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CHARACTERISATION OF THE POLES OF THE ℓ-MODULAR ASAI L-FACTOR

by Robert Kurinczuk & Nadir Matringe

Abstract. — Let F/F_{o} be a quadratic extension of non-archimedean local fields of odd residual characteristic, set $G = GL_n(F)$, $G_{\mathsf{o}} = GL_n(F_{\mathsf{o}})$ and let ℓ be a prime number different from the residual characteristic of F. For a complex cuspidal representation π of G, the Asai L-factor $L_{As}(X, \pi)$ has a pole at $X = 1$, if and only if π is G_{o}-distinguished. In this paper, we solve the problem of characterising the occurrence of a pole at $X = 1$ of $L_{As}(X, \pi)$ when π is an ℓ-modular cuspidal representation of G; we show that $L_{As}(X, \pi)$ has a pole at $X = 1$, if and only if π is a *relatively banal* distinguished representation, namely π is G_{o}-distinguished but not $|\det(\ )|_{F_{\mathsf{o}}}$-distinguished. This notion turns out to be an exact analogue for the symmetric space G/G_{o} of Mínguez and Sécherre's notion of banal cuspidal $\overline{\mathbb{F}_\ell}$-representation of G_{o}. Along the way, we compute the Asai L-factor of all cuspidal ℓ-modular representations of G in terms of type theory and prove new results concerning lifting and reduction modulo ℓ of distinguished cuspidal representations. Finally, we determine when the natural G_{o}-period on the Whittaker model of a distinguished cuspidal representation of G is non-zero.

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Résumé (*Caractérisation des pôles du facteur d'Asai ℓ-modulaire*). — Soit F/F_o une extension quadratique de corps locaux non archimédiens de caractéristique résiduelle impaire. Posons $G = GL_n(F)$, $G_o = GL_n(F_o)$ et soit ℓ un nombre premier différent de la caractéristique résiduelle de F. Pour une représentation cuspidale complexe π de G, le facteur L d'Asai $L_{As}(X, \pi)$ admet un pôle en $X = 1$ si et seulement si π est G_o-distinguée. Dans cet article nous résolvons le problème de l'occurence d'un pôle en $X = 1$ de $L_{As}(X, \pi)$ quand π est une représentation cuspidale ℓ-modulaire de G: dans ce cas $L_{As}(X, \pi)$ admet un pôle en $X = 1$ si et seulement si π est *relativement banale* distinguée; autrement dit π est G_o-distinguée mais pas $|\det(\ )|_{F_o}$-distinguée. Cette notion est l'analogue pour l'espace symétrique G/G_o de la notion de cuspidale banale introduite par Mínguez et Sécherre pour les $\overline{\mathbb{F}_\ell}$-représentations de G_o. En cours de route, on calcule le facteur L d'Asai des représentations cuspidales ℓ-modulaires de G par la théorie des types, et on prouve de nouveaux résultats concernant le relèvement et la réduction modulo ℓ des représentations cuspidales distinguées. Finalement, on détermine quand la G_o-période sur le modèle de Whittaker d'une représentation cuspidale distinguée de G est non nulle.

1. Introduction

Let F/F_o be a quadratic extension of non-archimedean local fields of residual characteristic $p \neq 2$ and set $G = GL_n(F)$ and $G_o = GL_n(F_o)$. An irreducible representation of G is said to be *distinguished by* G_o, if it possesses a non-zero G_o-invariant linear form. In the case of complex representations, the equality of the Asai L-factor defined by the Rankin–Selberg method and its Galois avatar ([3], [21]) provides a bridge between functorial lifting from the quasi-split unitary group $U_n(F/F_o)$ and G_o-distinction of discrete series representations of G; a discrete series of G is a (stable or unstable depending on the parity of n) lift of a (necessarily discrete series) representation of $U_n(F/F_o)$, if and only if the Asai L-factor of its Galois parameter has a pole at $X = 1$ ([25], [10]), whereas it is G_o-distinguished, if and only if its Asai L-factor obtained by the Rankin–Selberg method has a pole at $X = 1$ ([14], [1]).

Recently, motivated by the study of congruences between automorphic representations, there has been great interest in studying representations of G on vector spaces over fields of positive characteristic ℓ. There are two very different cases: when $\ell = p$ and when $\ell \neq p$. This article focuses on the latter $\ell \neq p$ case, where there is a theory of Haar measure that allows us to define Asai L-factors via the Rankin–Selberg method as in the complex case (Section 7).

The aim of this article is to show that in this case, a connection remains between the poles of Asai L-factor and distinction; however, this connection no longer characterises distinction, but a more subtle notion, which we call a *relatively banal* distinction. The easiest way to state that a cuspidal distinguished ℓ-modular representation is relatively banal is to say that it is not $|\det(\,\cdot\,)|_{F_o}$-distinguished, where $|\det(\,\cdot\,)|_{F_o}$ is considered as an $\overline{\mathbb{F}_\ell}$-valued character, but other compact definitions can also be given in terms of type theory, as well as in terms of its supercuspidal lifts:

PROPOSITION 1.1 (Definition 6.2, Theorem 6.11 and Corollary 6.3). — *Let π be an ℓ-modular cuspidal distinguished representation of* G*. Then, the following are equivalent, and when they are satisfied, we say that π is* relatively banal*:*

(i) *π is not $|\det(\cdot)|_{F_\circ}$-distinguished.*
(ii) *All supercuspidal lifts of π are distinguished by an unramified character of* $G_\circ$*.*
(iii) $q_\circ^{n/e_\circ(\pi)} \not\equiv 1[\ell]$*, where $e_\circ(\pi)$ denotes the invariant associated to π in* [4, Lemma 5.10] *(see Section 5.2).*

Relatively banal for $G_\circ$-distinguished cuspidal representations turns out to be the exact analogue of the definition of *banal cuspidal representations* of $G_\circ$ (see [24, Remark 8.15] and [23]) after one identifies the cuspidal (irreducible) representations of $G_\circ$ with the $\Delta(G_\circ)$-distinguished cuspidal (irreducible) representations of $G_\circ \times G_\circ$, where $\Delta : G_\circ \to G_\circ \times G_\circ$ is the diagonal embedding, as we explain in Section 8.3.

The main theorem of this paper characterises the poles of the Asai L-factor:

THEOREM 1.2 (Theorem 8.1). — *Let π be a cuspidal ℓ-modular representation, then $L_{As}(X, \pi)$ has a pole at $X = 1$, if and only if π is relatively banal distinguished.*

Note that the proof of the above theorem is completely different from the proof of characterisation theorem in the complex case (see Remark 8.2 for more on the comparison of the proofs). Here, we show that the Asai L-factor of a cuspidal ℓ-modular representation is equal to 1 whenever π is not the unramified twist of a relatively banal representation using Theorem 6.11, which is the characterisation of relatively banal in terms of supercuspidal lifts. Then, when π is the unramified twist of relatively banal representation, following our paper [18], we get an explicit formula for $L_{As}(X, \pi)$ in Theorem 7.8 from the test vector computation of [4], which due to the relatively banal assumption (more precisely its type theory version) reduces modulo ℓ without vanishing. We then deduce Theorem 1.2 from this computation, together with the computation of the group of unramified characters μ of $G_\circ$, such that π is μ-distinguished (Corollary 5.17).

Finally, denoting by N the unipotent radical of the group of upper triangular matrices in G, by $Z_\circ$ the centre of $G_\circ$ and by $N_\circ$ the group $N \cap G_\circ$, the most natural $G_\circ$-invariant linear form to consider on the Whittaker model of an ℓ-modular cuspidal representation π with respect to a distinguished non-degenerate character of N trivial on $N_\circ$ is the local period

$$\mathcal{L}_\pi : W \mapsto \int_{Z_\circ N_\circ \backslash G_\circ} W(h)dh.$$

In fact, this period plays an essential role in the proof of Theorem 1.2 over the field of complex numbers (see Remark 8.2). One of the main differences in the