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Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Fax: (33) 01 40 46 90 96
asterisque@smf.emath.fr • http://smf.emath.fr/

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Jeffrey D. Adams
Department of Mathematics
University of Maryland
jda@math.umd.edu

Marc A. A. van Leeuwen
Laboratoire de Mathématiques et Applications
Université de Poitiers
Marc.van-Leeuwen@math.univ-poitiers.fr

Peter E. Trapa
Department of Mathematics
University of Utah
ptrapa@math.utah.edu

David A. Vogan, Jr.
Department of Mathematics
Massachusetts Institute of Technology
dav@math.mit.edu

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UNITARY REPRESENTATIONS OF REAL REDUCTIVE GROUPS

by Jeffrey D. ADAMS, Marc A. A. van LEEUWEN,
Peter E. TRAPA & David A. VOGAN, Jr.

Abstract. — We present an algorithm for computing the irreducible unitary representations of a real reductive group G. The Langlands classification, as formulated by Knapp and Zuckerman, exhibits any representation with an invariant Hermitian form as a deformation of a unitary representation from the Plancherel formula. The behavior of these deformations was in part determined in the Kazhdan-Lusztig analysis of irreducible characters; more complete information comes from the Beilinson-Bernstein proof of the Jantzen conjectures.

Our algorithm traces the signature of the form through this deformation, counting changes at reducibility points. An important tool is a variant of Weyl's "unitary trick": replacing the classical invariant Hermitian form (where $\mathrm{Lie}(G)$ acts by skew-adjoint operators) by a new one (where a compact form of $\mathrm{Lie}(G)$ acts by skew-adjoint operators).

Résumé. **(Représentations unitaires des groupes de Lie réductifs)** — Nous présentons un algorithme pour le calcul des représentations unitaires irréductibles d'un groupe de Lie réductif réel G. La classification de Langlands, dans sa formulation par Knapp et Zuckerman, présente toute représentation hermitienne comme étant la déformation d'une représentation unitaire intervenant dans la formule de Plancherel. Le comportement de ces déformations est en partie déterminé par l'analyse de Kazhdan-Lusztig des caractères irréductibles; une information plus complète provient de la preuve par Beilinson-Bernstein des conjectures de Jantzen.

Notre algorithme trace à travers cette déformation les changements de la signature de la forme qui peuvent intervenir aux points de réductibilité. Un outil important est une variante de "l'astuce unitaire" de Weyl: on remplace la forme hermitienne classique (pour laquelle $\mathrm{Lie}(G)$ agit par des opérateurs antisymétriques) par une forme hermitienne nouvelle (pour laquelle c'est une forme compacte de $\mathrm{Lie}(G)$ qui agit par des opérateurs antisymétriques).

CONTENTS

INDEX OF NOTATION

(A,ν)	continuous parameter; Definition 6.5.
${}^{\delta}G(\mathbb{C}), {}^{\delta}G, {}^{\delta}K$	extended groups; Definition 12.3.
$\widehat{G}_u$	unitary dual of G; (1.1a).
$\widehat{G}$	nonunitary dual of G; Definition 2.8.
$G(\mathbb{R},\sigma)$	real points of a real form of complex Lie group $G(\mathbb{C})$; (3.1d).
$\mathscr{G}(\mathfrak{h}, L(\mathbb{C}))^{\sigma}$	Grothendieck group of finite length admissible $(\mathfrak{h}, L(\mathbb{C}))$ modules with nondegenerate σ-invariant forms; Definition 15.5.
$\Gamma = (H, \gamma, R^+_{i\mathbb{R}})$	(continued) Langlands parameter; Definitions 6.2, 6.3.
$\Gamma_1 = ({}^1H, \gamma, R^+_{i\mathbb{R}})$	extended Langlands parameter; Definition 13.7.
$\Gamma^{h,\sigma_0}, \Gamma^{h,\sigma_c}$	Hermitian dual (or c-Hermitian dual) of Langlands parameter; Definitions 10.3, 10.6.
$H \simeq T \times A$	Cartan decomposition of real torus; Proposition 4.3.
${}^1H = \langle H, \delta_1 \rangle$	extended maximal torus; Definition 13.5.
$(\mathfrak{h}, {}^{\delta}L(\mathbb{C}))$	(extended) pair; Definitions 8.1, 8.12.
θ	Cartan involution; (3.2a).
Θ_π	distribution character of π; Definition 5.11.
$I_{\mathrm{quo}}(\Gamma), I_{\mathrm{sub}}(\Gamma)$	standard quotient-type and sub-type modules; Theorem 9.2.
$\ell(\Gamma)$	integral length of Γ; Definition 18.1.
$\ell_o(\Gamma)$	orientation number of Γ; Definition 20.5.
$\Lambda = (T, \lambda, R^+_{i\mathbb{R}})$	discrete Langlands parameter; Definition 6.5.
$m_{\Xi,\Gamma}, M_{\Gamma,\Psi}$	entries of multiplicity matrices for standard modules (or character formulas for irreducible modules); (15.1).
$(\mathrm{pos}_V, \mathrm{neg}_V, \mathrm{rad}_V)$	signature character of σ-invariant Hermitian form on admissible $(\mathfrak{h}, L(\mathbb{C}))$ module V; Proposition 8.9(5).
$P_{\Gamma,\Psi}$	character polynomial; Definition 18.8.
$P^c_{\Gamma,\Psi}$	signature character polynomial; Definition 20.4.
$Q_{\Gamma,\Psi}$	multiplicity polynomial; Definition 18.9.
$Q^c_{\Gamma,\Psi}$	signature multiplicity polynomial; Definition 20.2.

$R_{\mathbb{R}}, R_{i\mathbb{R}}, R_{\mathbb{C}}$	real, imaginary, and complex roots; Definition 5.7.
$2\rho_{\mathrm{abs}}$	character of H; Lemma 5.9.
σ	real form of complex algebraic variety or group; (2.10), (3.1c).
σ_c	compact real form of complex reductive group; Theorem 2.11.
$(\pi^{h,\sigma}, V^{h,\sigma})$	σ-Hermitian dual of $(\mathfrak{h}, L(\mathbb{C}))$-module V; (8.5).
$\mathbb{W}$	signature ring; Definition 15.8.
W^{Λ}	stabilizer of Λ in real Weyl group; Proposition 6.6(1).
$W(G,H)$	real Weyl group; Definition 5.7, Proposition 5.8(4).
${}^{\delta}W(G,H)$	extended real Weyl group; Definition 13.5.
$X(\mathbb{R},\sigma)$	real points for real form σ of variety $X(\mathbb{C})$; (2.10).

CHAPTER 1

FIRST INTRODUCTION

The purpose of this paper is to give a finite algorithm for computing the set of irreducible unitary representations of a real reductive Lie group G. Before explaining the nature of the algorithm, it is worth recalling why this is an interesting question. A serious historical survey would go back at least to the work of Fourier (which can be understood in terms of the irreducible unitary representations of the circle).

Since we are not serious historians, we will begin instead with a formulation of "abstract harmonic analysis" arising from the work of Gelfand beginning in the 1930s. In Gelfand's formulation, one begins with a topological group G acting on a topological space X. A reasonable example to keep in mind is $G = GL(n, \mathbb{R})$ acting on the space X of lattices in $\mathbb{R}^n$. What makes such spaces difficult to study is that there is little scope for using algebra.

The first step in Gelfand's program is therefore to find a nice Hilbert space $\mathscr{H}$ (often of functions on X) on which G acts by unitary operators:

$$\pi \colon G \to U(\mathscr{H}).$$

For example, if G preserves a measure on X, one can take

$$\mathscr{H} = L^2(X).$$

Such a group homomorphism (assumed to be continuous, in the sense that the map

$$G \times \mathscr{H} \to \mathscr{H}, \qquad (g, v) \mapsto \pi(g)v$$

is continuous) is called a *unitary representation of* G. Gelfand's program says that questions about the action of G on X should be recast as questions about the unitary representation of G on $\mathscr{H}$, where one can bring to bear tools of linear algebra.

One of the most powerful tools of linear algebra is the theory of eigenvalues and eigenvectors, which allow some problems about linear transformations of a complex vector space to be understood by decomposing the space as a direct sum of one-dimensional invariant subspaces, in each of which the transformation is just multiplication by a complex number. These one-dimensional subspaces preserved by the linear transformation are those spanned by an eigenvector of the transformation, and the associated complex number its eigenvalue. In unitary representation theory the analogue of such a one-dimensional space is an *irreducible unitary representation*: a nonzero unitary representation having no proper closed subspaces invariant under $\pi(G)$. Just as a finite-dimensional complex vector space equipped with a (nice enough) linear transformation is a direct sum of subspaces spanned by eigenvectors, so any (nice enough) unitary representation is something like a direct sum of irreducible unitary sub-representations.

The assumption that we are looking at a *unitary* representation avoids the difficulties (like nilpotent matrices) attached to eigenvalue decompositions in the finite-dimensional case; but allowing infinite-dimensional Hilbert spaces introduces complications of other kinds. First, one must allow not only direct sums but also "direct integrals" of irreducible representations. This complication appears already in the case of the action of $\mathbb{R}$ on $L^2(\mathbb{R})$ by translation. The decomposition into one-dimensional irreducible representations is accomplished by the Fourier transform, and so involves integrals rather than sums.

For general groups there are more serious difficulties, described by von Neumann's theory of "types". But one of Harish-Chandra's fundamental theorems ([**14**, Theorem 7]) is that real reductive Lie groups are "type I," and therefore that any unitary representation of a real reductive group may be written uniquely as a direct integral of irreducible unitary representations. The second step in Gelfand's program is to recast questions about the (reducible) unitary representation π into questions about the irreducible representations into which it is decomposed.

The third step in Gelfand's program is to describe all of the irreducible unitary representations of G. This is the problem of "finding the unitary dual"

(1.1a) $$\widehat{G}_u =_{\text{def}} \{\text{equiv. classes of irreducible unitary representations of } G\}.$$

It is this problem for which we offer a solution (for real reductive G) in this paper. It is far from a completely satisfactory solution for Gelfand's program; for of course what Gelfand's program asks is that one should be able to answer interesting questions about all irreducible unitary representations. (Then these answers can be assembled into answers to the questions about the reducible representation π, and finally translated into answers to the original questions about the topological space X on which G acts.) We offer not a list of unitary representations but a method to calculate the list. To answer general questions about unitary representations in this way, one would need to study how the questions interact with our algorithm.

Which is to say that we may continue to write papers after this one.

Here is an outline of the algorithm for identifying unitary group representations. We will use a number of ideas to be introduced and explained only later. What we will actually describe is an algorithm for computing the signature of an invariant Hermitian form on any irreducible representation that admits one. The algorithm for calculating signatures of invariant Hermitian forms in the analogous setting of highest weight modules is treated by Yee in [**48, 49**]. We follow her ideas closely.

The Langlands classification Theorem 6.1 realizes each irreducible representation $\overline{\pi}_1$ of G as the unique irreducible quotient of an induced representation π_1. This induced representation is part of an analytic one-parameter family $\{\pi_t \mid t \in [0,\infty)\}$ of representations. (One simply replaces the real part ν_{Re} of the continuous parameter (cf. Proposition 4.3) by $t\nu_{\mathrm{Re}}$.) If $\overline{\pi}_1$ is tempered (and therefore unitary), then $\nu_{\mathrm{Re}} = 0$ and $\pi_t = \pi_1 = \overline{\pi}_1$ is a constant family. In all cases π_0 is tempered, and therefore unitary. The representation π_t is irreducible except for a discrete set of values of t.

In case $\overline{\pi}_1$ admits an invariant Hermitian form $\langle\, ,\rangle_1$, we can extend it to an analytic one-parameter family $\langle\, ,\rangle_t$ of invariant Hermitian forms for the representations π_t (Proposition 14.2). This result of Knapp and Stein is one of the earliest general tools for studying non-tempered unitary representations. Because of the analyticity, the signature of the form $\langle\, ,\rangle_t$ is *locally constant* on the open set where π_t is irreducible. Because π_0 is unitary, the form $\langle\, ,\rangle_0$ is definite.

Our algorithm calculates the signature of $\langle\, ,\rangle_t$ beginning with the definiteness at $t = 0$, and then calculating the *change* in the signature at each reducibility point $t' \in [0, 1]$. This idea, taken from **[46]**, is Corollary 14.8. The answer is described in terms of forms on subquotients of the *Jantzen filtration* of $\pi_{t'}$ (defined in (14.3)).

We have therefore described the signature of the invariant form $\langle\, ,\rangle_1$ in terms of various signatures of forms on subquotients π' of $\pi_{t'}$, for $t' \leq 1$. The reason this leads to an effective algorithm is that (as is evident from Langlands' arguments in **[29]**) the real part $\mathrm{Re}\,\nu'$ of the continuous parameter attached to π' satisfies

$$\|\mathrm{Re}\,\nu'\| < \|\mathrm{Re}\,\nu\|.$$

In order to carry out this computation, we first need to identify explicitly the irreducible representations π' appearing in the various levels of the Jantzen filtrations. This is done by the Jantzen Conjecture (Theorem 18.11, proved by Beilinson and Bernstein in **[5]**). The answer is phrased in terms of the Kazhdan-Lusztig polynomials defined and calculated in **[31]**.

All of these ideas were in place in the 1980s. The great difficulty is that each irreducible representation π' with an invariant Hermitian form actually has *two* inequivalent forms, one the negative of the other; and there is no way to specify a preferred form. (A convincing example of this phenomenon is provided by the two-dimensional irreducible representation $\overline{\pi}_1$ of $G = SU(1, 1)$. There is an invariant Hermitian form of signature $(1, 1)$ on $\overline{\pi}_1$. This form and its negative are exchanged by the action of the outer automorphism group of G.)

When π' appears in a Jantzen filtration (and therefore carries a form defined by Jantzen, related to our orginal $\langle\, ,\rangle_1$ by analytic continuation) we must decide which of the two forms on π' is the Jantzen form. This is the content of Theorem 20.6.

The main idea in the proof of Theorem 20.6 is evident in its statement: the theorem refers to Hermitian forms preserved not by the real Lie algebra $\mathfrak{g}_0$, but rather by a compact real form of $\mathfrak{g}_0$. We call these forms *c-invariant forms*. Their existence (which is quite easy) is Proposition 10.7. The great advantage of c-invariant forms is that they are automatically definite on the lowest K-types (see Proposition 10.7(5)). Consequently there is a natural choice of c-invariant form, the one which is *positive* on all the lowest K-types.

Of course we are ultimately interested in ordinary invariant Hermitian forms rather than c-invariant forms, so we need to be able to translate signature calculations from one to the other. This is done by Theorem 12.9. The translation depends on extending the representation to include an action of the Cartan involution. The Cartan involution is inner exactly when $\mathrm{rank}(G) = \mathrm{rank}(K)$; this setting has been called the *equal rank case* since the work of Harish-Chandra on discrete series representations in the 1960s. In the equal rank case, the translation of signatures between invariant and c-invariant forms is elementary (Theorem 11.2).

When the Cartan involution is *not* inner (the *unequal rank case*), we need to extend the representations we study to include an action of the Cartan involution. Formally this is a straightforward version of "Clifford theory"; the result is Theorem 13.15. But we also need to extend the theory of Kazhdan-Lusztig polynomials to these extended groups, and this is a serious matter. (The outer automorphisms act on the perverse sheaves calculated in [**31**], and we need to know the $+1$ and -1 eigenspaces of this action on each cohomology stalk.) The mathematical ideas are in [**32**], and the tracking of signs is done in [**2**]. One formulation of the answer is Corollary 21.4.

We offer more details about the algorithm in Chapter 7, after recalling results of Harish-Chandra, Langlands, and others in terms of which it is formulated.

Chapter 22 carries out the algorithm for the complex reductive group $SL(2, \mathbb{C})$, computing all invariant Hermitian forms and in particular identifying the unitary representations (first described in [**13**]).

The algorithm described in this paper has been implemented in the `atlas` software package [**11**]. There it has been tested on thousands of known unitary (and non-unitary) representations of groups of rank up to about eight. The software in its present form is *not* fast enough to determine the unitarity of an arbitrary representation of our favorite example, the split real form of E_8.

One general abuse of notation: we will deal with an enormous number of automorphisms σ (of many kinds of mathematical objects) with the property that σ^2 is the identity. We will express this property by saying that *σ has order two*, even though it is more precise to say that "σ has order one or two".

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We are deeply grateful to George Lusztig for producing the mathematics in [**32**], without which we would have been unable to complete this work.

Finally, we thank Wai Ling Yee, whose work first showed us that an analysis of unitary representations along these lines might be possible; indeed we hoped to persuade her to be a coauthor.