ISSN 0012-9593

quatrième série - tome 53 fascicule 5 septembre-octobre 2020

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Diletta MARTINELLI & Stefan SCHREIEDER & Luca TASIN

On the number and boundedness of log minimal models of general type

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 428 euros.
Abonnement avec supplément papier :
Europe : 576 €. Hors Europe : 657 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 53, 2020, p. 1183 à 1207

ON THE NUMBER AND BOUNDEDNESS OF LOG MINIMAL MODELS OF GENERAL TYPE

BY Diletta MARTINELLI, Stefan SCHREIEDER AND Luca TASIN

Abstract. – We show that the number of marked minimal models of an n-dimensional smooth complex projective variety of general type can be bounded in terms of its volume, and, if $n = 3$, also in terms of its Betti numbers. For an n-dimensional projective klt pair (X, Δ) with $K_X + \Delta$ big, we show more generally that the number of its weak log canonical models can be bounded in terms of the coefficients of Δ and the volume of $K_X + \Delta$. We further show that all n-dimensional projective klt pairs (X, Δ), such that $K_X + \Delta$ is big and nef of fixed volume and such that the coefficients of Δ are contained in a given DCC set, form a bounded family. It follows that in any dimension, minimal models of general type and bounded volume form a bounded family.

Résumé. – On montre que le nombre de modèles minimaux marqués sur une variété projective n-dimensionelle complexe et lisse de type général est borné en termes de son volume, et dans le cas $n = 3$ il est aussi borné en termes de ses nombres de Betti. Plus en général, soit (X, Δ) une paire projective klt avec $K_X + \Delta$ gros. On prouve que le nombre de modèles canoniques faibles peut être borné en termes du coefficient de Δ et du volume de $K_X + \Delta$. On montre aussi que toutes les paires projectives klt n-dimensionelles (X, Δ) telles que $K_X + \Delta$ est nef et gros et de volume fixe, et telles que les coefficients de Δ sont contenus dans un ensemble DCC donné, forment une famille bornée. Il s'ensuit que, en toute dimension, les modèles minimaux de type général et de volume borné forment une famille bornée.

1. Introduction

1.1. Number of minimal models

It is well known that starting from dimension three, a minimal model of a smooth complex projective variety X is in general not unique. Nevertheless, if X is of general type, even the number of marked minimal models of X is finite [2, 16]; that is, up to isomorphism, there are only finitely many pairs (Y, ϕ), where $\phi: X \dashrightarrow Y$ is a birational map and Y is a minimal model, cf. Section 2.2 below. Such a finiteness statement fails if X is not of general type [24, Example 6.8]. However, it is conjectured that the number of minimal models Y of X is always finite up to isomorphism; this is known for threefolds of positive Kodaira dimension [14].

doi:10.24033/asens.2443

In this paper we study the number of marked minimal models in families. In particular, we show that the corresponding function on any moduli space of complex projective varieties of general type is constructible; that is, it is constant on the strata of some stratification by locally closed subsets.

THEOREM 1. – *Let $\pi\colon \mathcal{X} \longrightarrow B$ be a family of complex projective varieties such that the resolution of each fiber is of general type. Then the function $f\colon B \longrightarrow \mathbb{N}$, which associates to $b \in B$ the number of marked minimal models of the fiber X_b, is constructible in the Zariski topology of B.*

In contrast to the above theorem, recall that the Picard number is in general not a constructible function on the base of families of varieties of general type.

Since smooth complex projective varieties of general type, of given dimension and bounded volume form a birationally bounded family [9, 26, 27], Theorem 1 implies the following.

COROLLARY 2. – *Let $n \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$. Then there is a positive constant $N(c)$, such that for any n-dimensional smooth complex projective variety X of general type and volume $\mathrm{vol}(X) \leq c$, the number of marked minimal models of X is at most $N(c)$.*

One of our original motivations for Theorem 1 (resp. Corollary 2) stems from [3], where Cascini and Lazić proved that the number of log minimal models of a certain class of three-dimensional log smooth pairs (X, Δ) of general type can be bounded by a constant that depends only on the homeomorphism type of the pair (X, Δ). This has its roots in earlier results that show that the topology governs the birational geometry to some extent, see for instance [5] and [17]. Motivated by their result, Cascini and Lazić [3] conjectured that the number of minimal models of a smooth complex projective threefold of general type is bounded in terms of the underlying topological space. As an immediate consequence of Corollary 2 and [4, Theorem 1.2], we solve this conjecture.

COROLLARY 3. – *The number of marked minimal models of a smooth complex projective threefold of general type can be bounded in terms of its Betti numbers.*

1.2. The case of klt pairs

A weak log canonical model of a projective klt pair (X, Δ) is a $(K_X + \Delta)$-non-positive birational contraction $f\colon (X, \Delta) \dashrightarrow (Y, \Gamma = f_*\Delta)$, such that (Y, Γ) is klt and $K_Y + \Gamma$ is nef. If $K_X + \Delta$ is big, then there are only finitely many such models by [2]. If $\Delta = 0$ and X is smooth (or terminal), then any marked minimal model of X is also a weak log canonical model in the above sense. The converse is not true, because weak log canonical models are not assumed to be $\mathbb{Q}$-factorial; in particular, the number of weak log canonical models is in general larger than the number of marked minimal models. Theorem 1 generalizes to families of klt pairs as follows.

THEOREM 4. – *Let $\pi\colon (\mathcal{X}, \Delta) \longrightarrow B$ be a projective family of klt pairs (X_b, Δ_b) with $K_{X_b} + \Delta_b$ big. Then the function $f\colon B \longrightarrow \mathbb{N}$, which associates to $b \in B$ the number of weak log canonical models of (X_b, Δ_b), is constructible in the Zariski topology of B.*

Theorem 4 remains true if we count only those weak log canonical models that are $\mathbb{Q}$-factorial, see Remark 32. Using the log birational boundedness result from [12] (cf. Theorem 11 below), Theorem 4 implies the following generalization of Corollary 2.

COROLLARY 5. – *The number of weak log canonical models of a projective klt pair* (X, Δ) *with* $K_X + \Delta$ *big, is bounded in terms of the dimension of* X*, the coefficients of* Δ *and the volume of* $K_X + \Delta$.

1.3. Boundedness of log minimal models of general type

Let $\mathfrak{F}$ be a collection of projective pairs (X, Δ). We recall that the pairs $(X, \Delta) \in \mathfrak{F}$ form a bounded family (or that $\mathfrak{F}$ is bounded), if there is a complex projective family of pairs $\pi: (\mathcal{X}, \Delta) \longrightarrow B$ over a scheme B of finite type, whose fibers belong to $\mathfrak{F}$ and such that any element of $\mathfrak{F}$ is isomorphic to some fiber of π. We call π a parametrizing family of $\mathfrak{F}$.

Hacon, McKernan and Xu proved the boundedness of the set $\mathfrak{F}_{slc}$ of all semi log canonical pairs (X, D), where X has given dimension, the coefficients of D belong to a DCC set $I \subset [0, 1]$, $K_X + D$ is ample and $(K_X + D)^n$ is fixed, see [10, 11, 12]. Here the DCC condition on I means that any non-increasing sequence in I becomes stationary at some point; this holds in particular for any finite set $I \subset [0, 1]$.

As a consequence of our study of (log) minimal models in families, we obtain the following partial generalization of that result. While we require the pair (X, D) to be klt, we relax the condition on $K_X + D$ to be only big and nef; our result relies on the boundedness theorem from [12]. The two-dimensional case goes back to Alexeev [1].

THEOREM 6. – *Let* n *be a natural number,* c *a positive rational number and* $I \subset [0, 1) \cap \mathbb{Q}$ *be a DCC set. Consider the set* $\mathfrak{F}$ *of all klt pairs* (X, D) *such that*

1. X *is a projective variety of dimension* n*,*
2. *the coefficients of* D *belong to* I*,*
3. $K_X + D$ *is big and nef,*
4. $(K_X + D)^n = c$.

Then the pairs $(X, D) \in \mathfrak{F}$ *form a bounded family. Moreover, the parametrizing family can be chosen as a disjoint union* $(\mathcal{X}^Q, D^Q) \sqcup (\mathcal{X}^{\neq Q}, D^{\neq Q})$*, where* $(\mathcal{X}^Q, D^Q)$ *is* $\mathbb{Q}$*-factorial and parametrizes exactly the* $\mathbb{Q}$*-factorial members of* $\mathfrak{F}$.

By Theorem 6, the subset $\mathfrak{F}^Q \subset \mathfrak{F}$ of $\mathbb{Q}$-factorial pairs is also bounded. We do not deduce this directly from the boundedness of $\mathfrak{F}$, but use some refined arguments to reduce to the fact that $\mathbb{Q}$-factoriality is an open condition for families of terminal varieties [22].

Theorem 6 has several interesting corollaries, which we collect next.

As it is for instance explained by Hacon and Kovács in [8, p. 9], the boundedness result for canonical models of surfaces implies that in dimension two, minimal models of general type and bounded volume form also a bounded family. That argument fails in higher dimensions, because it uses in an essential way that minimal models of surfaces are unique. The following corollary settles the boundedness question for minimal models of general type in arbitrary dimensions.