h_Y(r) \mid r \in F_Y^t(\mathbf{N}),\ r \notin F(f)_*(F_X(\mathbf{N}))\}.$$

(If the set in the right hand side is empty, we take $m_t = 0$.)

For each $s \in F_X$ mapping to $t \in F_Y$ define

$$m_{s,t} = \min\{h_Y(r) \mid r \in F(f)_*(F_X^s(\mathbf{N})) \subset F_Y^t(\mathbf{N})\}.$$

Finally, let

$$m = \max\{m_t, m_{s,t}\},$$

where $t \in F_Y$ and where $s \in F_X$ maps to $t \in F_Y$.

Let us prove that m satisfies the conclusion of the proposition. One implication is obvious, so we prove the other one: if every $P \in V(k_v)$ with $h_Y(F(\mathscr{P})) \leq m$ is contained in $f(X(k_v))$, we need to show that $Y(k_v) = f(X(k_v))$.

It suffices to show that $V(k_v) \subseteq f(U(k_v))$. Indeed, for the topology of k_v the set $V(k_v)$ is dense $Y(k_v)$, whereas $f(X(k_v))$ is closed in $Y(k_v)$ because f is proper.

Take any $P \in V(k_v)$. There exists an $r \in F_Y(\mathbf{N})$, $h_Y(r) \leq m$, such that the images of $F(\mathscr{P})$ and r are both contained in some F_Y^t and, furthermore, either both are in the image of some $F_X^s(\mathbf{N})$ or neither is contained in $F(f)_*(F_X(\mathbf{N}))$.

Let $(E_i)_{i \in I}$ be the irreducible components of E, such that the stratum $U(t)$ defined by $t \in F_Y$ is a connected component of the locally closed subscheme (with reduced structure)

$$\left(\bigcap_{i \in I} E_i\right) \setminus \left(\bigcup_{i \notin I} E_i\right) \subset Y.$$

Write $r = \sum_{i \in I} r_i u_i$, where r_i is a non-negative integer and $u_i \in F_Y(\mathbf{N})$ is the primitive generator of the one-dimensional cone corresponding to E_i, so that $r_i \neq 0$ if and only if $i \in I$. The height of r is then given by the formula $h_Y(r) = \sum_{i \in I} r_i$.

We now construct a commutative diagram of log schemes

$$
\begin{array}{ccc}
(\mathrm{Spec}\ \mathbf{F}_v)^\dagger & \longrightarrow & (\mathcal{Y}_v, \mathcal{E}_v) \\
\downarrow & & \downarrow \\
(\mathrm{Spec}\ \mathcal{O}_v)^\dagger & \longrightarrow & (\mathrm{Spec}\ \mathcal{O}_v)^{\mathrm{tr}}.
\end{array}
\tag{6.1}
$$

Here $(\mathrm{Spec}\ \mathcal{O}_v)^{\mathrm{tr}}$ stands for the scheme $\mathrm{Spec}\ \mathcal{O}_v$ equipped with the trivial log structure $\mathcal{O}_v^* \to O_v$. The morphism $(\mathrm{Spec}\ \mathcal{O}_v)^\dagger \to (\mathrm{Spec}\ \mathcal{O}_v)^{\mathrm{tr}}$ is the forgetful morphism defined by the identity morphism of the underlying schemes and by the natural morphism of monoids $\mathcal{O}_v^* \to \mathcal{O}_v \setminus \{0\}$. The right hand vertical arrow is induced by the structure morphism $\mathcal{Y}_v \to \mathrm{Spec}\ \mathcal{O}_v$. Next, $(\mathrm{Spec}\ \mathbf{F}_v)^\dagger$ is the standard log point defined by the monoid $(\mathcal{O}_v \setminus \{0\})/(1+\pi_v \mathcal{O}_v)$. The left hand vertical arrow is the natural morphism of log schemes. On the underlying schemes the top horizontal arrow sends $\mathrm{Spec}\ \mathbf{F}_v$ to $P \bmod \pi_v$. The commutativity of (6.1) as a diagram of schemes is clear.

We need to define the top horizontal arrow as a morphism of log schemes. Let A be an $\mathcal{O}_v$-algebra such that $\mathrm{Spec}\ A$ is an affine neighborhood of $P \bmod \pi_v$ in $\mathcal{Y}_v$. We can assume that A contains a local equation π_i for the Zariski closure of E_i in $\mathcal{Y}_v$, where $i \in I$. Let $S \subset A$ be the multiplicative system generated by the π_i and let $S^{-1}A$ be the localisation of A with respect to S. The log scheme $(\mathcal{Y}, \mathcal{E})$ is defined by the subsheaf of $\mathcal{O}_{\mathcal{Y}}$ consisting of functions invertible outside $\mathcal{E}$. Hence the log scheme $(\mathrm{Spec}\, A, A \cap (S^{-1}A)^*)$ is an open subscheme of $(\mathcal{Y}, \mathcal{E})$. Locally the diagram of monoids attached to (6.1) is

$$
\begin{array}{ccc}
(\mathcal{O}_v \setminus \{0\})/(1+\pi_v \mathcal{O}_v) & \longleftarrow & A \cap (S^{-1}A)^* \\
{\scriptstyle \varphi}\uparrow & & \uparrow \\
\mathcal{O}_v \setminus \{0\} & \longleftarrow & \mathcal{O}_v^*.
\end{array}
\tag{6.2}
$$

Here φ is the canonical surjective morphism of monoids [(1)]

$$\varphi : \mathcal{O}_v \setminus \{0\} \longrightarrow (\mathcal{O}_v \setminus \{0\})/(1+\pi_v \mathcal{O}_v) \cong \mathbf{F}_v^* \oplus \mathbf{N},$$

where the isomorphism depends on the choice of π_v. Let us denote by $\overline{\varphi}$ the composition of φ with the projection to $\mathbf{F}_v^*$. The choice of π_i, $i \in I$, gives an isomorphism

$$A \cap (S^{-1}A)^* \cong A^* \times \mathbf{N}^I.$$

To complete the definition of $(\mathrm{Spec}\ \mathbf{F}_v)^\dagger \to (\mathcal{Y}_v, \mathcal{E}_v)$ we define

$$A^* \times \mathbf{N}^I \longrightarrow \mathbf{F}_v^* \oplus \mathbf{N}$$

as the morphism of monoids that sends $\alpha \in A^*$ to $(\alpha(P \bmod \pi_v), 0)$ and sends the canonical generator $1_i \in \mathbf{N}^I$ to $(\overline{\varphi}(\pi_i(\mathcal{P})), r_i)$. Checking the commutativity of the diagram (6.2) is straightforward. The factor $\overline{\varphi}(\pi_i(\mathcal{P}))$ is irrelevant for the commutativity of this diagram, but will play a role later in the proof.

(1) This is related to Denef's notion of *multiplicative residue* [13, Definition 3.2]. The projection to $\mathbf{F}_v^*$ is called the *angular component* in [13, Definition 3.5].

Since $\mathscr{Y} \to \operatorname{Spec} \mathscr{O}_S$ is smooth, the morphism of log schemes $(\mathscr{Y}, \mathscr{E}) \to (\operatorname{Spec} \mathscr{O}_v)^{\mathrm{tr}}$ is log smooth. Applying Proposition 5.13 to (6.1) produces a morphism $(\operatorname{Spec} \mathscr{O}_v)^{\dagger} \to (\mathscr{Y}, \mathscr{E})$ such that the two resulting triangles commute. This gives a point $Q \in V(k_v)$ such that

$$Q \bmod \pi_v = P \bmod \pi_v, \quad F(\mathscr{Q}) = r, \quad \overline{\varphi}(\pi_i(\mathscr{P})) = \overline{\varphi}(\pi_i(\mathscr{Q})), \quad i \in I.$$

Since $h_Y(F(\mathscr{Q})) \leq m$, our assumptions imply that $Q = f(R)$ for some $R \in U(k_v)$. In particular, $P \bmod \pi_v = f(R \bmod \pi_v)$ and $F(\mathscr{P}) = F(f)_*(a)$ for some $a \in F_X^s(\mathbf{N})$, where $s \in F_X$ maps to $t \in F_Y$ and $F(\mathscr{R}) \in F_X^s(\mathbf{N})$. Let $(D_j)_{j \in J}$ be the irreducible components of D such that the stratum $U(s)$ defined by $s \in F_X$ is a connected component of

$$\Big(\bigcap_{j \in J} D_j\Big) \setminus \Big(\bigcup_{j \notin J} D_j\Big) \subset X.$$

Write $a = \sum_{j \in J} a_j u'_j$, where a_j is an integer and $u'_j \in F_X(\mathbf{N})$ is the primitive generator of the one-dimensional cone corresponding to D_j, so that $a_j \neq 0$ if and only if $j \in J$.

Let us now construct a commutative diagram of log schemes

$$\begin{array}{ccc} (\operatorname{Spec} \mathbf{F}_v)^{\dagger} & \longrightarrow & (\mathscr{X}, \mathscr{D}) \\ \downarrow & & \downarrow \\ (\operatorname{Spec} \mathscr{O}_v)^{\dagger} & \longrightarrow & (\mathscr{Y}, \mathscr{E}). \end{array} \tag{6.3}$$

Here the vertical arrows are the obvious morphisms, and the lower horizontal arrow is given by $\mathscr{P} \in \mathscr{Y}_v(\mathscr{O}_v)$. The upper horizontal arrow sends $\operatorname{Spec} \mathbf{F}_v$ to $R \bmod \pi_v$. The commutativity of (6.3) as a diagram of schemes is clear since $P \bmod \pi_v = f(R \bmod \pi_v)$.

Choose an A-algebra B such that $\operatorname{Spec} B$ is an affine neighborhood of $R \bmod \pi_v$ in $\mathscr{X}_v$. We can assume that B contains a local equation ϖ_j for the closure of D_j in $\mathscr{X}_v$, where $j \in J$. Define $(\operatorname{Spec} \mathbf{F}_v)^{\dagger} \to (\mathscr{X}, \mathscr{D})$ via the morphism of monoids

$$B \cap (S^{-1}B)^* \cong B^* \oplus \mathbf{N}^J \longrightarrow \mathbf{F}_v^* \oplus \mathbf{N},$$

such that $\beta \in B^*$ goes to $(\beta(R \bmod \pi_v), 0)$ and

$$1_j \mapsto (\overline{\varphi}(\varpi_j(\mathscr{R})), a_j).$$

To check the commutativity of (6.3) as a diagram of log schemes we need to check the commutativity of the diagram of monoids

$$\begin{array}{ccc} \mathbf{F}_v^* \oplus \mathbf{N} & \longleftarrow & B^* \oplus \mathbf{N}^J \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ \mathscr{O}_v \setminus \{0\} & \longleftarrow & A^* \oplus \mathbf{N}^I. \end{array}$$

For $(\alpha, 0) \in A^* \oplus \mathbf{N}^I$ the commutativity follows from $P \bmod \pi_v = f(R \bmod \pi_v)$. Write

$$\pi_i = b_i \prod \varpi_j^{m_{ij}},$$

where $b_i \in B^*$. The right hand vertical map sends the element $(0, 1_i) \in A^* \oplus \mathbf{N}^I$ to the element $(b_i, \sum_{j \in J} m_{ij} 1_j)$ of $B^* \oplus \mathbf{N}^J$, which then goes to

$$\Big(b_i(R \bmod \pi_v) \prod_{j \in J} \overline{\varphi}(\varpi_j(\mathscr{R}))^{m_{ij}}, \sum_{j \in J} m_{ij} a_j\Big). \tag{6.4}$$

The lower horizontal arrow sends 1_i to $\pi_i(\mathscr{P})$ and then the left hand vertical arrow gives $(\overline{\varphi}(\pi_i(\mathscr{P})), \operatorname{val}_v(\pi_i(\mathscr{P})))$. This coincides with (6.4). Indeed, on the one hand, $\operatorname{val}_v(\pi_i(\mathscr{P}))$ is the i-th coordinate of $F(\mathscr{P}) = F(f)_*(a)$, which equals $\sum_{j\in J} m_{ij}a_j$. On the other hand, $\overline{\varphi}(\pi_i(\mathscr{P})) = \overline{\varphi}(\pi_i(\mathscr{Q}))$ and $Q = f(R)$. This proves the commutativity of (6.3).

The morphism $(\mathscr{X}, \mathscr{D}) \to (\mathscr{Y}, \mathscr{E})$ is log smooth, hence we can apply Proposition 5.13 to the diagram (6.3). We deduce that there is a morphism $(\mathrm{Spec}\ \mathcal{O}_v)^\dagger \to (\mathscr{X}, \mathscr{D})$ whose composition with $f : (\mathscr{X}, \mathscr{D}) \to (\mathscr{Y}, \mathscr{E})$ is the morphism $(\mathrm{Spec}\ \mathcal{O}_v)^\dagger \to (\mathscr{Y}, \mathscr{E})$ given by P. Therefore $P \in f(X(k_v))$. This proves that $Y(k_v) = f(X(k_v))$. □

6.3. Further modifications of the log smooth model

We now use the bound obtained in Proposition 6.1 to construct a specific modification of the morphism f.

PROPOSITION 6.2. – *Let $f : X \to Y$ and $m \geq 1$ be as above. There exists a commutative diagram of morphisms of log regular schemes*

$$\begin{array}{ccc} (X', D') & \xrightarrow{\sigma_X} & (X, D) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (Y', E') & \xrightarrow{\sigma_Y} & (Y, E), \end{array} \tag{6.5}$$

where X' and Y' are smooth, proper and geometrically integral, f' is dominant and log smooth, σ_X and σ_Y are log blow-ups and the following holds: if $P \in V(k_v)$ satisfies $1 \leq h_Y(F(\mathscr{P})) \leq m$, then $\sigma_Y^{-1}(P) \bmod \pi_v$ is a smooth point of the reduction of E'.

The last statement of the proposition implies that $\sigma_Y^{-1}(P) \bmod \pi_v$ belongs to exactly one geometric irreducible component of E'.

Proof. – We let $\sigma_Y : (Y', E') \to (Y, E)$ be the log blow-up defined by the subdivision of the Kato fan of (Y, E) as in Lemma 5.10. We see that Y' is smooth and proper and E' is a strict normal crossings divisor (so (Y', E') is log regular).

Recall that in the category of fs log schemes, log blow-ups are stable under composition [27, Corollary 4.11] and under base change [27, Proposition 4.5, Corollary 4.8].

Indeed, Nizioł shows in [27, Corollary 4.8] that the fs fibered product of (X, D) and (Y', E') over (Y, E) is a log blow-up $\sigma_X : (X', D') \to (X, D)$ making (6.5) commute. More precisely, if (Y', E') is the log blow-up of (Y, E) in a coherent ideal $\mathscr{J} \subset \mathscr{M}_Y$, then (X', D') is the log blow-up of (X, D) in the inverse image ideal $\mathscr{J}\mathscr{M}_X$.

We note that (X', D') is a log regular scheme by [22, Theorem 8.2]. Since log smooth morphisms are stable under base change, the morphism $f' : (X', D') \to (Y', E')$ is log smooth. There is a further log blow-up $(X,'' D'') \to (X', D')$ such that X'' is smooth as a scheme (see [22, (10.4)] or [27, Theorem 5.8]). The composition $(X,'' D'') \to (Y', E')$ is still log smooth. Therefore replacing (X', D') by $(X,'' D'')$ yields the result. □

Recall that the codimension 1 strata of (Y', E') are precisely the irreducible components of the smooth locus of E'. The morphisms σ_X and σ_Y in Proposition 6.2 are log blow-ups, so they induce isomorphisms $X' \setminus D' \cong U = X \setminus D$ and $Y' \setminus E' \cong V = Y \setminus E$.

COROLLARY 6.3. – *Let $f' : (X', D') \to (Y', E')$ be as in Proposition 6.2. Then there exists a finite set $S \subset \Omega_{k,\mathrm{f}}$ such that for $v \notin S$ we have $f(X(k_v)) = Y(k_v)$ if and only if $s_{f',\kappa(Z)}(v) = 1$ for each codimension 1 stratum Z of (Y', E').*

Proof. – Let $f' : (\mathcal{X}', \mathcal{D}') \to (\mathcal{Y}', \mathcal{E}')$ be a model of $(X', D') \to (Y', E')$ over $\mathcal{O}_{k,S}$. We enlarge the finite set of places S to ensure that $f' : (\mathcal{X}', \mathcal{D}') \to (\mathcal{Y}', \mathcal{E}')$ is a proper, log smooth morphism of log regular schemes over $\mathcal{O}_{k,S}$ such that the induced morphism of Kato fans is the same as for $(X', D') \to (Y', E')$.

Let $W = Y' \setminus E'_{\text{sing}}$ and let $\mathcal{W} \subset \mathcal{Y}'$ be the complement to the Zariski closure of E'_{sing} in $\mathcal{Y}'$. For a codimension 1 stratum Z of (Y', E') we denote by $\mathcal{Z}$ the Zariski closure of Z in $\mathcal{W}$. We now further enlarge S to ensure that the following properties hold.

(1) For any $v \notin S$ and for any $x \in \mathcal{W}(\mathbf{F}_v)$ such that the fiber $(f')^{-1}(x)$ is split, there is a smooth $\mathbf{F}_v$-point in $(f')^{-1}(x)$. This is arranged via the Lang-Weil inequality.

(2) For each codimension 1 stratum Z of (Y', E'), for any $v \notin S$ such that $s_{f',\kappa(Z)}(v) = 1$ and for any $x \in \mathcal{Z}(\mathbf{F}_v)$ the fiber $(f')^{-1}(x)$ is split. This is achieved by applying Corollary 3.15 to the morphism $(f')^{-1}(Z) \to Z$. Indeed, in this case the assumption of this corollary is satisfied by Proposition 5.18.

(3) For any $v \notin S$, if there exists a codimension 1 stratum Z of (Y', E') such that $s_{f',\kappa(Z)}(v) < 1$, then $Y(k_v) \neq f(X(k_v))$. This follows from Theorem 4.2.

Let us now prove the statement of the corollary. Fix any $v \notin S$. One implication is immediate: if there exists a codimension 1 stratum Z of (Y', E') with $s_{f',\kappa(Z)}(v) < 1$, then (3) implies that $Y(k_v) \neq f(X(k_v))$.

Conversely, assume that $s_{f',Z}(v) = 1$ for each codimension 1 stratum Z of (Y', E'). By Propositions 6.1 and 6.2, it suffices to prove that if $P \in V(k_v)$ is such that $P \bmod \pi_v$ is in $\mathcal{W} \times_{\operatorname{Spec} O_S} \operatorname{Spec} \mathbf{F}_v$, then $P \in f(X(k_v))$. Write $\mathcal{U} = \mathcal{X}' \setminus \mathcal{D}'$ and $\mathcal{V} = \mathcal{Y}' \setminus \mathcal{E}'$.

Let us first consider the case when $P \bmod \pi_v \in \mathcal{V}(\mathbf{F}_v)$. The morphism $f'|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ is smooth and proper, with geometrically integral generic fiber. This implies that all fibers of $f'|_{\mathcal{U}}$ are smooth and geometrically integral; in particular, such is $(f')^{-1}(P \bmod \pi_v)$. By (1) this fiber has a smooth $\mathbf{F}_v$-point, which is clearly also a smooth point of f'.

In the case when $P \bmod \pi_v \in \mathcal{Z}(\mathbf{F}_v)$ for some codimension 1 stratum Z of (Y', E'), a combination of (1) and (2) implies that the fiber $(f')^{-1}(P \bmod \pi_v)$ has a smooth $\mathbf{F}_v$-point. But a smooth point in this fiber is actually a smooth point of the morphism f' as well, since $f'|_{(f')^{-1}(\mathcal{W})}$ is a flat morphism, by [21, Corollary 4.4.(ii), Corollary 4.5].

An application of the classical version of Hensel's lemma now allows one to lift such an $\mathbf{F}_v$-point to a k_v-point of X' over P in both of the cases considered above, as required. □

6.4. Proofs of the main theorems

Theorem 1.5 now follows immediately from Corollary 6.3, and Theorem 1.4 follows from Lemma 3.11, Corollary 4.4 and Theorem 1.5. Finally, Theorem 1.6 is a formal consequence of Lemma 3.10 and Theorem 1.5, because the intersection of finitely many Frobenian sets is Frobenian (see [31, §3.3.1]). □

REMARK 6.4. – Let us finish by explaining how to recover Denef's result (Theorem 1.2) from our Theorem 1.4. The subtlety is the following. Denef imposes a condition on modifications $f' : X' \to Y'$ of f such that the generic fibers of f and f' are isomorphic. We consider *arbitrary* modifications $f' : X' \to Y'$ of f, since using Theorem 5.11 forces us to consider modifications for which the generic fiber changes birationally, and we impose a *weaker* condition for these modifications (pseudo-splitness instead of splitness).

But in fact we could just as well have imposed our pseudo-splitness condition only for the modifications $f' : X' \to Y'$ with generic fiber isomorphic to the generic fiber of f, as does Denef. It turns out that this is enough to guarantee that the same condition holds for arbitrary modifications, as we will now explain.

The argument is the following. Let $f : X \to Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over k, with geometrically integral generic fiber. Assume that for every modification $f' : X' \to Y'$ of f with X' and Y' smooth such that the generic fibers of f and f' are isomorphic and for every $D \in (Y')^{(1)}$, the fiber $(f')^{-1}(D)$ is a pseudo-split $\kappa(D)$-variety. Let us check that the same property then holds for an arbitrary modification $f' : X' \to Y'$ of f with X' and Y' smooth.

So let $f' : X' \to Y'$ be such an arbitrary modification. Let Z be the unique irreducible component of $X \times_Y Y'$ which dominates X, equipped with the natural morphism $Z \to Y'$. Note that the generic fiber of $Z \to Y'$ is the same as the generic fiber of f. Choose a desingularisation $\widetilde{Z} \to Z$ such that the composition $\widetilde{Z} \to Y'$ still has the same generic fiber as f. Let $D \in (Y')^{(1)}$ and let $R = \mathcal{O}_{Y',D}$ be the corresponding discrete valuation ring. Consider the R-scheme $\widetilde{Z} \times_{Y'} \operatorname{Spec} R$. This is a regular scheme with the same generic fiber as f. The assumptions imply that its special fiber is pseudo-split.

However the R-scheme $X' \times_{Y'} \operatorname{Spec} R$ is regular and its generic fiber is smooth and birational to the generic fiber of f. Therefore Lemma 2.12 implies that its special fiber is pseudo-split as well, which proves our claim.

REMARK 6.5. – Yongqi Liang observed that exactly the same proofs yield in fact a slightly stronger version of Theorem 1.4. Indeed, the conclusion that $f(X(k_v)) = Y(k_v)$ for almost all places v (assuming the pseudo-splitness assumption) may be replaced by the following stronger conclusion: there exists a finite set of places S of k such that if K/k is a finite field extension and if w is a place of K which does not lie above any of the places in S, then $f(X(K_w)) = Y(K_w)$.

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MIXED HODGE STRUCTURES AND FORMALITY OF SYMMETRIC MONOIDAL FUNCTORS

BY Joana CIRICI AND Geoffroy HOREL

ABSTRACT. – We use mixed Hodge theory to show that the functor of singular chains with rational coefficients is formal as a lax symmetric monoidal functor, when restricted to complex varieties whose weight filtration in cohomology satisfies a certain purity property. This has direct applications to the formality of operads or, more generally, of algebraic structures encoded by a colored operad. We also prove a dual statement, with applications to formality in the context of rational homotopy theory. In the general case of complex varieties with non-pure weight filtration, we relate the singular chains functor to a functor defined via the first term of the weight spectral sequence.

RÉSUMÉ. – Nous utilisons la théorie de Hodge mixte pour montrer que le foncteur des chaînes singulières à coefficients rationnels est formel, comme foncteur symétrique monoïdal lax, lorsqu'on le restreint aux variétés complexes dont la filtration par le poids en cohomologie satisfait une certaine propriété de pureté. Ce résultat a des applications directes à la formalité d'opérades ou plus généralement à des structures algébriques encodées par une opérade colorée. Nous prouvons aussi le résultat dual, avec des applications à la formalité dans le contexte de la théorie de l'homotopie rationnelle. Dans le cas général d'une variété dont la filtration par le poids n'est pas pure, nous relions le foncteur des chaînes singulières à un foncteur défini par la première page de la suite spectrale des poids.

1. Introduction

There is a long tradition of using Hodge theory as a tool for proving formality results. The first instance of this idea can be found in [18] where the authors prove that compact Kähler manifolds are formal (i.e., the commutative differential graded algebra of differential forms is quasi-isomorphic to its cohomology). In the introduction of that paper, the authors explain that their intuition came from the theory of étale cohomology and the fact that the degree n étale cohomology group of a smooth projective variety over a finite field is pure

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of weight n. This purity is what heuristically prevents the existence of non-trivial Massey products. In the setting of complex algebraic geometry, Deligne introduced in [16, 17] a filtration on the rational cohomology of every complex algebraic variety X, called the *weight filtration*, with the property that each of the successive quotients of this filtration behaves as the cohomology of a smooth projective variety, in the sense that it has a Hodge-type decomposition. Deligne's mixed Hodge theory was subsequently promoted to the rational homotopy of complex algebraic varieties (see [35], [27], [36]). This can then be used to make the intuition of the introduction of [18] precise. In [22] and [12], it is shown that purity of the weight filtration in cohomology implies formality, in the sense of rational homotopy, of the underlying topological space. However, the treatment of the theory in these references lacks functoriality and is restricted to smooth varieties in the first paper and to projective varieties in the second.

In another direction, in the paper [26], the authors elaborate on the method of [18] and prove that operads (as well as cyclic operads, modular operads, etc.) internal to the category of compact Kähler manifolds are formal. Their strategy is to introduce the functor of de Rham currents which is a functor from compact Kähler manifolds to chain complexes that is lax symmetric monoidal and quasi-isomorphic to the singular chain functor as a lax symmetric monoidal functor. Then they show that this functor is formal as a lax symmetric monoidal functor. Recall that, if $\mathcal{C}$ is a symmetric monoidal category and $\mathcal{A}$ is an abelian symmetric monoidal category, a lax symmetric monoidal functor $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ is said to be formal if it is weakly equivalent to $H \circ F$ in the category of lax symmetric monoidal functors. It is then straightforward to see that such functors send operads in $\mathcal{C}$ to formal operads in $\mathrm{Ch}_*(\mathcal{A})$. The functoriality also immediately gives us that a map of operads in $\mathcal{C}$ is sent to a formal map of operads or that an operad with an action of a group G is sent to a formal operad with a G-action. Of course, there is nothing specific about operads in these statements and they would be equally true for monoids, cyclic operads, modular operads, or more generally any algebraic structure that can be encoded by a colored operad.

The purpose of this paper is to push this idea of formality of symmetric monoidal functors from complex algebraic varieties in several directions in order to prove the most general possible theorem of the form "purity implies formality". Before explaining our results more precisely, we need to introduce a bit of terminology.

Let X be a complex algebraic variety. Deligne's weight filtration on the rational n-th cohomology vector space of X is bounded by

$$0 = W_{-1}H^n(X,\mathbb{Q}) \subseteq W_0H^n(X,\mathbb{Q}) \subseteq \cdots \subseteq W_{2n}H^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q}).$$

If X is smooth then $W_{n-1}H^n(X,\mathbb{Q}) = 0$, while if X is projective $W_nH^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q})$. In particular, if X is smooth and projective then we have

$$0 = W_{n-1}H^n(X,\mathbb{Q}) \subseteq W_nH^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q}).$$

In this case, the weight filtration on $H^n(X,\mathbb{Q})$ is said to be *pure of weight n*. More generally, for α a rational number and X a complex algebraic variety, we say that the weight filtration on $H^*(X,\mathbb{Q})$ is *α-pure* if, for all $n \geq 0$, we have

$$\mathrm{Gr}^W_p H^n(X,\mathbb{Q}) := \frac{W_pH^n(X,\mathbb{Q})}{W_{p-1}H^n(X,\mathbb{Q})} = 0 \text{ for all } p \neq \alpha n.$$

The bounds on the weight filtration tell us that this makes sense only when $0 \leq \alpha \leq 2$. Note as well that if we write $\alpha = a/b$ with $(a, b) = 1$, α-purity implies that the cohomology is concentrated in degrees that are divisible by b, and that $H^{bn}(X, \mathbb{Q})$ is pure of weight an.

Aside from smooth projective varieties, some well-known examples of varieties with 1-pure weight filtration are: projective varieties whose underlying topological space is a $\mathbb{Q}$-homology manifold ([17, Theorem 8.2.4]) and the moduli spaces $\mathcal{M}_{Dol}$ and $\mathcal{M}_{dR}$ appearing in the non-abelian Hodge correspondence ([28]). Some examples of varieties with 2-pure weight filtration are: complements of hyperplane arrangements ([33]), which include the moduli spaces $\mathcal{M}_{0,n}$ of smooth projective curves of genus 0 with n marked points, and complements of toric arrangements ([22]). As we shall see in Section 8, complements of codimension d subspace arrangements are examples of smooth varieties whose weight filtration in cohomology is $2d/(2d-1)$-pure. For instance, this includes configuration spaces of points in $\mathbb{C}^d$.

Our main result is Theorem 7.3. We show that, for a non-zero rational number α, the singular chains functor

$$S_*(-, \mathbb{Q}) : \mathrm{Var}_{\mathbb{C}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to complex varieties whose weight filtration in cohomology is α-pure. Here $\mathrm{Var}_{\mathbb{C}}$ denotes the category of complex algebraic varieties (i.e the category of schemes over $\mathbb{C}$ that are reduced, separated and of finite type). This generalizes the main result of [26] on the formality of $S_*(X, \mathbb{Q})$ for any operad X in smooth projective varieties, to the case of operads in possibly singular and/or non-compact varieties with pure weight filtration in cohomology.

As direct applications of the above result, we prove formality of the operad of singular chains of some operads in complex varieties, such as the noncommutative analog of the (framed) little 2-discs operad introduced in [19] and the monoid of self-maps of the complex projective line studied by Cazanave in [11] (see Theorems 7.4 and 7.7). We also reinterpret in the language of mixed Hodge theory the proofs of the formality of the little disks operad and Getzler's gravity operad appearing in [38] and [23]. These last two results do not fit directly in our framework, since the little disks operad and the gravity operad do not quite come from operads in algebraic varieties. However, the action of the Grothendieck-Teichmüller group provides a bridge to mixed Hodge theory.

In Theorem 8.1 we prove a dual statement of our main result, showing that Sullivan's functor of piecewise linear forms

$$\mathcal{A}^*_{PL} : \mathrm{Var}_{\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to varieties whose weight filtration in cohomology is α-pure, where α is a non-zero rational number.

This gives functorial formality in the sense of rational homotopy for such varieties, generalizing both "purity implies formality" statements appearing in [22] for smooth varieties and in [12] for singular projective varieties. Our generalization is threefold: we allow α-purity (instead of just 1-and 2-purity), we obtain functoriality and we study possibly singular and open varieties simultaneously.

Theorems 7.3 and 8.1 deal with situations in which the weight filtration is pure. In the general context with mixed weights, it was shown by Morgan [35] for smooth varieties and in [13] for possibly singular varieties, that the first term of the multiplicative weight spectral sequence carries all the rational homotopy information of the variety. In Theorem 7.8 we provide the analogous statement for the lax symmetric monoidal functor of singular chains. A dual statement for Sullivan's functor of piecewise linear forms is proven in Theorem 8.11, enhancing the results of [35] and [13] with functoriality.

We now explain the structure of this paper. The first four sections are purely algebraic. In Section 2 we collect the main properties of formal lax symmetric monoidal functors that we use. In particular, in Theorem 2.3 we recall a recent theorem of rigidification due to Hinich that says that, over a field of characteristic zero, formality of functors can be checked at the level of ∞-functors. We also introduce the notion of α-purity for complexes of bigraded objects in a symmetric monoidal abelian category and show that, when restricted to α-pure complexes, the functor defined by forgetting the degree is formal.

The connection of this result with mixed Hodge structures is done in Section 3, where we prove a symmetric monoidal version of Deligne's weak splitting of mixed Hodge structures over $\mathbb{C}$. Such splitting is a key tool towards formality. In Section 4 we study lax symmetric monoidal functors to vector spaces over a field of characteristic zero equipped with a compatible filtration. We show, in Theorem 4.3, that the existence of a lax symmetric monoidal splitting for such functors can be verified after extending the scalars to a larger field. As a consequence, we obtain splittings for the weight filtration over $\mathbb{Q}$. This result enables us to bypass the theory of descent of formality for operads of [26], which assumes the existence of minimal models. Putting the above results together we are able to show that the forgetful functor

$$\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}}) \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

induced by sending a rational mixed Hodge structure to its underlying vector space is formal when restricted to those complexes whose mixed Hodge structure in homology is α-pure.

In order to obtain a symmetric monoidal functor from the category of complex varieties to an algebraic category encoding mixed Hodge structures, we have to consider more flexible objects than complexes of mixed Hodge structures. This is the content of Section 5, where we study the category $\mathrm{MHC}_{\mathbb{k}}$ of mixed Hodge complexes. In Theorem 5.4 we construct an equivalence of symmetric monoidal ∞-categories between mixed Hodge complexes and complexes of mixed Hodge structures. This result is a lift of Beilinson's equivalence of triangulated categories $D^b(\mathrm{MHS}_{\mathbb{k}}) \longrightarrow \mathrm{ho}(\mathrm{MHC}_{\mathbb{k}})$ (see also [20], [9]).

The geometric character of this paper comes in Section 6, where we construct a symmetric monoidal functor from complex varieties to mixed Hodge complexes. This is done in two steps. First, for smooth varieties, we dualize Navarro's construction [36] of functorial mixed Hodge complexes to obtain a symmetric monoidal ∞-functor

$$\mathscr{D}_* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

such that its composite with the forgetful functor $\mathbf{MHC}_{\mathbb{Q}} \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$ is naturally weakly equivalent to $S_*(-,\mathbb{Q})$ as a symmetric monoidal ∞-functor (see Theorem 6.5). Note that in order to obtain monoidality, we move to the world of ∞-categories, denoted in boldface letters. In the second step, we extend this functor from smooth, to singular varieties, by standard cohomological descent arguments.

The main results of this paper are stated and proven in Section 7, where we also explain several applications to operad formality. Lastly, Section 8 contains applications to the rational homotopy theory of complex varieties.

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Notations

As a rule, we use boldface letters to denote ∞-categories and normal letters to denote 1-categories. For $\mathcal{C}$ a 1-category, we denote by $\mathbf{N}(\mathcal{C})$ its nerve seen as an ∞-category.

For $\mathscr{A}$ an additive category, we will denote by $\mathrm{Ch}_*^?(\mathscr{A})$ the category of (homologically graded) chain complexes in $\mathscr{A}$, where "?" denotes the boundedness condition: nothing for unbounded, b for bounded below and above and ≥ 0 (resp. ≤ 0) for non-negatively (resp. non-positively) graded complexes. We denote by $\mathbf{Ch}_*^?(\mathscr{A})$ the ∞-category obtained from $\mathrm{Ch}_*^?(\mathscr{A})$ by inverting the quasi-isomorphisms.

2. Formal symmetric monoidal functors

The main result of this section is a "purity implies formality" statement in the setting of symmetric monoidal functors.

Let $(\mathscr{A}, \otimes, \mathbf{1})$ be an abelian symmetric monoidal category with infinite direct sums. The homology functor $H : \mathrm{Ch}_*(\mathscr{A}) \longrightarrow \prod_{n\in\mathbb{Z}} \mathscr{A}$ is a lax symmetric monoidal functor, via the usual Künneth morphism. In the cases that will interest us, all the objects of $\mathscr{A}$ will be flat and the homology functor is in fact strong symmetric monoidal. We will also make the small abuse of identifying the category $\prod_{n\in\mathbb{Z}} \mathscr{A}$ with the full subcategory of $\mathrm{Ch}_*(\mathscr{A})$ spanned by the chain complexes with zero differential.

We recall the following definition from [26].

Definition 2.1. – Let $\mathcal{C}$ be a symmetric monoidal category and $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathscr{A})$ a lax symmetric monoidal functor. Then F is said to be a *formal lax symmetric monoidal* functor if F and $H \circ F$ are *weakly equivalent* in the category of lax symmetric monoidal functors: there is a string of natural transformations of lax symmetric monoidal functors

$$F \xleftarrow{\Phi_1} F_1 \longrightarrow \cdots \longleftarrow F_n \xrightarrow{\Phi_n} H \circ F,$$

such that for every object X of $\mathcal{C}$, the morphisms $\Phi_i(X)$ are quasi-isomorphisms.

Definition 2.2. – Let $\mathcal{C}$ be a symmetric monoidal category and $F : \mathbf{N}(\mathcal{C}) \to \mathbf{Ch}_*(\mathscr{A})$ a lax symmetric monoidal functor (in the ∞-categorical sense). We say that F is a *formal lax symmetric monoidal ∞-functor* if F and $H \circ F$ are equivalent in the ∞-category of lax symmetric monoidal functors from $\mathbf{N}(\mathcal{C})$ to $\mathbf{Ch}_*(\mathscr{A})$.

Clearly a formal lax symmetric monoidal functor $\mathcal{C} \to \mathrm{Ch}_*(\mathscr{A})$ induces a formal lax symmetric monoidal ∞-functor $\mathbf{N}(\mathcal{C}) \to \mathbf{Ch}_*(\mathscr{A})$. The following theorem and its corollary give a partial converse.

THEOREM 2.3 (Hinich). – *Let $\Bbbk$ be a field of characteristic* 0*. Let $\mathcal{C}$ be a small symmetric monoidal category. Let F and G be two lax symmetric monoidal functors $\mathcal{C} \to \mathrm{Ch}_*(\Bbbk)$. If F and G are equivalent as lax symmetric monoidal ∞-functors $\mathbf{N}(\mathcal{C}) \longrightarrow \mathbf{Ch}_*(\Bbbk)$, then F and G are weakly equivalent as lax symmetric monoidal functors.*

Proof. – This theorem is true more generally if $\mathcal{C}$ is a colored operad. Indeed recall that any symmetric monoidal category has an underlying colored operad whose category of algebras is equivalent to the category of lax symmetric monoidal functors out of the original category.

Now since we are working in characteristic zero, the operad underlying $\mathcal{C}$ is homotopically sound (following the terminology of [29]). Therefore, [29, Theorem 4.1.1] gives us an equivalence of ∞-categories

$$\mathbf{N}(\mathrm{Alg}_{\mathcal{C}}(\mathrm{Ch}_*(\Bbbk)) \xrightarrow{\sim} \mathbf{Alg}_{\mathcal{C}}(\mathbf{Ch}_*(\Bbbk))$$

where we denote by $\mathrm{Alg}_{\mathcal{C}}$ (resp. $\mathbf{Alg}_{\mathcal{C}}$) the category of lax symmetric monoidal functors (resp. the ∞-category of lax symmetric monoidal functors) out of $\mathcal{C}$. Now, the two functors F and G are two objects in the source of the above map that become weakly equivalent in the target. Hence, they are already equivalent in the source, which is precisely saying that they are connected by a zig-zag of weak equivalences of lax symmetric monoidal functors. □

COROLLARY 2.4. – *Let $\Bbbk$ be a field of characteristic* 0*. Let $\mathcal{C}$ be a small symmetric monoidal category. Let $F : \mathcal{C} \to \mathrm{Ch}_*(\Bbbk)$ be a lax symmetric monoidal functor. If F is formal as lax symmetric monoidal ∞-functor $\mathbf{N}(\mathcal{C}) \longrightarrow \mathbf{Ch}_*(\Bbbk)$, then F is formal as a lax symmetric monoidal functor.*

Proof. – It suffices to apply Theorem 2.3 to F and $H \circ F$. □

The following proposition whose proof is trivial is the reason we are interested in formal lax monoidal functors.

PROPOSITION 2.5 ([26], Proposition 2.5.5). – *If $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathscr{A})$ is a formal lax symmetric monoidal functor then F sends operads in $\mathcal{C}$ to formal operads in $\mathrm{Ch}_*(\mathscr{A})$.*

In rational homotopy, there is a criterion of formality in terms of weight decompositions which proves to be useful in certain situations (see for example [6] and [5]). We next provide an analogous criterion in the setting of symmetric monoidal functors.

Denote by $\mathrm{gr}\,\mathscr{A}$ the category of graded objects of $\mathscr{A}$. It inherits a symmetric monoidal structure from that of $\mathscr{A}$, with the tensor product defined by

$$(A \otimes B)^n := \bigoplus_p A^p \otimes B^{p-n}.$$

The unit in $\mathrm{gr}\,\mathscr{A}$ is given by $\mathbf{1}$ concentrated in degree zero. The functor $U : \mathrm{gr}\,\mathscr{A} \longrightarrow \mathscr{A}$ obtained by forgetting the degree is strong symmetric monoidal. The category of graded complexes $\mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A})$ inherits a symmetric monoidal structure via a graded Künneth morphism.

Definition 2.6. – Given a rational number α, denote by $\mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A})^{\alpha\text{-pure}}$ the full subcategory of $\mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A})$ given by those graded complexes $A = \bigoplus A_n^p$ with *α-pure homology*:

$$H_n(A)^p = 0 \text{ for all } p \neq \alpha n.$$

Note that if $\alpha = a/b$, with a and b coprime, then the above condition implies that $H_*(A)$ is concentrated in degrees that are divisible by b, and in degree kb, it is pure of weight ka:

$$H_{kb}(A)^p = 0 \text{ for all } p \neq ka.$$

Proposition 2.7. – *Let $\mathscr{A}$ be an abelian category and α a non-zero rational number. The functor $U : \mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A})^{\alpha\text{-pure}} \longrightarrow \mathrm{Ch}_*(\mathscr{A})$ defined by forgetting the degree is formal as a lax symmetric monoidal functor.*

Proof. – We will define a functor $\tau : \mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A}) \longrightarrow \mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A})$ together with natural transformations

$$\Phi : U \circ \tau \Rightarrow U \text{ and } \Psi : U \circ \tau \Rightarrow H \circ U$$

giving rise to weak equivalences when restricted to chain complexes with α-pure homology.

Consider the truncation functor $\tau : \mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A}) \longrightarrow \mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A})$ defined by sending a graded chain complex $A = \bigoplus A_n^p$ to the graded complex given by:

$$(\tau A)_n^p := \begin{cases} A_n^p & n > \lceil p/\alpha \rceil, \\ \mathrm{Ker}(d : A_n^p \to A_{n-1}^p) & n = \lceil p/\alpha \rceil, \\ 0 & n < \lceil p/\alpha \rceil, \end{cases}$$

where $\lceil p/\alpha \rceil$ denotes the smallest integer greater than or equal to p/α. Note that for each p, $\tau(A)_*^p$ is the chain complex given by the canonical truncation of A_*^p at $\lceil p/\alpha \rceil$, which satisfies

$$H_n(\tau(A)_*^p) \cong H_n(A_*^p) \text{ for all } n \geq \lceil p/\alpha \rceil.$$

To prove that τ is a lax symmetric monoidal functor it suffices to see that

$$\tau(A)_n^p \otimes \tau(B)_m^q \subseteq \tau(A \otimes B)_{n+m}^{p+q}$$

for all $A, B \in \mathrm{Ch}_*(\mathrm{gr}\,\mathscr{A})$. By symmetry in A and B, it suffices to consider the following three cases :

1. If $n > \lceil p/\alpha \rceil$ and $m \geq \lceil q/\alpha \rceil$ then $n + m > \lceil p/\alpha \rceil + \lceil q/\alpha \rceil \geq \lceil (p+q)/\alpha \rceil$. Therefore we have $\tau(A \otimes B)_{n+m}^{p+q} = (A \otimes B)_{n+m}^{p+q}$ and the above inclusion is trivially satisfied.
2. If $n = \lceil p/\alpha \rceil$ and $m = \lceil q/\alpha \rceil$ then $n + m = \lceil p/\alpha \rceil + \lceil q/\alpha \rceil \geq \lceil (p+q)/\alpha \rceil$. Now, if $n + m > \lceil (p+q)/\alpha \rceil$ then again we have $\tau(A \otimes B)_{n+m}^{p+q} = (A \otimes B)_{n+m}^{p+q}$. If $n + m = \lceil (p+q)/\alpha \rceil$ then the above inclusion reads

$$\mathrm{Ker}(d : A_n^p \to A_{n-1}^p) \otimes \mathrm{Ker}(d : B_m^q \to B_{m-1}^q) \subseteq \mathrm{Ker}(d : (A \otimes B)_{n+m}^{p+q} \to (A \otimes B)_{n+m-1}^{p+q}).$$

   This is verified by the Leibniz rule.
3. Lastly, if $n < \lceil p/\alpha \rceil$ then $\tau(A)_n^p = 0$ and there is nothing to verify.

The projection $\mathrm{Ker}(d : A_n^p \to A_{n-1}^p) \twoheadrightarrow H_n(A)^p$ defines a morphism $\tau A \to H(A)$ by

$$(\tau A)_n^p \mapsto \begin{cases} 0 & n \neq \lceil p/\alpha \rceil, \\ H_n(A)^p & n = \lceil p/\alpha \rceil. \end{cases}$$

This gives a symmetric monoidal natural transformation $\Psi : U \circ \tau \Rightarrow H \circ U = U \circ H$. Likewise, the inclusion $\tau A \hookrightarrow A$ defines a symmetric monoidal natural transformation $\Phi : U \circ \tau \Rightarrow U$.

Let A be a complex of $\mathrm{Ch}_*(\mathrm{gr}\,\mathcal{A})^{\alpha\text{-pure}}$. Then both morphisms

$$\Psi(A) : \tau \circ U(A) \to H \circ U(A) \text{ and } \Phi(A) : U \circ \tau(A) \to U(A)$$

are clearly quasi-isomorphisms. □

For graded chain complexes whose homology is pure up to a certain degree, we obtain a result of partial formality as follows.

Definition 2.8. – Let $q \geq 0$ be an integer. A morphism of chain complexes $f : A \to B$ is called a *q-quasi-isomorphism* if the induced morphism in homology $H_i(f) : H_i(A) \to H_i(B)$ is an isomorphism for all $i \leq q$.

Remark 2.9. – There is a notion of q-quasi-isomorphism in rational homotopy which asks in addition that the map induced in degree $(q+1)$-cohomology is a monomorphism. Dually, for chain complexes one could ask to have an epimorphism in degree $(q+1)$-homology. Note that we don't consider this extra condition here, since we work with possibly negatively and positively graded complexes and such a condition would break the symmetry. In addition, in our subsequent work on formality with torsion coefficients [15], the notion of partial formality as defined below plays a fundamental role.

Definition 2.10. – Let $q \geq 0$ be an integer. A functor $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ is a *q-formal lax symmetric monoidal* functor if there are natural transformations Φ_i as in Definition 2.1 such that $\Phi_i(X)$ are q-quasi-isomorphisms for all $X \in \mathcal{C}$ and all $1 \leq i \leq n$.

Proposition 2.11. – *Let $\mathcal{A}$ be an abelian category. Given a non-zero rational number α and an integer $q \geq 0$, denote by $\mathrm{Ch}_*(\mathrm{gr}\,\mathcal{A})_q^{\alpha\text{-pure}}$ the full subcategory of $\mathrm{Ch}_*(\mathrm{gr}\,\mathcal{A})$ given by those graded complexes $A = \bigoplus A_n^p$ whose homology in degrees $\leq q$ is α-pure: for all $n \leq q$,*

$$H_n(A)^p = 0 \text{ for all } p \neq \alpha n.$$

Then the functor $U : \mathrm{Ch}_*(\mathrm{gr}\,\mathcal{A})_q^{\alpha\text{-pure}} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ defined by forgetting the degree is q-formal.

Proof. – The proof is parallel to that of Proposition 2.7 by noting that, if $H_n(A)$ is α-pure for $n \leq q+1$, then the morphisms

$$\Psi(A) : \tau \circ U(A) \to H \circ U(A) \text{ and } \Phi(A) : U \circ \tau(A) \to U(A)$$

are q-quasi-isomorphisms. □

3. Mixed Hodge structures

We next collect some main definitions and properties on mixed Hodge structures and prove a symmetric monoidal version of Deligne's splitting for the weight filtration.

Denote by $\mathcal{F}\mathcal{A}$ the category of filtered objects of an abelian symmetric monoidal category $(\mathcal{A}, \otimes, \mathbf{1})$. All filtrations will be assumed to be of finite length and exhaustive. With the tensor product

$$W_p(A \otimes B) := \sum_{i+j=p} \mathrm{Im}(W_i A \otimes W_j B \longrightarrow A \otimes B),$$

and the unit given by $\mathbf{1}$ concentrated in weight zero, $\mathcal{F}\mathcal{A}$ is a symmetric monoidal category. The functor $U^{\mathrm{fil}} : \mathrm{gr}\,\mathcal{A} \longrightarrow \mathcal{F}\mathcal{A}$ defined by $A = \bigoplus A^p \mapsto W_m A := \bigoplus_{q \le m} A^q$ is strong symmetric monoidal. The category of filtered complexes $\mathrm{Ch}_*(\mathcal{F}\mathcal{A})$ inherits a symmetric monoidal structure via a filtered Künneth morphism and we have a strong symmetric monoidal functor

$$U^{\mathrm{fil}} : \mathrm{Ch}_*(\mathrm{gr}\,\mathcal{A}) \longrightarrow \mathrm{Ch}_*(\mathcal{F}\mathcal{A}).$$

Let $\Bbbk \subset \mathbb{R}$ be a subfield of the real numbers.

Definition 3.1. – A *mixed Hodge structure* on a finite dimensional $\Bbbk$-vector space V is given by an increasing filtration W of V, called the *weight filtration*, together with a decreasing filtration F on $V_{\mathbb{C}} := V \otimes \mathbb{C}$, called the *Hodge filtration*, such that for all $m \ge 0$, each $\Bbbk$-vector space $\mathrm{Gr}^W_m V := W_m V / W_{m-1} V$ carries a pure Hodge structure of weight m given by the filtration induced by F on $\mathrm{Gr}^W_m V \otimes \mathbb{C}$, that is, there is a direct sum decomposition

$$\mathrm{Gr}_W^m V \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q} \text{ where } V^{p,q} = F^p(\mathrm{Gr}^W_m V \otimes \mathbb{C}) \cap \overline{F}^q(\mathrm{Gr}^W_m V \otimes \mathbb{C}) = \overline{V}^{q,p}.$$

Morphisms of mixed Hodge structures are given by morphisms $f : V \to V'$ of $\Bbbk$-vector spaces compatible with filtrations: $f(W_m V) \subset W_m V'$ and $f(F^p V_{\mathbb{C}}) \subset F^p V'_{\mathbb{C}}$. Denote by $\mathrm{MHS}_{\Bbbk}$ the category of mixed Hodge structures over $\Bbbk$. It is an abelian category by [16, Theorem 2.3.5].

Remark 3.2. – Given mixed Hodge structures V and V', then $V \otimes V'$ carries a mixed Hodge structure with the filtered tensor product. This makes $\mathrm{MHS}_{\Bbbk}$ into a symmetric monoidal category. Also, $\mathsf{Hom}(V, V')$ carries a mixed Hodge structure with the weight filtration given by

$$W_p \mathsf{Hom}(V, V') := \{f : V \to V';\, f(W_q V) \subset W_{q+p} V',\ \forall q\}$$

and the Hodge filtration defined in the same way. In particular, the dual of a mixed Hodge structure is again a mixed Hodge structure.

Let $\Bbbk \subset \mathbb{K}$ be a field extension. The functors

$$\Pi_{\mathbb{K}} : \mathrm{MHS}_{\Bbbk} \longrightarrow \mathrm{Vect}_{\mathbb{K}} \text{ and } \Pi^W_{\mathbb{K}} : \mathrm{MHS}_{\Bbbk} \longrightarrow \mathcal{F}\mathrm{Vect}_{\mathbb{K}}$$

defined by sending a mixed Hodge structure (V, W, F) to $V_{\mathbb{K}} := V_{\Bbbk} \otimes \mathbb{K}$ and $(V_{\mathbb{K}}, W)$ respectively, are strong symmetric monoidal functors.

Deligne introduced a global decomposition of $V_{\mathbb{C}} := V \otimes \mathbb{C}$ into subspaces $I^{p,q}$, for any mixed Hodge structure (V, W, F) which generalizes the decomposition of pure Hodge structures of a given weight. In this case, one has a congruence $I^{p,q} \equiv \overline{I}^{q,p}$ modulo W_{p+q-2}.

From this decomposition, Deligne deduced that morphisms of mixed Hodge structures are strictly compatible with filtrations and that the category of mixed Hodge structures is abelian (see [16, Section 1], see also [37, Section 3.1]). We next study this decomposition in the context of symmetric monoidal functors.

LEMMA 3.3 (Deligne's splitting). – *The functor* $\Pi_{\mathbb{C}}^{W}$ *admits a factorization*

$$\begin{tikzcd} \mathrm{MHS}_{\Bbbk} \arrow[r, "G"] \arrow[dr, "\Pi_{\mathbb{C}}^{W}"'] & \mathrm{grVect}_{\mathbb{C}} \arrow[d, "U^{\mathrm{fil}}"] \\ & \mathscr{F}\mathrm{Vect}_{\mathbb{C}} \end{tikzcd}$$

into strong symmetric monoidal functors. In particular, there is an isomorphism of functors

$$U^{\mathrm{fil}} \circ \mathrm{gr} \circ \Pi_{\mathbb{C}}^{W} \cong \Pi_{\mathbb{C}}^{W},$$

where $\mathrm{gr} : \mathscr{F}\mathrm{Vect}_{\mathbb{C}} \longrightarrow \mathrm{grVect}_{\mathbb{C}}$ *is the graded functor given by* $\mathrm{gr}(V_{\mathbb{C}}, W)^p = \mathrm{Gr}_p^W V_{\mathbb{C}}$.

Proof. – Let (V, W, F) be a mixed Hodge structure. By [16, 1.2.11] (see also [25, Lemma 1.12]), there is a direct sum decomposition $V_{\mathbb{C}} = \bigoplus I^{p,q}(V)$ where

$$I^{p,q}(V) = (F^p W_{p+q} V_{\mathbb{C}}) \cap \left(\overline{F}^q W_{p+q} V_{\mathbb{C}} + \sum_{i>0} \overline{F}^{q-i} W_{p+q-1-i} V_{\mathbb{C}} \right).$$

This decomposition is functorial for morphisms of mixed Hodge structures and satisfies

$$W_m V_{\mathbb{C}} = \bigoplus_{p+q \leq m} I^{p,q}(V).$$

Define G by letting $G(V, W, F)^n := \bigoplus_{p+q=n} I^{p,q}(V)$ for any mixed Hodge structure. Since $f(I^{p,q}(V)) \subset I^{p,q}(V')$ for every morphism $f : (V, W, F) \to (V', W, F)$ of mixed Hodge structures, G is functorial. To see that G is strong symmetric monoidal it suffices to use the definition of $I^{p,q}$ together with the tensor product mixed Hodge structure defined via the filtered tensor product, to obtain isomorphisms

$$\sum_{\substack{p+q=n \\ p'+q'=n'}} I^{p,q}(V) \otimes I^{p',q'}(V') \cong \sum_{i+j=n+n'} I^{i,j}(V \otimes V')$$

showing that the splittings $I^{*,*}$ are compatible with tensor products (see also [35, Proposition 1.9]).

The functor $U^{\mathrm{fil}} : \mathrm{grVect} \longrightarrow \mathscr{F}\mathrm{Vect}$ is the strong symmetric monoidal functor given by

$$\bigoplus_n V^n \mapsto (V, W), \text{ with } W_m V := \bigoplus_{n \leq m} V^n.$$

Therefore we have $U^{\mathrm{fil}} \circ G = \Pi_{\mathbb{C}}^{W}$. In order to prove the isomorphism $U^{\mathrm{fil}} \circ \mathrm{gr} \circ \Pi_{\mathbb{C}}^{W} \cong \Pi_{\mathbb{C}}^{W}$ it suffices to note that there is an isomorphism of functors $\mathrm{gr} \circ U^{\mathrm{fil}} \cong \mathrm{Id}$. □

4. Descent of splittings of lax symmetric monoidal functors

In this section, we study lax symmetric monoidal functors to vector spaces over a field of characteristic zero $\Bbbk$ equipped with a compatible filtration. More precisely, we are interested in lax symmetric monoidal maps $\mathcal{C} \longrightarrow \mathcal{F}\mathrm{Vect}_{\Bbbk}$. Our goal is to prove that the existence of a lax symmetric monoidal splitting for such a functor (i.e., of a lift of this map to $\mathcal{C} \longrightarrow \mathrm{grVect}_{\Bbbk}$) can be checked after extending the scalars to a larger field. Our proof follows similar arguments to those appearing in [13, Section 2.4], see also [26] and [40]. A main advantage of our approach with respect to these references is that, in proving descent at the level of functors, we avoid the use of minimal models (and thus restrictions to, for instance, operads with trivial arity 0).

It will be a bit more convenient to study a more general situation where $\mathcal{C}$ is allowed to be a colored operad instead of a symmetric monoidal category. Indeed recall that any symmetric monoidal category can be seen as an operad whose colors are the objects of $\mathcal{C}$ and where a multimorphism from $(c_1, \dots, c_n)$ to d is just a morphism in $\mathcal{C}$ from $c_1 \otimes \cdots \otimes c_n$ to d. Then, given another symmetric monoidal category $\mathcal{D}$, there is an equivalence of categories between the category of lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and the category of $\mathcal{C}$-algebras in the symmetric monoidal category $\mathcal{D}$.

We fix (V, W) a map of colored operads $\mathcal{C} \longrightarrow \mathcal{F}\mathrm{Vect}_{\Bbbk}$ such that for each color c of $\mathcal{C}$, the vector space $V(c)$ is finite dimensional. We denote by $\mathsf{Aut}_W(V)$ the set of its automorphisms in the category of $\mathcal{C}$-algebras in $\mathcal{F}\mathrm{Vect}_{\Bbbk}$ and by $\mathsf{Aut}(\mathrm{Gr}^W V)$ the set of automorphisms of $\mathrm{Gr}^W V$ in the category of $\mathcal{C}$-algebras in $\mathrm{grVect}_{\Bbbk}$. We have a morphism $\mathrm{gr} : \mathsf{Aut}_W(V) \to \mathsf{Aut}(\mathrm{Gr}^W V)$.

Let $\Bbbk \to R$ be a commutative $\Bbbk$-algebra. The correspondence

$$R \mapsto \underline{\mathsf{Aut}}_W(V)(R) := \mathsf{Aut}_W(V \otimes_{\Bbbk} R)$$

defines a functor $\underline{\mathsf{Aut}}_W(V) : \mathrm{Alg}_{\Bbbk} \longrightarrow \mathrm{Gps}$ from the category $\mathrm{Alg}_{\Bbbk}$ of commutative $\Bbbk$-algebras, to the category Gps of groups. Clearly, we have $\underline{\mathsf{Aut}}_W(V)(\Bbbk) = \mathsf{Aut}_W(V)$. We define in a similar fashion a functor $\underline{\mathsf{Aut}}(\mathrm{Gr}^W V)$ from $\mathrm{Alg}_{\Bbbk}$ to Gps.

We recall the following properties:

Proposition 4.1. – *Let (V, W) be as above.* 1.1.

1. $\underline{\mathsf{Aut}}_W(V)$ is a group scheme whose group of $\Bbbk$-points is $\mathsf{Aut}_W(V)$.
2. The functor Gr^W induces a morphism $\underline{\mathrm{gr}} : \underline{\mathsf{Aut}}_W(V) \to \underline{\mathsf{Aut}}(\mathrm{Gr}^W V)$ of group schemes.
3. The kernel $U := \mathrm{Ker}\left(\underline{\mathrm{gr}} : \underline{\mathsf{Aut}}_W(V) \to \underline{\mathsf{Aut}}(\mathrm{Gr}^W V)\right)$ is a unipotent group scheme over $\Bbbk$.

Proof. – We first observe that there is an isomorphism

$$\underline{\mathsf{Aut}}_W(V) \cong \lim_S \underline{\mathsf{Aut}}_W(V_S),$$

in which the limit is taken over the poset of finite sets S of objects of $\mathcal{C}$ and V_S denotes the restriction of V to those objects. We can write the groups $\mathsf{Aut}_W(V)$ and $\underline{\mathsf{Aut}}(\mathrm{Gr}^W V)$ as similar limits. Therefore we may restrict to the case when $\mathcal{C}$ has finitely many objects and prove that in this case, the above objects live in the category of algebraic groups.

Let N be such that the vector space $\bigoplus_{c\in\mathcal{C}} V(c)$ can be linearly embedded in $\Bbbk^N$. Then $\mathsf{Aut}_W(V)$ is the closed subgroup of $\mathrm{GL}_N(\Bbbk)$ defined by the polynomial equations that express the data of a filtration preserving $\mathcal{C}$-algebra automorphism. Similarly, inside the functor of linear automorphisms $\bigoplus_{c\in\mathcal{C}} V(c)\otimes_\Bbbk R \longrightarrow \bigoplus_{c\in\mathcal{C}} V(c)\otimes_\Bbbk R$, let $F(R)$ be those preserving the structure of V as a $\mathcal{C}$-algebra in filtered vector spaces. The condition of preserving the filtration and the algebra structure is given by polynomial equations in the matrix entries and so F is representable (this is also explained in Section 7.6 of [41]). It follows that $\underline{\mathsf{Aut}}_W(V)$ is an algebraic group and its group of $\Bbbk$-points is $\mathsf{Aut}_W(V)$. Hence (1) is satisfied.

For every commutative $\Bbbk$-algebra R, the map

$$\underline{\mathsf{Aut}}_W(V)(R) = \mathsf{Aut}_W(V\otimes_\Bbbk R) \longrightarrow \mathsf{Aut}(\mathrm{Gr}^W(V\otimes_\Bbbk R)) = \underline{\mathsf{Aut}}(\mathrm{Gr}^W V)(R)$$

is a morphism of groups which is natural in R. Thus (2) follows and hence the kernel U is an algebraic group. It now suffices to take a basis of $\bigoplus_{c\in\mathcal{C}} V(c)$ compatible with W. Then we may view U as a subgroup of the group of upper-triangular matrices with 1's on the diagonal. Hence (3) is satisfied. □

Lemma 4.2. – *Let (V, W) be as above. The following assertions are equivalent:* 1.1.

1. The pair (V, W) admits a lax symmetric monoidal splitting: $W_pV \cong \bigoplus_{q\le p} \mathrm{Gr}^W_q V$.
2. The morphism $\mathrm{gr} : \mathsf{Aut}_W(V) \to \mathsf{Aut}(\mathrm{Gr}^W V)$ is surjective.
3. There exists $\alpha \in \Bbbk^*$ which is not a root of unity together with an automorphism $\Phi \in \mathsf{Aut}_W(V)$ such that $\mathrm{gr}(\Phi) = \psi_\alpha$ is the grading automorphism of $\mathrm{Gr}^W V$ associated with α, defined by
$$\psi_\alpha(a) = \alpha^p a,\ \text{ for } a \in \mathrm{Gr}^W_p V.$$

Proof. – The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. We show that (3) implies (1). Let $\Phi \in \mathsf{Aut}_W(V)$ be such that $\mathrm{gr}\Phi = \psi_\alpha$. We will first produce a decomposition $\Phi = \Phi_s.\Phi_u$ which is such that for any object c of $\mathcal{C}$, the restrictions $(\Phi_s(c), \Phi_u(c))$ is a Jordan decomposition for $\Phi(c)$. In order to do that, recall that we have an isomorphism

$$\mathsf{Aut}_W(V) = \lim_S \mathsf{Aut}_W(V_S)$$

where the limit is taken over the poset of finite subsets S of objects of $\mathcal{C}$ and V_S denotes the restriction of V to the subset S. For each of the groups $\mathsf{Aut}_W(V_S)$ we can find a Jordan decomposition of the image of Φ in each of them. The transition maps between those groups preserve this decomposition and it follows that this decomposition induces a decomposition of Φ with the desired property.

By [7, Theorem 4.4], there is a decomposition of the form $V(c) = V'(c) \oplus V''(c)$, where

$$V'(c) = \bigoplus V_p(c) \text{ with } V_p(c) := \mathrm{Ker}(\Phi_s(c) - \alpha^p I)$$

and $V''(c)$ is the complementary subspace corresponding to the remaining factors of the characteristic polynomial of $\Phi_s(c)$. By assumption, $\mathrm{Gr}^W V(c)$ contains nothing but eigenspaces of eigenvalue α^p. Therefore we have $\mathrm{Gr}^W V''(c) = 0$ and one concludes that $V''(c) = 0$.

In order to show that $W_p V = \bigoplus_{i \leq p} V_p$ it suffices to prove it objectwise. Let c be an object of $\mathcal{C}$. For $x \in V_p(c)$, let q be the smallest integer such that $x \in W_q V(c)$. Then x defines a class $x + W_{q+1} V(c) \in \mathrm{gr} V(c)$, and

$$\psi_\alpha(x + W_{q-1}V(c)) = \alpha^q x + W_{q-1}V(c) = \Phi(x) + W_{q-1}V(c) = \alpha^p x + W_{q-1}V(c).$$

Then $(\alpha^q - \alpha^p)x \in W_{q-1}V(c)$. Since $x \notin W_{q-1}V(c)$ we have $q = p$, hence $x \in W_p V$. $\square$

We may now state and prove the main theorem of this section.

THEOREM 4.3. – *Let (V, W) be a map of colored operads $\mathcal{C} \longrightarrow \mathcal{F}\mathrm{Vect}_{\Bbbk}$ such that for each color c of $\mathcal{C}$, the vector space $V(c)$ is finite dimensional. Let $\Bbbk \subset \mathbb{K}$ be a field extension. Then V admits a lax symmetric monoidal splitting if and only if $V_{\mathbb{K}} := V \otimes_{\Bbbk} \mathbb{K} : \mathcal{C} \longrightarrow \mathrm{Vect}_{\mathbb{K}}$ admits a lax symmetric monoidal splitting.*

Proof. – We may assume without loss of generality that $\mathbb{K}$ is algebraically closed. If $V_{\mathbb{K}}$ admits a splitting, the map

$$\underline{\mathsf{Aut}}_W(V)(\mathbb{K}) \longrightarrow \underline{\mathsf{Aut}}(\mathrm{Gr}^W V)(\mathbb{K})$$

is surjective by Lemma 4.2. Our goal is to prove surjectivity of

$$\underline{\mathsf{Aut}}_W(V)(\Bbbk) \longrightarrow \underline{\mathsf{Aut}}(\mathrm{Gr}^W V)(\Bbbk).$$

As in Proposition 4.1, we can write those groups as filtered limits. Since an inverse limit of surjections is a surjection, it is enough to prove the result when $\mathcal{C}$ has finitely many objects.

From [41, Section 18.1] there is an exact sequence of groups

$$1 \longrightarrow U(\Bbbk) \longrightarrow \underline{\mathsf{Aut}}_W(V)(\Bbbk) \longrightarrow \underline{\mathsf{Aut}}(\mathrm{Gr}^W V)(\Bbbk) \longrightarrow H^1(\mathbb{K}/\Bbbk, U) \longrightarrow \cdots$$

where U is a unipotent algebraic group by Proposition 4.1 and our assumption that $\mathcal{C}$ has finitely many objects. Since $\Bbbk$ has characteristic zero the group $H^1(\mathbb{K}/\Bbbk, U)$ is trivial (see [41, Example 18.2.e]) and we deduce the desired surjectivity. $\square$

From this theorem we deduce that Deligne's splitting holds over $\mathbb{Q}$. We record this fact in the following Lemma.

LEMMA 4.4 (Deligne's splitting over $\mathbb{Q}$). – *The forgetful functor $\Pi^W_{\mathbb{Q}} : \mathrm{MHS}_{\mathbb{Q}} \longrightarrow \mathcal{F}\mathrm{Vect}_{\mathbb{Q}}$ given by $(V, W, F) \mapsto (V, W)$ admits a factorization*

$$\begin{tikzcd} \mathrm{MHS}_{\mathbb{Q}} \arrow[r, "G"] \arrow[dr, "\Pi^W_{\mathbb{Q}}"'] & \mathrm{grVect}_{\mathbb{Q}} \arrow[d, "U^{\mathrm{fil}}"] \\ & \mathcal{F}\mathrm{Vect}_{\mathbb{Q}} \end{tikzcd}$$

into lax symmetric monoidal functors. In particular, there is an isomorphism of functors

$$U^{\mathrm{fil}} \circ \mathrm{gr} \circ \Pi^W_{\mathbb{Q}} \cong \Pi^W_{\mathbb{Q}},$$

where $\mathrm{gr} : \mathcal{F}\mathrm{Vect}_{\mathbb{Q}} \longrightarrow \mathrm{grVect}_{\mathbb{Q}}$ *is the graded functor given by* $\mathrm{gr}(V_{\mathbb{Q}}, W)^p = \mathrm{Gr}^W_p V_{\mathbb{Q}}$.

Proof. – We apply Theorem 4.3 to the lax symmetric monoidal functor $\Pi_{\mathbb{Q}}^W$ using the fact that $\Pi_{\mathbb{Q}}^W \otimes_{\mathbb{Q}} \mathbb{C}$ admits a splitting by Lemma 3.3. □

REMARK 4.5. – We want to emphasize that Theorem 4.3 does not say that the splitting of the previous lemma recovers the splitting of Lemma 3.3 after tensoring with $\mathbb{C}$. In fact, it can probably be shown that such a splitting cannot exist. Nevertheless, the existence of Deligne's splitting over $\mathbb{C}$ abstractly forces the existence of a similar splitting over $\mathbb{Q}$ which is all this Lemma is saying. Note as well that these are not splittings of mixed Hodge structures, but only of the weight filtration. They are also referred to as *weak splittings* of mixed Hodge structures (see for example [37, Section 3.1]). As is well-known, mixed Hodge structures do not split in general.

The above splitting over $\mathbb{Q}$ yields the following "purity implies formality" statement in the abstract setting of functors defined from the category of complexes of mixed Hodge structures. Given a rational number α, denote by $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})^{\alpha\text{-pure}}$ the full subcategory of $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})$ of complexes with pure weight α homology: an object (K, W, F) in $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})^{\alpha\text{-pure}}$ is such that $\mathrm{Gr}_W^p H_n(K) = 0$ for all $p \neq \alpha n$.

COROLLARY 4.6. – *The restriction of the functor* $\Pi_{\mathbb{Q}} : \mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}}) \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$ *to the category* $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})^{\alpha\text{-pure}}$ *is formal for any non-zero rational number* α.

Proof. – This follows from Proposition 2.7 together with Lemma 4.4. □

5. Mixed Hodge complexes

In this section, we construct an equivalence of symmetric monoidal ∞-categories between mixed Hodge complexes and complexes of mixed Hodge structures, lifting Beilinson's equivalence of triangulated categories.

We first recall the notion of mixed Hodge complex introduced by Deligne in [17] in its chain complex version (with homological degree). Note that, in contrast with the classical setting of mixed Hodge theory, in the homological version of a mixed Hodge complex, the weight filtration W will be decreasing while the Hodge filtration F will be increasing.

Let $\Bbbk \subset \mathbb{R}$ be a subfield of the real numbers.

DEFINITION 5.1. – A *mixed Hodge complex over* $\Bbbk$ is given by a filtered chain complex $(K_{\Bbbk}, W)$ over $\Bbbk$, a bifiltered chain complex $(K_{\mathbb{C}}, W, F)$ over $\mathbb{C}$, together with a finite string of filtered quasi-isomorphisms of filtered complexes of $\mathbb{C}$-vector spaces

$$(K_{\Bbbk}, W) \otimes \mathbb{C} \xrightarrow{\alpha_1} (K_1, W) \xleftarrow{\alpha_2} \cdots \xrightarrow{\alpha_{l-1}} (K_{l-1}, W) \xrightarrow{\alpha_l} (K_{\mathbb{C}}, W).$$

We call l the *length* of the mixed Hodge complex. The following axioms must be satisfied:

(MH_0) The homology $H_*(K_{\Bbbk})$ is bounded and finite-dimensional.

(MH_1) The differential of $\mathrm{Gr}_W^p K_{\mathbb{C}}$ is strictly compatible with F.

(MH_2) The filtration on $H_n(\mathrm{Gr}_W^p K_{\mathbb{C}})$ induced by F makes $H_n(\mathrm{Gr}_W^p K_{\Bbbk})$ into a pure Hodge structure of weight $p + n$.

Such a mixed Hodge complex will be denoted by $\mathscr{K} = \{(K_{\Bbbk}, W), (K_{\mathbb{C}}, W, F)\}$, omitting the data of the comparison morphisms α_i.

The above axioms imply that for all $n \geq 0$ the triple $(H_n(K_{\Bbbk}), W[n], F)$ is a mixed Hodge structure over $\Bbbk$, where $W[n]$ is the shifted weight filtration given by

$$W[n]^p H_n(K_{\Bbbk}) := W^{p-n} H_n(K_{\Bbbk}).$$

Morphisms of mixed Hodge complexes are given by levelwise bifiltered morphisms of complexes making the corresponding diagrams commute. Denote by $\mathrm{MHC}_{\Bbbk}$ the category of mixed Hodge complexes of a certain fixed length, which we omit in the notation. The tensor product of mixed Hodge complexes is again a mixed Hodge complex (see [37, Lemma 3.20]). This makes $\mathrm{MHC}_{\Bbbk}$ into a symmetric monoidal category, with a filtered variant of the Künneth formula.

DEFINITION 5.2. – A morphism $f : K \to L$ in $\mathrm{MHC}_{\Bbbk}$ is said to be a *weak equivalence* if $H_*(f_{\Bbbk})$ is an isomorphism of $\Bbbk$-vector spaces.

Since the category of mixed Hodge structures is abelian, the homology of every complex of mixed Hodge structures is a graded mixed Hodge structure. We have a functor

$$\mathcal{T} : \mathrm{Ch}_*^b(\mathrm{MHS}_{\Bbbk}) \longrightarrow \mathrm{MHC}_{\Bbbk}$$

given on objects by $(K, W, F) \mapsto \{(K, TW), (K \otimes \mathbb{C}, TW, F)\}$, where TW is the shifted filtration $(TW)^p K_n := W^{p+n} K_n$. The comparison morphisms α_i are the identity. Also, $\mathcal{T}$ is the identity on morphisms. This functor clearly preserves weak equivalences.

LEMMA 5.3. – *The shift functor* $\mathcal{T} : \mathrm{Ch}_*^b(\mathrm{MHS}_{\Bbbk}) \longrightarrow \mathrm{MHC}_{\Bbbk}$ *is strong symmetric monoidal.*

Proof. – It suffices to note that given filtered complexes (K, W) and (L, W), we have:

$$T(W \otimes W)^p (K \otimes L)_n = (TW \otimes TW)^p (K \otimes L)_n. \qquad \square$$

Beilinson [2] gave an equivalence of categories between the derived category of mixed Hodge structures and the homotopy category of a shifted version of mixed Hodge complexes. We will require a finer version of Beilinson's equivalence, in terms of symmetric monoidal ∞-categories. Denote by $\mathbf{MHC}_{\Bbbk}$ the ∞-category obtained by inverting weak equivalences of mixed Hodge complexes, omitting the length in the notation. As explained in [20, Theorem 2.7.], this object is canonically a symmetric monoidal stable ∞-category. Note that in loc. cit., mixed Hodge complexes have fixed length 2 and are polarized. The results of [20] as well as Beilinson's equivalence, are equally valid for the category of mixed Hodge complexes of an arbitrary fixed length.

THEOREM 5.4. – *The shift functor induces an equivalence* $\mathbf{Ch}_*^b(\mathrm{MHS}_{\Bbbk}) \longrightarrow \mathbf{MHC}_{\Bbbk}$ *of symmetric monoidal ∞-categories.*

Proof. – A proof in the polarizable setting appears in [20]. Also, in [9], a similar statement is proven for a shifted version of mixed Hodge complexes. We sketch a proof in our setting.

We first observe as in Lemma 2.6 of [9] that both ∞-categories are stable and that the shift functor is exact. The stability of $\mathbf{MHC}_{\Bbbk}$ follows from the observation that this ∞-category is the Verdier quotient at the acyclic complexes of the ∞-category of mixed Hodge complexes with the homotopy equivalences inverted. This last ∞-category underlies a dg-category that can easily be seen to be stable. The stability of $\mathbf{Ch}_*^b(\mathrm{MHS}_{\Bbbk})$ follows in a similar way. Since a complex of mixed Hodge structures is acyclic if and only if the underlying complex of $\mathbb{k}$-vector spaces is acyclic, and $\mathscr{T}$ is the identity on the underlying complexes of $\mathbb{k}$-vector spaces, it follows that $\mathscr{T}$ is exact. Therefore, in order to prove that $\mathscr{T}$ is an equivalence of ∞-categories, it suffices to show that it induces an equivalence of homotopy categories

$$D^b(\mathrm{MHS}_{\mathbb{k}}) \longrightarrow \mathrm{ho}(\mathrm{MHC}_{\mathbb{k}}).$$

In [2, Lemma 3.11], it is proven that the shift functor $\mathscr{T} : \mathrm{Ch}_*^b(\mathrm{MHS}_{\mathbb{k}}^p) \longrightarrow \mathrm{MHC}_{\mathbb{k}}^p$ induces an equivalence at the level of homotopy categories. Here the superindex p indicates that the mixed Hodge objects are polarized. One may verify that Beilinson's proof is equally valid if we remove the polarization (see also [14, Theorem 4.10] where Beilinson's equivalence is proven in the non-polarized version via other methods). The fact that $\mathscr{T}$ can be given the structure of a strong symmetric monoidal ∞-functor follows from the work of Drew in [20]. $\square$

6. Logarithmic de Rham currents

The goal of this section is to construct a strong symmetric monoidal ∞-functor from algebraic varieties over $\mathbb{C}$ to mixed Hodge complexes which computes the correct mixed Hodge structure after passing to homology. The construction for smooth varieties is relatively straightforward. It suffices to take a functorial mixed Hodge complex model for the cochains as constructed for instance in [36] and dualize it. The monoidality of that functor is slightly tricky as one has to move to the world of ∞-categories for it to exist. Once one has constructed this functor for smooth varieties, it can be extended to more general varieties by standard descent arguments.

We denote by $\mathrm{Var}_{\mathbb{C}}$ the category of complex schemes that are reduced, separated and of finite type. We use the word variety for an object of this category. We denote by $\mathrm{Sm}_{\mathbb{C}}$ the subcategory of smooth schemes. Both of these categories are essentially small (i.e., there is a set of isomorphisms classes of objects) and symmetric monoidal under the cartesian product.

We will make use of the following very simple observation.

Proposition 6.1. – *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with finite products seen as symmetric monoidal categories with respect to the product. Then any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has a preferred oplax symmetric monoidal structure.*

Proof. – We need to construct comparison morphisms $F(c \times c') \longrightarrow F(c) \times F(c')$. By definition of the product, there is a unique such functor whose composition with the first projection is the map $F(c \times c') \longrightarrow F(c)$ induced by the first projection $c \times c' \longrightarrow c$ and whose composition with the second projection is the map $F(c \times c') \longrightarrow F(c')$ induced by the second projection $c \times c' \longrightarrow c'$. Similarly, one has a unique map $F(*) \longrightarrow *$. One checks easily that these two maps give F the structure of an oplax symmetric monoidal functor. $\square$

6.1. For smooth varieties

In this section, we construct a lax symmetric monoidal functor

$$\mathscr{D}_* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

such that its composition with the forgetful functor $\mathbf{MHC}_{\mathbb{Q}} \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$ is naturally weakly equivalent to $S_*(-, \mathbb{Q})$ as a lax symmetric monoidal functor.

We will adapt Navarro-Aznar's construction of mixed Hodge diagrams [36]. Let X be a smooth projective complex variety and $j : U \hookrightarrow X$ an open subvariety such that $D := X - U$ is a normal crossing divisor. Denote by $\mathscr{A}_X^*$ the analytic de Rham complex of the underlying real analytic variety of X and let $\mathscr{A}_X^*(\log D)$ denote the subsheaf of $j_* \mathscr{A}_U^*$ of logarithmic forms in D. Note that in Deligne's approach to mixed Hodge theory, the sheaf $\Omega_X^*(\log D)$ of holomorphic forms on X with logarithmic poles along D is used instead. As explained in 8.5 of [36], the main advantage to consider analytic forms is the natural real structure obtained, together with a decomposition of the form

$$\mathscr{A}_X^n(\log D) \otimes \mathbb{C} = \bigoplus_{p+q=n} \mathscr{A}_X^{p,q}(\log D).$$

Also, there is an inclusion $\Omega_X^*(\log D) \hookrightarrow \mathscr{A}_X^*(\log D) \otimes \mathbb{C}$ which is a quasi-isomorphism and $\mathscr{A}_X^*(\log D)$ may be naturally endowed with a multiplicative weight filtration W (see 8.3 of [36]). Proposition 8.4 of loc. cit. gives a string of quasi-isomorphisms of sheaves of filtered cdga's over $\mathbb{R}$:

$$(\mathbf{R}_{\mathrm{TW}} j_* \underline{\mathbb{Q}}_U, \tau) \otimes \mathbb{R} \xrightarrow{\sim} (\mathbf{R}_{\mathrm{TW}} j_* \mathscr{A}_U^*, \tau) \xleftarrow{\sim} (\mathscr{A}_X^*(\log D), \tau) \xrightarrow{\sim} (\mathscr{A}_X^*(\log D), W),$$

where τ is the canonical filtration. In this diagram,

$$\mathbf{R}_{\mathrm{TW}} j_* : \mathrm{Sh}(U, \mathrm{Ch}_*^{\leq 0}(\Bbbk)) \longrightarrow \mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\Bbbk))$$

is the functor defined by

$$\mathbf{R}_{\mathrm{TW}} j_* := \mathbf{s}_{\mathrm{TW}} \circ j_* \circ G^+$$

where

$$G^+ : \mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\Bbbk)) \longrightarrow \Delta\mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\Bbbk))$$

is the Godement canonical cosimplicial resolution functor and

$$\mathbf{s}_{\mathrm{TW}} : \Delta\mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\Bbbk)) \longrightarrow \mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\Bbbk))$$

is the Thom-Whitney simple functor introduced by Navarro in Section 2 of loc. cit. Both functors are lax symmetric monoidal and hence $\mathbf{R}_{\mathrm{TW}} j_*$ is a lax symmetric monoidal functor (see [39, Section 3.2]).

The complex $\mathscr{A}_X^*(\log D) \otimes \mathbb{C}$ carries a natural multiplicative Hodge filtration F (see 8.6 of [36]). The above string of quasi-isomorphisms gives a commutative algebra object in (cohomological) mixed Hodge complexes after taking global sections. Specifically, the composition

$$\mathbf{R}_{\mathrm{TW}} \Gamma(X, -) := \mathbf{s}_{\mathrm{TW}} \circ \Gamma(X, -) \circ G^+$$

gives a derived global sections functor

$$\mathbf{R}_{\mathrm{TW}} \Gamma(X, -) : \mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\mathbb{Q})) \longrightarrow \mathrm{Ch}_*^{\leq 0}(\mathbb{Q})$$

which again is lax symmetric monoidal. There is also a filtered version of this functor defined via the filtered Thom-Whitney simple (see Section 6 of [36]). Theorem 8.15 of loc. cit. asserts that by applying the (bi)filtered versions of $\mathbf{R}_{\mathrm{TW}}\Gamma(X,-)$ to each of the pieces of the above string of quasi-isomorphisms, one obtains a commutative algebra object in mixed Hodge complexes $\mathscr{H}dg(X,U)$ whose cohomology gives Deligne's mixed Hodge structure on $H^*(U,\mathbb{Q})$ and such that

$$\mathscr{H}dg(X,U)_{\mathbb{Q}} = \mathbf{R}_{\mathrm{TW}}\Gamma(X, \mathbf{R}_{\mathrm{TW}} j_* \underline{\mathbb{Q}}_U)$$

is naturally quasi-isomorphic to $S^*(U,\mathbb{C})$ (as a cochain complex). This construction is functorial for morphisms of pairs $f:(X,U)\to(X',U')$. The definition of $\mathscr{H}dg(f)$ follows as in the additive setting (see [32, Lemma 6.1.2] for details), by replacing the classical additive total simple functor with the Thom-Whitney simple functor.

Now we wish to get rid of the dependence on the compactification. For this purpose, we define for U a smooth variety over $\mathbb{C}$, a category R_U whose objects are pairs (X,U) where X is smooth and proper variety containing U as an open subvariety and $X-U$ is a normal crossing divisor. Morphisms in R_U are morphisms of pairs. We then define $\mathscr{D}^*(U)$ by the formula

$$\mathscr{D}^*(U) := \operatorname{colim}_{R_U^{\mathrm{op}}} \mathscr{H}dg(X,U)$$

By theorems of Hironaka and Nagata, the category R_U^{op} is a non-empty filtered category. Note that we have to be slightly careful here as the category of mixed Hodge complexes does not have all filtered colimits. However, we can form this colimit in the category of pairs $(K_{\mathbb{Q}},W),(K_{\mathbb{C}},W,F)$ having the structure required in Definition 5.1 but not necessarily satisfying the axioms MH_0, MH_1 and MH_2. Since taking filtered colimit is an exact functor, we deduce from the classical isomorphism between sheaf cohomology and singular cohomology that there is a quasi-isomorphism

$$\mathscr{D}^*(U)_{\mathbb{Q}} \to S^*(U,\mathbb{Q})$$

This shows that the cohomology of $\mathscr{D}^*(U)$ is of finite type and hence, that $\mathscr{D}^*(U)$ satisfies axiom MH_0. The other axioms are similarly easily seen to be satisfied. Moreover, filtered colimits preserve commutative algebra structures, therefore the functor $\mathscr{D}^*$ is a functor from $\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}$ to commutative algebras in $\mathrm{MHC}_{\mathbb{Q}}$.

Since the coproduct in commutative algebras is the tensor product, we deduce from the dual of Proposition 6.1 that $\mathscr{D}^*$ is canonically a lax symmetric monoidal functor from $\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}$ to $\mathrm{MHC}_{\mathbb{Q}}$. But since the comparison map

$$\mathscr{D}^*(U)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathscr{D}^*(V)_{\mathbb{Q}} \longrightarrow \mathscr{D}^*(U\times V)_{\mathbb{Q}}$$

is a quasi-isomorphism, this functor extends to a strong symmetric monoidal ∞-functor

$$\mathscr{D}^* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}}.$$

Remark 6.2. – A similar construction for real mixed Hodge complexes is done in [10, Section 3.1]. There is also a similar construction in [20] that includes polarizations.

Now, the category $\mathrm{MHC}_{\mathbb{Q}}$ is equipped with a duality functor. It sends a mixed Hodge complex $\{(K_{\mathbb{Q}},W),(K_{\mathbb{C}},W,F)\}$ to the linear duals $\{(K_{\mathbb{Q}}^{\vee},W^{\vee}),(K_{\mathbb{C}}^{\vee},W^{\vee},F^{\vee})\}$ where the dual of a filtered complex is defined as in 3.2. One checks easily that this dual object satisfies the axioms of a mixed Hodge complex. Moreover, the duality functor $\mathrm{MHC}_{\mathbb{Q}}^{\mathrm{op}} \longrightarrow \mathrm{MHC}_{\mathbb{Q}}$ is lax symmetric monoidal and preserves weak equivalences of mixed Hodge complexes, therefore it induces a lax symmetric monoidal ∞-functor

$$\mathbf{MHC}_{\mathbb{Q}}^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}},$$

but in fact, we have the following proposition.

PROPOSITION 6.3. – *The dualization ∞-functor*

$$\mathbf{MHC}_{\mathbb{Q}}^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

is strong symmetric monoidal.

Proof. – It suffices to observe that the canonical map

$$K^{\vee} \otimes L^{\vee} \longrightarrow (K \otimes L)^{\vee}$$

is a weak equivalence. This follows from the fact that mixed Hodge complexes are assumed to have finite type cohomology. □

Composing the duality functor with $\mathscr{D}^*$, we get a strong symmetric monoidal ∞-functor

$$\mathscr{D}_* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}.$$

REMARK 6.4. – One should note that $\mathscr{D}^*$ comes from a lax symmetric monoidal functor from $\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}$ to $\mathrm{MHC}_{\mathbb{Q}}$. On the other hand, $\mathscr{D}_*$ is induced by a strict functor which is neither lax nor oplax. Indeed, it is obtained as the composition of $(\mathscr{D}^*)^{\mathrm{op}}$ which is an oplax symmetric monoidal functor $\mathrm{Sm}_{\mathbb{C}} \longrightarrow (\mathrm{MHC}_{\mathbb{Q}})^{\mathrm{op}}$ and the duality functor which is a lax symmetric monoidal functor. Thus, the symmetric monoidal structure on $\mathscr{D}_*$ only exists at the ∞-categorical level.

To conclude this construction, it remains to compare the functor $\mathscr{D}_*(-)_{\mathbb{Q}}$ with the singular chains functor. These two functors are naturally quasi-isomorphic as shown in [36] but we will need to know that they are quasi-isomorphic as symmetric monoidal ∞-functors. We denote by $S_*(-, R)$ the singular chain complex functor from the category of topological spaces to the category of chain complexes over a commutative ring R. The functor $S_*(-, R)$ is lax symmetric monoidal. Moreover, the natural map

$$S_*(X, R) \otimes S_*(Y, R) \to S_*(X \times Y, R)$$

is a quasi-isomorphism. This implies that $S_*(-, R)$ induces a strong symmetric monoidal ∞-functor from the category of topological spaces to the ∞-category $\mathbf{Ch}_*(R)$ of chain complexes over R. We still use the symbol $S_*(-, R)$ to denote this ∞-functor.

THEOREM 6.5. – *The functors $\mathscr{D}_*(-)_{\mathbb{Q}}$ and $S_*(-, \mathbb{Q})$ are weakly equivalent as strong symmetric monoidal ∞-functors from $\mathbf{N}(\mathrm{Sm}_{\mathbb{C}})$ to $\mathbf{Ch}_*(\mathbb{Q})$.*

Proof. – We introduce the category Man of smooth real manifolds. We consider the ∞-category $\mathbf{PSh}(\mathrm{Man})$ of presheaves of spaces on the ∞-category $\mathbf{N}(\mathrm{Man})$. This is a symmetric monoidal ∞-category under the product. We can consider the reflective subcategory $\mathbf{T}$ spanned by presheaves $\mathscr{G}$ satisfying the following two conditions:

1. Given a hypercover $U_\bullet \to M$ of a manifold M, the induced map
$$\mathcal{G}(M) \to \lim_\Delta \mathcal{G}(U_\bullet)$$
is an equivalence.
2. For any manifold M, the map $\mathcal{G}(M) \to \mathcal{G}(M \times \mathbb{R})$ induced by the projection $M \times \mathbb{R} \to M$ is an equivalence.

The presheaves satisfying these conditions are stable under product, hence the ∞-category $\mathbf{T}$ inherits the structure of a symmetric monoidal locally presentable ∞-category. It has a universal property that we now describe.

Given another symmetric monoidal locally presentable ∞-category $\mathbf{D}$, we denote by $\mathrm{Fun}^{L,\otimes}(\mathbf{T},\mathbf{D})$ the ∞-category of colimit preserving strong symmetric monoidal functors $\mathbf{T} \to \mathbf{D}$. Then, we can consider the composition
$$\mathrm{Fun}^{L,\otimes}(\mathbf{T},\mathbf{D}) \to \mathrm{Fun}^{L,\otimes}(\mathbf{PSh}(\mathrm{Man}),\mathbf{D}) \to \mathrm{Fun}^{\otimes}(\mathbf{N}\mathrm{Man},\mathbf{D})$$
where the first map is induced by precomposition with the left adjoint to the inclusion $\mathbf{T} \to \mathbf{PSh}(\mathrm{Man})$ and the second map is induced by precomposition with the Yoneda embedding. We claim that the above composition is fully faithful and that its essential image is the full subcategory of $\mathrm{Fun}^{\otimes}(\mathbf{N}\mathrm{Man},\mathbf{D})$ spanned by the functors F that satisfy the following two properties:

1. Given a hypercover $U_\bullet \to M$ of a manifold M, the map
$$\mathrm{colim}_{\Delta^{\mathrm{op}}} F(U_\bullet) \to F(M)$$
is an equivalence.
2. For any manifold M, the map $F(M \times \mathbb{R}) \to F(M)$ induced by the projection $M \times \mathbb{R} \to M$ is an equivalence.

This statement can be deduced from the theory of localizations of symmetric monoidal ∞-categories (see [30, Section 3]).

In particular, there exists an essentially unique strong symmetric monoidal and colimit preserving functor from $\mathbf{T}$ to $\mathbf{S}$ (the ∞-category of spaces) that is determined by the fact that it sends a manifold M to the simplicial set $\mathrm{Sing}(M)$. This functor is an equivalence of ∞-categories. This is a folklore result. A proof of a model category version of this fact can be found in [21, Proposition 8.3.].

The ∞-category $\mathbf{S}$ is the unit of the symmetric monoidal ∞-category of presentable ∞-categories. It follows that it has a commutative algebra structure (which corresponds to the symmetric monoidal structure coming from the cartesian product) and that it is the initial symmetric monoidal presentable ∞-category. Since $\mathbf{T}$ is equivalent to $\mathbf{S}$ as a symmetric monoidal presentable ∞-category, we deduce that, up to equivalence, there is a unique functor $\mathbf{T} \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$ that is strong symmetric monoidal and colimit preserving. But, using the universal property of $\mathbf{T}$, we easily see that $S_*(-,\mathbb{Q})$ and $\mathscr{D}_*(-)_\mathbb{Q}$ can be extended to strong symmetric monoidal and colimits preserving functors from $\mathbf{T}$ to $\mathbf{Ch}_*(\mathbb{Q})$. It follows that they must be equivalent. $\square$

6.2. For varieties

In this subsection, we extend the construction of the previous subsection to the category of varieties.

We have the site $(\mathrm{Var}_{\mathbb{C}})_{\mathrm{pro}}$ of varieties over $\mathbb{C}$ with the proper topology and the site $(\mathrm{Sm}_{\mathbb{C}})_{\mathrm{pro}}$ which is the restriction of this site to the category of smooth varieties (see [4, Section 3.5] for the definition of the proper topology).

PROPOSITION 6.6 (Blanc). – *Let* $\mathbf{C}$ *be a symmetric monoidal presentable* ∞*-category. We denote by* $\mathrm{Fun}^{\otimes}_{\mathrm{pro}}(\mathrm{Var}_{\mathbb{C}}, \mathbf{C})$ *the* ∞*-category of strong symmetric monoidal functors from* $\mathrm{Var}_{\mathbb{C}}$ *to* $\mathbf{C}$ *whose underlying functor satisfies descent with respect to proper hypercovers. Similarly, we denote by* $\mathrm{Fun}^{\otimes}_{\mathrm{pro}}(\mathrm{Sm}_{\mathbb{C}}, \mathbf{C})$ *the* ∞*-category of strong symmetric monoidal functors from* $\mathrm{Sm}_{\mathbb{C}}$ *to* $\mathbf{C}$ *whose underlying functor satisfies descent with respect to proper hypercovers. The restriction functor*

$$\mathrm{Fun}^{\otimes}_{\mathrm{pro}}(\mathrm{Var}_{\mathbb{C}}, \mathbf{C}) \longrightarrow \mathrm{Fun}^{\otimes}_{\mathrm{pro}}(\mathrm{Sm}_{\mathbb{C}}, \mathbf{C})$$

is an equivalence.

Proof. – We have the categories $\mathrm{Fun}(\mathrm{Var}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet})$ and $\mathrm{Fun}(\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet})$ of presheaves of simplicial sets over $\mathrm{Var}_{\mathbb{C}}$ and $\mathrm{Sm}_{\mathbb{C}}$ respectively. These categories are related by an adjunction

$$\pi^* : \mathrm{Fun}(\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet}) \leftrightarrows \mathrm{Fun}(\mathrm{Var}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet}) : \pi_*$$

where the right adjoint π_* is just the restriction of a presheaf to smooth varieties. Both sides of this adjunction have a symmetric monoidal structure by taking objectwise product. The functor π_* is obviously strong symmetric monoidal. We can equip both sides with the local model structure with respect to the proper topology. We obtain a Quillen adjunction

$$\pi^* : \mathrm{Fun}_{\mathrm{pro}}(\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet}) \leftrightarrows \mathrm{Fun}_{\mathrm{pro}}(\mathrm{Var}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet}) : \pi_*$$

between symmetric monoidal model categories in which the right adjoint is a strong symmetric monoidal functor. In [4, Proposition 3.22], it is proved that this is a Quillen equivalence. The model category $\mathrm{Fun}_{\mathrm{pro}}(\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet})$ presents the ∞-topos of hypercomplete sheaves over the proper site on $\mathrm{Sm}_{\mathbb{C}}$ and similarly for model category $\mathrm{Fun}_{\mathrm{pro}}(\mathrm{Var}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet})$. Therefore, this Quillen equivalence implies that these two ∞-topoi are equivalent. Moreover, as in the proof of 6.5, these topoi, seen as symmetric monoidal presentable ∞-categories under the cartesian product, represent the functor $\mathbf{C} \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{pro}}(\mathrm{Sm}_{\mathbb{C}}, \mathbf{C})$ (resp. $\mathbf{C} \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{pro}}(\mathrm{Var}_{\mathbb{C}}, \mathbf{C})$). The result immediately follows. □

THEOREM 6.7. – *Up to weak equivalences, there is a unique strong symmetric monoidal functor*

$$\mathscr{D}_* : \mathbf{N}(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

which satisfies descent with respect to proper hypercovers and whose restriction to $\mathrm{Sm}_{\mathbb{C}}$ *is equivalent to the functor* $\mathscr{D}_*$ *constructed in the previous subsection.*

There is also a unique strong symmetric monoidal functor

$$\mathscr{D}^* : \mathbf{N}(\mathrm{Var}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

which satisfies descent with respect to proper hypercovers and whose restriction to $\mathrm{Sm}_{\mathbb{C}}$ *is equivalent to the functor* $\mathscr{D}^*$ *constructed in the previous subsection.*

Proof. – Let $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$ be the Ind-category of the ∞-category of mixed Hodge complexes. This is a stable presentable ∞-category. We first prove that the composite

$$\mathscr{D}_* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}} \longrightarrow \mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$$

satisfies descent with respect to proper hypercovers. Let Y be a smooth variety and $X_\bullet \to Y$ be a hypercover for the proper topology. We wish to prove that the map

$$\alpha : \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathscr{D}_*(X_\bullet) \longrightarrow \mathscr{D}_*(Y)$$

is an equivalence in $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$. By [4, Proposition 3.24] and the fact that taking singular chains commutes with homotopy colimits in spaces, we see that the map

$$\beta : \mathrm{colim}_{\Delta^{\mathrm{op}}} S_*(X_\bullet, \mathbb{Q}) \longrightarrow S_*(Y, \mathbb{Q})$$

is an equivalence. On the other hand, writing $\mathbf{Ch}_*(\mathbb{Q})^\omega$ for the ∞-category of chain complexes whose homology is finite dimensional, the forgetful functor

$$U : \mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}}) \longrightarrow \mathrm{Ind}(\mathbf{Ch}_*(\mathbb{Q})^\omega) \simeq \mathbf{Ch}_*(\mathbb{Q})$$

preserves colimits and by Theorem 6.5, the composite $U \circ \mathscr{D}_*$ is weakly equivalent to $S_*(-, \mathbb{Q})$. Therefore, the map β is weakly equivalent to the map $U(\alpha)$ in particular, we deduce that the source of α is in $\mathbf{MHC}_{\mathbb{Q}}$ (as opposed to $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$). And since the functor $U : \mathbf{MHC}_{\mathbb{Q}} \to \mathbf{Ch}_*(\mathbb{C})$ is conservative, it follows that α is an equivalence as desired.

Hence, by Proposition 6.6, there is a unique extension of $\mathscr{D}_*$ to a strong symmetric monoidal functor $\mathbf{N}(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$ that has proper descent. Moreover, by the first paragraph of this proof, if Y is an object of $\mathrm{Var}_{\mathbb{C}}$ and $X_\bullet \longrightarrow Y$ is a proper hypercover by smooth varieties, then $\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathscr{D}_*(X_\bullet, \mathbb{Q})$ has finitely generated homology. It follows that this unique extension of $\mathscr{D}_*$ to $\mathrm{Var}_{\mathbb{C}}$ lands in $\mathbf{MHC}_{\mathbb{Q}} \subset \mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$.

For the case of $\mathscr{D}^*$, we know from Proposition 6.3 that dualization induces a strong symmetric monoidal equivalence of ∞-categories $\mathbf{MHC}_{\mathbb{Q}}^{\mathrm{op}} \simeq \mathbf{MHC}_{\mathbb{Q}}$ (we emphasize that, as a functor, dualization is only lax symmetric monoidal but as an ∞-functor it is strong symmetric monoidal). Thus, we see that we have no other choice but to define $\mathscr{D}^*$ as the composite

$$\mathbf{N}(\mathrm{Var})^{\mathrm{op}} \xrightarrow{(\mathscr{D}_*)^{\mathrm{op}}} \mathbf{MHC}_{\mathbb{Q}}^{\mathrm{op}} \xrightarrow{(-)^\vee} \mathbf{MHC}_{\mathbb{Q}}$$

and this will be the unique strong symmetric monoidal functor

$$\mathscr{D}^* : \mathbf{N}(\mathrm{Var}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}},$$

which satisfies descent with respect to proper hypercovers and whose restriction to $\mathrm{Sm}_{\mathbb{C}}$ is equivalent to the functor $\mathscr{D}^*$ constructed in the previous subsection. □

PROPOSITION 6.8. – 1. *There is a weak equivalence* $\mathscr{D}_*(-)_{\mathbb{Q}} \simeq S_*(-, \mathbb{Q})$ *in the category of strong symmetric monoidal ∞-functors* $\mathbf{N}(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$.

2. *There is a weak equivalence* $\mathscr{A}^*_{PL}(-) \simeq \mathscr{D}^*(-)_{\mathbb{Q}} \simeq S^*(-, \mathbb{Q})$ *in the ∞-category of strong symmetric monoidal ∞-functors* $\mathbf{N}(\mathrm{Var}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$.

Proof. – We prove the first claim. By construction $\mathscr{D}_*(-)_{\mathbb{Q}}$ is a symmetric monoidal functor that satisfies proper descent. By [4, Proposition 3.24], the same is true for $S_*(-,\mathbb{Q})$. Since these two functors are moreover weakly equivalent when restricted to $\mathrm{Sm}_{\mathbb{C}}$, they are equivalent by Proposition 6.6.

The linear dual functor is strong symmetric monoidal when restricted to chain complexes whose homology is of finite type. Moreover, both $S_*(-,\mathbb{Q})$ and $\mathscr{D}_*(-)_{\mathbb{Q}}$ land in the ∞-category of such chain complexes. Therefore, the equivalence $S^*(-,\mathbb{Q}) \simeq \mathscr{D}^*(-)_{\mathbb{Q}}$ follows from the first part. The equivalence $\mathscr{A}^*_{PL}(-) \simeq S^*(-,\mathbb{Q})$ is classical. □

7. Formality of the singular chains functor

In this section, we prove the main results of the paper on the formality of the singular chains functor. We also explain some applications to operad formality.

Definition 7.1. – Let X be a complex variety and let α be a rational number. We say that the weight filtration on $H^*(X,\mathbb{Q})$ is *α-pure* if for all $n \geq 0$ we have

$$\mathrm{Gr}^W_p H^n(X,\mathbb{Q}) = 0 \text{ for all } p \neq \alpha n.$$

Remark 7.2. – Note that since the weight filtration on $H^n(-,\mathbb{Q})$ has weights in the interval $[0,2n] \cap \mathbb{Z}$, the above definition makes sense only for $\alpha \in [0,2] \cap \mathbb{Q}$. For $\alpha = 1$ we recover the purity property shared by the cohomology of smooth projective varieties. A very simple example of a variety whose filtration is α-pure, with α not integer, is given by $\mathbb{C}^2 \setminus \{0\}$. Its reduced cohomology is concentrated in degree 3 and weight 4, so its weight filtration is 4/3-pure. We refer to Proposition 8.6 in the following section for more elaborate examples.

Here is our main theorem.

Theorem 7.3. – *Let α be a non-zero rational number. The singular chains functor*

$$S_*(-,\mathbb{Q}) : \mathrm{Var}_{\mathbb{C}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to varieties whose weight filtration in cohomology is α-pure.

Proof. – By Corollary 2.4, it suffices to prove that this functor is formal as an ∞-lax symmetric monoidal functor. By Proposition 6.8, it is equivalent to prove that $\mathscr{D}_*(-)_{\mathbb{Q}}$ is formal. We denote by $\bar{\mathscr{D}}_*$ the composite of $\mathscr{D}_*$ with a strong symmetric monoidal inverse of the equivalence of Theorem 5.4. Because of that theorem, $\mathscr{D}_*(-)_{\mathbb{Q}}$ is weakly equivalent to $\Pi_{\mathbb{Q}} \circ \bar{\mathscr{D}}_*$. The restriction of $\bar{\mathscr{D}}_*$ to $\mathrm{Var}_{\mathbb{C}}^{\alpha\text{-pure}}$ lands in $\mathbf{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})^{\alpha\text{-pure}}$, the full subcategory of $\mathbf{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})$ spanned by chain complexes whose homology is α-pure. By Corollary 4.6, the ∞-functor $\Pi_{\mathbb{Q}}$ from $\mathbf{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})^{\alpha\text{-pure}}$ to $\mathbf{Ch}_*(\mathbb{Q})$ is formal and hence so is $\Pi_{\mathbb{Q}} \circ \bar{\mathscr{D}}_*$. □

We now list a few applications of this result.

7.1. Noncommutative little disks operad

The authors of [19] introduce two nonsymmetric topological operads $\mathcal{A}s_{S^1}$ and $\mathcal{A}s_{S^1} \rtimes S^1$. In each arity, these operads are given by a product of copies of $\mathbb{C} - \{0\}$ and the operad maps can be checked to be algebraic maps. It follows that the operads $\mathcal{A}s_{S^1}$ and $\mathcal{A}s_{S^1} \rtimes S^1$ are operads in the category $\mathrm{Sm}_{\mathbb{C}}$ and that the weight filtration on their cohomology is 2-pure. Therefore, by 7.3 we have the following result.

THEOREM 7.4. – *The operads $S_*(\mathcal{A}s_{S^1}, \mathbb{Q})$ and $S_*(\mathcal{A}s_{S^1} \rtimes S^1, \mathbb{Q})$ are formal.*

REMARK 7.5. – The fact that the operad $S_*(\mathcal{A}s_{S^1}, \mathbb{Q})$ is formal is proved in [19, Proposition 7] by a more elementary method and it is true even with integral coefficients. The other formality result was however unknown to the authors of [19].

7.2. Self-maps of the projective line

We denote by F_d the algebraic variety of degree d algebraic maps from $\mathbf{P}^1_{\mathbb{C}}$ to itself that send the point ∞ to the point 1. Explicitly, a point in F_d is a pair (f, g) of degree d monic polynomials without any common roots. Sending a monic polynomial to its set of coefficients, we may see the variety F_d as a Zariski open subset of $\mathbf{A}^{2d}_{\mathbb{C}}$. See [31, Section 5] for more details.

PROPOSITION 7.6. – *The weight filtration on $H^*(F_d, \mathbb{Q})$ is 2-pure.*

Proof. – The variety F_d is denoted $\mathrm{Poly}_1^{d,2}$ in [24, Definition 1.1.]. It is explained in Step 4 of the proof of Theorem 1.2 in that paper, that the variety F_d is the quotient of the complement of a hyperplane arrangement H in $\mathbf{A}^{2d}_{\mathbb{C}}$ by the group $\Sigma_d \times \Sigma_d$ acting by permuting the coordinates. The quotient map

$$\pi : \mathbf{A}^{2d}_{\mathbb{C}} - H \to F_d$$

is algebraic and thus induces a morphism of mixed Hodge structures $\pi^* : H^*(F_d, \mathbb{Q}) \to H^*(\mathbf{A}^{2d}_{\mathbb{C}} - H, \mathbb{Q})$. Moreover, it is classical that π^* is injective (see e.g., [8, Theorem III.2.4]). Since the mixed Hodge structure of $H^k(\mathbf{A}^{2d}_{\mathbb{C}} - H, \mathbb{Q})$ is pure of weight $2k$ (by Proposition 8.6 or by [33]), the desired result follows. □

In [11, Proposition 3.1.], Cazanave shows that the variety $\bigsqcup_d F_d$ has the structure of a graded monoid in $\mathrm{Sm}_{\mathbb{C}}$. The structure of a graded monoid can be encoded by a colored operad. Thus the following result follows from Theorem 7.3.

THEOREM 7.7. – *The graded monoid in chain complexes $\bigoplus_d S_*(F_d, \mathbb{Q})$ is formal.*

7.3. The little disks operad

In [38], Petersen shows that the operad of little disks $\mathscr{D}$ is formal. The method of proof is to use the action of a certain group $\mathrm{GT}(\mathbb{Q})$ on $S_*(\mathscr{PAB}_{\mathbb{Q}}, \mathbb{Q})$ which follows from work of Drinfeld. Here the operad $\mathscr{PAB}_{\mathbb{Q}}$ is rationally equivalent to $\mathscr{D}$ and $\mathrm{GT}(\mathbb{Q})$ is the group of $\mathbb{Q}$-points of the pro-algebraic Grothendieck-Teichmüller group. We can reinterpret this proof using the language of mixed Hodge structures. Indeed, the group GT receives a map from the group $\mathrm{Gal}(\mathrm{MT}(\mathbb{Z}))$, the Galois group of the Tannakian category of mixed Tate motives over $\mathbb{Z}$ (see [1, 25.9.2.2]). Moreover there is a map $\mathrm{Gal}(\mathrm{MHTS}_{\mathbb{Q}}) \to \mathrm{Gal}(\mathrm{MT}(\mathbb{Z}))$ from the Tannakian Galois group of the abelian category of mixed Hodge Tate structures (the full subcategory of $\mathrm{MHS}_{\mathbb{Q}}$ generated under extensions by the Tate twists $\mathbb{Q}(n)$ for all n) which is Tannaka dual to the tensor functor

$$\mathrm{MT}(\mathbb{Z}) \longrightarrow \mathrm{MHTS}_{\mathbb{Q}}$$

sending a mixed Tate motive to its Hodge realization. This map of Galois group allows us to view $S_*(\mathscr{PAB}_{\mathbb{Q}}, \mathbb{Q})$ as an operad in $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})$ which moreover has a 2-pure weight filtration (as follows from the computation in [38]). Therefore by Corollary 4.6, the operad $S_*(\mathscr{PAB}_{\mathbb{Q}}, \mathbb{Q})$ is formal and hence also $S_*(\mathscr{D}, \mathbb{Q})$.

7.4. The gravity operad

In [23], Dupont and the second author prove the formality of the gravity operad of Getzler. It is an operad structure on the collection of graded vector spaces $\{H_{*-1}(\mathscr{M}_{0,n+1}), n \in \mathbb{N}\}$. It can be defined as the homotopy fixed points of the circle action on $S_*(\mathscr{D}, \mathbb{Q})$. The method of proof in [23] can also be interpreted in terms of mixed Hodge structures. Indeed, a model $\mathscr{G}rav^{W'}$ of gravity is constructed in 2.7 of loc. cit. This model comes with an action of $\mathrm{GT}(\mathbb{Q})$ and a $\mathrm{GT}(\mathbb{Q})$-equivariant map $\iota : \mathscr{G}rav^{W'} \longrightarrow S_*(\mathscr{PAB}_{\mathbb{Q}}, \mathbb{Q})$ which is injective on homology. As in the previous subsection, this action of $\mathrm{GT}(\mathbb{Q})$ allows us to interpret $\mathscr{G}rav^{W'}$ as an operad in $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})$. Moreover, the injectivity of ι implies that $\mathscr{G}rav^{W'}$ also has a 2-pure weight filtration. Therefore by Corollary 4.6, we deduce the formality of $\mathscr{G}rav^{W'}$. In fact, we obtain the stronger result that the map

$$\iota : \mathscr{G}rav^{W'} \longrightarrow S_*(\mathscr{PAB}_{\mathbb{Q}}, \mathbb{Q})$$

is formal as a map of operads (i.e., it is connected to the induced map in homology by a zig-zag of maps of operads).

7.5. E^1-formality

The above results deal with objects whose weight filtration is pure. In general, for mixed weights, the singular chains functor is not formal, but it is E^1-formal as we now explain.

The r-stage of the spectral sequence associated to a filtered complex is an r-bigraded complex with differential of bidegree $(-r, r-1)$. By taking its total degree and considering the column filtration we obtain a filtered complex. Denote by

$$E^r : \mathbf{Ch}_*(\mathscr{F}\mathbb{Q}) \longrightarrow \mathbf{Ch}_*(\mathscr{F}\mathbb{Q})$$

the resulting strong symmetric monoidal ∞-functor. Denote by

$$\tilde{\Pi}^W_{\mathbb{Q}} : \mathbf{MHC}_{\mathbb{Q}} \longrightarrow \mathbf{Ch}_*(\mathscr{F}\mathbb{Q})$$

the forgetful functor defined by sending a mixed Hodge complex to its rational component together with the weight filtration. Note that, since the weight spectral sequence of a mixed Hodge complex degenerates at the second stage, the homology of $E^1 \circ \tilde{\Pi}_{\mathbb{Q}}^W$ gives the weight filtration on the homology of mixed Hodge complexes. We have:

THEOREM 7.8. – *Denote by $S_*^{\mathrm{fil}} : \mathbf{N}(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$ the composite functor*

$$\mathbf{N}(\mathrm{Var}_{\mathbb{C}}) \xrightarrow{\mathscr{D}_*} \mathbf{MHC}_{\mathbb{Q}} \xrightarrow{\tilde{\Pi}_{\mathbb{Q}}^W} \mathbf{Ch}_*(\mathscr{F}\mathbb{Q}).$$

There is an equivalence of strong symmetric monoidal ∞-functors $E^1 \circ S_*^{\mathrm{fil}} \simeq S_*^{\mathrm{fil}}$.

Proof. – It suffices to prove an equivalence $\tilde{\Pi}_{\mathbb{Q}}^W \simeq E^1 \circ \tilde{\Pi}_{\mathbb{Q}}^W$. We have a commutative diagram of strong symmetric monoidal ∞-functors.

$$\begin{tikzcd}
\mathbf{Ch}_*(\mathrm{MHS}_{\mathbb{Q}}) \arrow[r, "\mathscr{T}"] \arrow[d, "\Pi_{\mathbb{Q}}^W"'] & \mathbf{MHC}_{\mathbb{Q}} \arrow[d, "\tilde{\Pi}_{\mathbb{Q}}^W"] \\
\mathbf{Ch}_*(\mathscr{F}\mathbb{Q}) \arrow[r, "T"] \arrow[d, "E^0"'] & \mathbf{Ch}_*(\mathscr{F}\mathbb{Q}) \arrow[d, "E^1"] \\
\mathbf{Ch}_*(\mathscr{F}\mathbb{Q}) \arrow[r, "T"] & \mathbf{Ch}_*(\mathrm{gr}\mathbb{Q}).
\end{tikzcd}$$

The commutativity of the top square follows from the definition of $\mathscr{T}$. We prove that the bottom square commutes. Recall that $T(K, W)$ is the filtered complex (K, TW) defined by $TW^pK_n := W^{p+n}K_n$. It satisfies $d(TW^pK_p) \subset TW^{p+1}K_{n-1}$. In particular, the induced differential on $\mathrm{Gr}_{TW}K$ is trivial. Therefore we have:

$$E^1_{-p,q}(K, TW) \cong H_{q-p}(\mathrm{Gr}^p_{TW}K) \cong \mathrm{Gr}^p_{TW}K_{q-p} = \mathrm{Gr}^q_W K_{q-p} = E^0_{-q,2q-p}(K, W).$$

This proves that the above diagram commutes.

Since $\mathscr{T}$ is an equivalence of ∞-categories, it is enough to prove that $E^1 \circ \tilde{\Pi}_{\mathbb{Q}}^W \circ \mathscr{T}$ is equivalent to $\tilde{\Pi}_{\mathbb{Q}}^W \circ \mathscr{T}$. By the commutation of the above diagram it suffices to prove that there is an equivalence $E^0 \circ \Pi_{\mathbb{Q}}^W \cong \Pi_{\mathbb{Q}}^W$. This follows from Lemma 4.4, since $E^0 = U^{\mathrm{fil}} \circ \mathrm{gr}$. □

8. Rational homotopy of varieties and formality

For X a space, we denote by $\mathscr{A}^*_{PL}(X)$, Sullivan's algebra of piecewise linear differential forms. This is a commutative dg-algebra over $\mathbb{Q}$ that captures the rational homotopy type of X. A contravariant version of Theorem 7.3 gives:

THEOREM 8.1. – *Let α be a non-zero rational number. The functor*

$$\mathscr{A}^*_{PL} : \mathrm{Var}_{\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to varieties whose weight filtration in cohomology is α-pure.

Proof. – The proof is the same as the proof of Theorem 7.3 using $\mathscr{D}^*$ instead of $\mathscr{D}_*$ and using the fact that $\mathscr{D}^*(-)_{\mathbb{Q}}$ is quasi-isomorphic to $\mathscr{A}^*_{PL}$ as a lax symmetric monoidal functor (see [36, Théorème 5.5]). □

Recall that a topological space X is said to be *formal* if there is a string of quasi-isomorphisms of commutative dg-algebras from $\mathscr{A}^*_{PL}(X)$ to $H^*(X,\mathbb{Q})$, where $H^*(X,\mathbb{Q})$ is considered as a commutative dg-algebra with trivial differential. Likewise, a continuous map of topological spaces $f : X \longrightarrow Y$ is *formal* if there is a string of homotopy commutative diagrams of morphisms

$$\begin{tikzcd}
\mathscr{A}^*_{PL}(Y) \arrow[d, "f^*"] & * \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[r] & * \arrow[r] \arrow[d] & H^*(Y,\mathbb{Q}) \arrow[d, "H^*(f)"] \\
\mathscr{A}^*_{PL}(X) & * \arrow[l] & \cdots \arrow[l] \arrow[r] & * \arrow[r] & H^*(X,\mathbb{Q}),
\end{tikzcd}$$

where the horizontal arrows are quasi-isomorphisms. Note that if $f : X \to Y$ is a map of topological spaces and X and Y are both formal spaces, then it is not always true that f is a formal map. Also, in general, the composition of formal morphisms is not formal.

Theorem 8.1 gives functorial formality for varieties with pure weight filtration in cohomology, generalizing both "purity implies formality" statements appearing in [22] for smooth varieties and in [12] for singular projective varieties. We also get a result of partial formality as done in these references, via Proposition 2.11. Our generalization is threefold, as explained in the following three subsections.

8.1. Rational purity

To our knowledge, in the existing references where α-purity of the weight filtration is discussed, only the cases $\alpha = 1$ and $\alpha = 2$ are considered, whereas we obtain formality for varieties with α-pure cohomology, for α an arbitrary non-zero rational number. We now show that certain complements of subspace arrangements give examples of such varieties.

Definition 8.2. – Let V be a finite dimensional $\mathbb{C}$-vector space. We say that a finite set $\{H_i\}_{i\in I}$ of subspaces of V is a *good arrangement of codimension d subspaces* if

(i) For each $i \in I$, the subspace H_i is of codimension d.

(ii) For each $i \in I$, the set of subspaces $\{H_i \cap H_j\}_{j\neq i}$ of H_i form a good arrangement of codimension d subspaces.

Remark 8.3. – In particular the empty set of subspaces is a good arrangement of codimension d subspaces. By induction on the size of I, we see that this condition is well-defined.

Example 8.4. – Recall that a set of subspaces of codimension d of an n-dimensional space is said to be in general position if the intersection of k of those subspaces is of codimension $\min(n, dk)$. One easily checks that a set of codimensions d subspaces in general position is a good arrangement. However, the converse does not hold as shown in the following example.

EXAMPLE 8.5. – Take $V = (\mathbb{C}^d)^m$ and define, for (i,j) an unordered pair of distinct elements in $\{1,\dots,m\}$, the subspace

$$W_{(i,j)} = \{(x_1,\dots,x_n) \in (\mathbb{C}^d)^m, x_i = x_j\}.$$

This collection of codimension d subspaces of V is a good arrangement. However, these subspaces are not in general position if m is at least 3. Indeed, the codimension of $W_{(1,2)} \cap W_{(1,3)} \cap W_{(2,3)}$ is $2d$. The complement $V - \bigcup_{(i,j)} W_{(i,j)}$ is exactly $F_m(\mathbb{C}^d)$, the space of configurations of m points in $\mathbb{C}^d$.

PROPOSITION 8.6. – *Let $H = \{H_1,\dots,H_k\}$ be a good arrangement of codimension d subspaces of $\mathbb{C}^n$. Then the mixed Hodge structure on $H^*(\mathbb{C}^n - \bigcup_i H_i, \mathbb{Q})$ is pure of weight $2d/(2d-1)$.*

Proof. – We proceed by induction on k. This is obvious for $k = 0$. Now, we consider the variety $X = \mathbb{C}^n - \bigcup_i^{k-1} H_i$, It contains an open subvariety $U = \mathbb{C}^n - \bigcup_i^k H_i$ and its closed complement $Z = H_k - \bigcup_i^{k-1} H_i \cap H_k$ which has codimension d. Therefore the purity long exact sequence on cohomology groups has the form

$$\cdots \longrightarrow H^{r-2d}(Z)(-d) \longrightarrow H^r(X) \longrightarrow H^r(U) \longrightarrow H^{r+1-2d}(Z)(-d) \longrightarrow \cdots$$

By the induction hypothesis, the Hodge structures on $H^{r+1-2d}(Z)(-d)$ and on $H^r(X)$ are pure of weight $2dr/(2d-1)$ and hence it is also the case for $H^r(U)$ as desired. $\square$

REMARK 8.7. – This proposition is well-known for $d = 1$ and is proved for instance in [33]. Note that the space $F_m(\mathbb{C}^d)$ of configurations of m points in $\mathbb{C}^d$ fits in the above proposition, so we recover formality of these spaces by purely Hodge theory arguments.

8.2. Functoriality

Every morphism of smooth complex projective varieties is formal. However, if $f : X \to Y$ is an algebraic morphism of complex varieties (possibly singular and/or non-projective), and both X and Y are formal, the morphism f need not be formal.

EXAMPLE 8.8. – Consider the algebraic Hopf fibration $f : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbf{P}^1_{\mathbb{C}}$ defined by $(x_0, x_1) \mapsto [x_0 : x_1]$. Both spaces $\mathbb{C}^2 \setminus \{0\} \simeq S^3$ and $\mathbf{P}^1_{\mathbb{C}} \simeq S^2$ are formal. The morphism induced by f in cohomology is trivial in all positive degrees. Therefore, if f were formal, this would mean that f is nullhomotopic. However, it is well-known that f generates the one dimensional vector space $\pi_3(S^2) \otimes \mathbb{Q}$. Note in fact, that $\mathbf{P}^1_{\mathbb{C}}$ has 1-pure weight filtration while $\mathbb{C}^2 \setminus \{0\}$ has $4/3$-pure weight filtration.

Theorem 8.1 tells us that if $f : X \longrightarrow Y$ is a morphism of algebraic varieties and both X and Y have α-pure cohomology, with α a non-zero rational number (the same α for X and Y), then f is a formal morphism. This generalizes the formality of holomorphic morphisms between compact Kähler manifolds of [18] and enhances the results of [22] and [12] by providing them with functoriality. In fact, we have:

PROPOSITION 8.9. – *Let $f : X \longrightarrow Y$ be a morphism between connected complex varieties. Assume that the weight filtration on the cohomology of X (resp. Y) is α-pure (resp. β-pure). Then:*

1. *If $\alpha = \beta$, then f is formal.*
2. *If $\alpha \neq \beta$, then f is formal only if it is rationally nullhomotopic.*

Proof. – Let us first give the precise definition that we will use of a rationally nullhomotopic map. We say that a map $g : U \longrightarrow V$ between topological spaces is rationally nullhomotopic if the induced map

$$\mathscr{A}_{PL}(g) : \mathscr{A}^*_{PL}(V) \longrightarrow \mathscr{A}^*_{PL}(U)$$

is equal in the homotopy category of cdga's to a map that factors through the initial cdga $\mathbb{Q}$.

When $\alpha = \beta$, Theorem 8.1 ensures that f is formal.

If $\alpha \neq \beta$, then we claim that $H^*(f)$ is zero in positive degree. Indeed, since $H^*(f)$ is strictly compatible with the weight filtration, it suffices to show that the morphism

$$\mathrm{Gr}^W_p H^n(Y, \mathbb{Q}) \longrightarrow \mathrm{Gr}^W_p H^n(X, \mathbb{Q})$$

is trivial for all $p \in \mathbb{Z}$ and all $n > 0$ which follows from the purity conditions. Therefore, if f is formal, the map $\mathscr{A}^*_{PL}(f)$ coincides with $H^*(f)$ in the homotopy category of cdga's and the latter map factors through $\mathbb{Q}$. □

8.3. Non-projective singular varieties

The following result of formality of non-projective singular complex varieties with pure Hodge structure seems to be a new result.

EXAMPLE 8.10. – Let X be an irreducible singular projective variety of dimension $n > 0$ with 1-pure weight filtration in cohomology. Let $p \in X$ be a smooth point of X. Then, we claim that the complement $X - p$ has 1-pure weight filtration in cohomology. Indeed, we can consider the long exact sequence of cohomology groups for the pair $(X, X - p)$.

$$\cdots \to H^{i-1}(X-p) \to H^i(X, X-p) \to H^i(X) \to H^i(X-p) \to H^{i+1}(X, X-p) \to \cdots$$

Since p is a smooth point, there exists a neighborhood U of p that is homeomorphic to $\mathbb{R}^{2n}$, therefore excision gives us an isomorphism

$$H^k(X, X-p) \cong H^k(U, U-p).$$

Since $H^k(U, U-p)$ is non-zero only when $k = 2n$, we deduce that the map $H^k(X) \to H^k(X-p)$ is an isomorphism for all $k < 2n-1$. Moreover, since X is irreducible, we have $H^{2n}(X) = \mathbb{Q}$ and this vector space has a generator, the fundamental class, which is in the image of $H^{2n}(X, X-q) \to H^{2n}(X)$ for any smooth point q. Together with the above long exact sequence, this implies that $H^{2n-1}(X-p) \cong H^{2n-1}(X)$ and $H^{2n}(X-p) = 0$. To summarize, we have proved that the inclusion $X - p \to X$ induces an isomorphism on all cohomology groups except on the top one where $H^{2n}(X) = \mathbb{Q}$ while $H^{2n}(X-p) = 0$. This proves that the weight filtration of $X - p$ is 1-pure. As a consequence, the space $X - p$ is formal and the inclusion $X - p \hookrightarrow X$ is formal.

8.4. E_1-formality

We also have a contravariant version of Theorem 7.8.

THEOREM 8.11. – *Denote by* $\mathscr{A}^*_{\mathrm{fil}} : \mathbf{N}(\mathrm{Var}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{Ch}_*(\mathscr{F}\mathbb{Q})$ *the composite functor*

$$\mathbf{N}(\mathrm{Var}_{\mathbb{C}})^{\mathrm{op}} \xrightarrow{\mathscr{D}^*} \mathbf{MHC}_{\mathbb{Q}} \xrightarrow{\tilde{\Pi}^W_{\mathbb{Q}}} \mathbf{Ch}_*(\mathscr{F}\mathbb{Q}).$$

Then

1. *The lax symmetric monoidal* ∞*-functors* $\mathscr{A}^*_{\mathrm{fil}}$ *and* $E_1 \circ \mathscr{A}^*_{\mathrm{fil}}$ *are weakly equivalent.*
2. *Let* $U : \mathrm{Ch}_*(\mathscr{F}\mathbb{Q}) \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$ *denote the forgetful functor. The lax symmetric monoidal* ∞*-functor* $U \circ E_1 \circ \mathscr{A}^*_{\mathrm{fil}} : \mathbf{N}(\mathrm{Var}_{\mathbb{C}})^{\mathrm{op}} \to \mathbf{Ch}_*(\mathbb{Q})$ *is weakly equivalent to Sullivan's functor* $\mathscr{A}^*_{PL}$ *of piecewise linear forms.*
3. *The lax symmetric monoidal functor* $U \circ E_1 \circ \mathscr{A}^*_{\mathrm{fil}} : \mathrm{Sm}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{Ch}_*(\mathbb{Q})$ *is weakly equivalent to Sullivan's functor* $\mathscr{A}^*_{PL}$ *of piecewise linear forms.*

Proof. – The first part is proven as Theorem 7.8 replacing $\mathscr{D}^*$ by $\mathscr{D}_*$. The second part follows from the first part and the fact that $\mathscr{A}^*_{PL}(-)$ is naturally weakly equivalent to $\mathscr{D}^*(-)_{\mathbb{Q}} \simeq U \circ \mathscr{A}^*_{\mathrm{fil}}$ (Proposition 6.8). The third part follows from the second part and Theorem 2.3, using the fact that both functors are ordinary lax symmetric monoidal functors when restricted to smooth varieties. □

REMARK 8.12. – In [35] it is proven that the complex homotopy type of every smooth complex variety is E_1-formal. This is extended to possibly singular varieties and their morphisms in [13]. Then, a descent argument is used to prove that for nilpotent spaces (with finite type minimal models), this result descends to the rational homotopy type. Theorem 8.11 enhances the contents of [13] in two ways: first, since descent is done at the level of functors, we obtain E_1-formality over $\mathbb{Q}$ for any complex variety, without nilpotency conditions (the only property needed is finite type cohomology). Second, the functorial nature of our statement makes E_1-formality at the rational level, compatible with composition of morphisms.

8.5. Formality of Hopf cooperads

Our main theorem takes two dual forms, one covariant and one contravariant. The covariant theorem yields formality for algebraic structures (like monoids, operads, etc.), the contravariant theorem yields formality for coalgebraic structure (like the comonoid structure coming from the diagonal $X \to X \times X$ for any variety X). One might wonder if there is a way to do both at the same time. For example, if M is a topological monoid, then $H^*(M, \mathbb{Q})$ is a Hopf algebra where the multiplication comes from the diagonal of M and the comultiplication comes from the multiplication of M. One may ask whether $S^*(M, \mathbb{Q})$ is formal as a Hopf algebra. This question is not well-posed because $S^*(M, \mathbb{Q})$ is not a Hopf algebra on the nose. The problem is that there does not seem to exist a model for singular chains or cochains that is strong symmetric monoidal: the standard singular chain functor $S_*(-, \mathbb{Q})$ is lax symmetric monoidal and Sullivan's functor $\mathscr{A}^*_{PL}$ is oplax symmetric monoidal functor from Top to $\mathrm{Ch}_*(\mathbb{Q})^{\mathrm{op}}$.

Nevertheless, the functor $\mathcal{A}^*_{PL}$ is strong symmetric monoidal "up to homotopy". It follows that, if M is a topological monoid, $\mathcal{A}^*_{PL}(M)$ has the structure of a cdga with a comultiplication up to homotopy and it makes sense to ask if it has formality as such an object. In order to formulate this more precisely, we introduce the notion of an algebraic theory. The following is inspired by Section 3 of [34].

DEFINITION 8.13. – An algebraic theory is a small category T with finite products. For $\mathcal{C}$ a category with finite products, a T-algebra in $\mathcal{C}$ is a finite product preserving functor $T \longrightarrow \mathcal{C}$.

There exist algebraic theories for which the T-algebras are monoids, groups, rings, operads, cyclic operads, modular operads etc.

REMARK 8.14. – Definitions of algebraic theories in the literature are usually more restrictive. This definition will be sufficient for our purposes.

DEFINITION 8.15. – Let T be an algebraic theory. Let $\mathbb{k}$ be a field. Then a dg Hopf T-coalgebra over $\mathbb{k}$ is a finite coproduct preserving functor from T^{op} to the category of cdga's over $\mathbb{k}$.

REMARK 8.16. – Recall that the coproduct in the category of cdga's is the tensor product. It follows that a dg Hopf T-coalgebra for T the algebraic theory of monoids is a dg Hopf algebra whose multiplication is commutative. A dg Hopf T-coalgebra for T the theory of operads is what is usually called a dg Hopf cooperad in the literature.

DEFINITION 8.17. – Let T be an algebraic theory and $\mathcal{C}$ a category with products and with a notion of weak equivalences. A weak T-algebra in $\mathcal{C}$ is a functor $F : T \longrightarrow \mathcal{C}$ such that for each pair (s,t) of objects of T, the canonical map

$$F(t \times s) \longrightarrow F(t) \times F(s)$$

is a weak equivalence. A weak T-algebra in the opposite category of $\mathrm{CDGA}_{\mathbb{k}}$ is called a weak dg Hopf T-coalgebra.

Observe that if $X : T \longrightarrow \mathrm{Top}$ is a T-algebra in topological spaces (or even a weak T-algebra), then $\mathcal{A}^*_{PL}(X)$ is a weak dg Hopf T-coalgebra. Our main theorem for Hopf T-coalgebras is the following.

THEOREM 8.18. – *Let α be a rational number different from zero. Let $X : T \longrightarrow \mathrm{Var}_{\mathbb{C}}$ be a T-algebra such that for all $t \in T$, the weight filtration on the cohomology of $X(t)$ is α-pure. Then $\mathcal{A}^*_{PL}(X)$ is formal as a weak dg Hopf T-coalgebra.*

Proof. – Being a weak T-coalgebra is a property of a functor $T^{\mathrm{op}} \to \mathrm{CDGA}_{\mathbb{k}}$ that is invariant under quasi-isomorphism. Thus the result follows immediately from Theorem 8.1. □

It should be noted that knowing that $\mathscr{A}^*_{PL}(X)$ is formal as a dg Hopf T-coalgebra implies that the data of $H^*(X, \mathbb{Q})$ is enough to reconstruct X as a T-algebra up to rational equivalence. Indeed, recall the Sullivan spatial realization functor

$$\langle - \rangle : \mathrm{CDGA}_{\Bbbk} \longrightarrow \mathrm{Top}.$$

Applying this functor to a weak dg Hopf T-coalgebra yields a weak T-algebra in rational spaces. Specializing to $\mathscr{A}^*_{PL}(X)$ where X is a T-algebra in spaces, we get a rational model for X in the sense that the map

$$X \longrightarrow \langle \mathscr{A}^*_{PL}(X) \rangle$$

is a rational weak equivalence of weak T-algebras whose target is objectwise rational. It should also be noted that for reasonable algebraic theories T (including in particular the theory for monoids, commutative monoids, operads, cyclic operads), the homotopy theory of T-algebras in spaces is equivalent to that of weak T-algebras by the main theorem of [3]. In particular our weak T-algebra $\langle \mathscr{A}^*_{PL}(X) \rangle$ can be strictified to a strict T-algebra that models the rationalization of X. If $\mathscr{A}^*_{PL}(X)$ is formal, one also gets a rational model for X by applying the spatial realization to the strict Hopf T-coalgebra $H^*(X, \mathbb{Q})$. Thus the rational homotopy type of X as a T-algebra is a formal consequence of $H^*(X, \mathbb{Q})$ as a Hopf T-coalgebra.

Example 8.19. – Applying this theorem to the non-commutative little disks operad and framed little disks operad of subsection 7.1, we deduce that $\mathscr{A}^*_{PL}(\mathscr{A}s_{S^1})$ and $\mathscr{A}^*_{PL}(\mathscr{A}s_{S^1} \rtimes S^1)$ are formal as a weak Hopf non-symmetric cooperads. Similarly applying this to the monoid of self maps of the projective line of subsection 7.2, we deduce that $\mathscr{A}^*_{PL}(\bigsqcup_d F_d)$ is formal as a weak Hopf graded comonoid.

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