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A VARIATIONAL PROOF OF PARTIAL REGULARITY FOR OPTIMAL TRANSPORTATION MAPS

BY Michael GOLDMAN AND Felix OTTO

ABSTRACT. – We provide a new proof of the known partial regularity result for the optimal transportation map (Brenier map) between two Hölder continuous densities. Contrary to the existing regularity theory for the Monge-Ampère equation, which is based on the maximum principle, our approach is purely variational. By constructing a competitor on the level of the Eulerian (Benamou-Brenier) formulation, we show that locally, the velocity is close to the gradient of a harmonic function provided the transportation cost is small. We then translate back to the Lagrangian description and perform a Campanato iteration to obtain an ε-regularity result.

RÉSUMÉ. – Nous donnons une nouvelle preuve d'un résultat connu concernant la régularité partielle des applications de transport optimal (application de Brenier) dans le cas du transport entre deux densités Hölder continues. À l'inverse de la théorie existante pour la régularité de l'équation de Monge-Ampère, basée sur le principe du maximum, notre preuve est purement variationnelle. En construisant un compétiteur pour la formulation eulerienne (Benamou-Brenier), nous montrons que localement, le champ de vitesses est proche du gradient d'une fonction harmonique si l'énergie de transport est assez petite. En traduisant cela dans la formulation lagrangienne, nous obtenons un résultat d'ε-régularité à travers un schéma itératif à la Campanato.

1. Introduction

For $\alpha \in (0,1)$, let ρ_0 and ρ_1 be two probability densities with bounded support which are $C^{0,\alpha}$ continuous, bounded and bounded away from zero on their support and let T be the solution of the optimal transportation problem

$$\min_{T\sharp\rho_0=\rho_1} \int_{\mathbb{R}^d} |T(x)-x|^2 \rho_0(x)dx, \tag{1.1}$$

where with a slight abuse of notation $T\sharp\rho_0$ denotes the push-forward by T of the measure $\rho_0 dx$ (existence and characterization of T as the gradient of a convex function ψ are given by Brenier's Theorem, see [21, Th. 2.12]). Our main result is a partial regularity theorem for T:

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THEOREM 1.1. – *There exist open sets $E \subseteq \operatorname{spt}\rho_0$ and $F \subseteq \operatorname{spt}\rho_1$ of full measure such that T is a $C^{1,\alpha}$-diffeomorphism between E and F.*

This theorem is a consequence of Alexandrov Theorem [22, Th. 14.25] and the following ε-regularity theorem:

THEOREM 1.2. – *Let T be the minimizer of* (1.1) *and assume that* $\rho_0(0) = \rho_1(0) = 1$. *There exists $\varepsilon(\alpha, d)$ such that if*[(1)]

$$\frac{1}{(2R)^{d+2}}\int_{B_{2R}} |T-x|^2\rho_0 dx + R^{2\alpha}[\rho_0]^2_{\alpha,2R} + R^{2\alpha}[\rho_1]^2_{\alpha,2R} \le \varepsilon,$$

then, T is $C^{1,\alpha}$ inside B_R.

Theorem 1.1 was already obtained by Figalli [11] in the case of planar transportation between sets and is a slightly weaker version of a result obtained by Figalli and Kim [12] (see also [9, 15] for a far-reaching generalization), but our proof departs from the usual scheme for proving regularity for the Monge-Ampère equation. Indeed, while most proofs use some variants of the maximum principle, our proof is variational. The classical approach operates on the level of the convex potential ψ and the ground-breaking paper in that respect is Caffarelli's [5]: by comparison with simple barriers it is shown that an Alexandrov (and thus viscosity) solution ψ of the Monge-Ampère equation is C^1, provided its convexity does not degenerate along a line crossing the entire domain of definition. The same author shows in [6] by similar arguments that the potential ψ of the Brenier map is a strictly convex Alexandrov solution, and thus regular, provided the target domain $\operatorname{spt}\rho_1$ is convex. The challenge in [12] is to follow the above line of arguments while avoiding the notion of Alexandrov solution, more precisely, without having access to the Upper Alexandrov estimate. The ε-regularity theorem in [12] in turn is used by Figalli and De Philippis as the core for a generalization to general cost functions by means of a Campanato iteration. On the contrary to these papers, we work directly at the level of the optimal transportation map T, and besides the L^∞ bound (see (3.11)) given by McCann's displacement convexity, we only use variational arguments.

Let us now give an outline of the proof. As in many ε-regularity results, it goes through a Campanato iteration (see Proposition 3.7). This scheme relies on two ingredients. The first is an 'improvement of flatness by tilting', see Proposition 3.6. This means that if the energy in a given ball is small then, up to a change of coordinates, the energy has a geometric decay on a smaller scale. The second ingredient is the invariance of the variational problem under affine transformations.

The main idea behind the proof of the one-step improvement Proposition 3.6 is the well-known fact that the linearization of the Monge-Ampère equation gives rise to the Laplace equation [21, Sec. 7.6]. In Proposition 3.5, we make this statement more quantitative and prove that if the energy in a given ball is small enough, then in the half-sized ball and up to an error which is super-linear in the energy, T is equal to the gradient of a harmonic function. Once we have this approximation result, using that by classical elliptic regularity, harmonic functions are close to their second-order Taylor expansion, we obtain Proposition 3.6.

The core of our proof is thus Proposition 3.5, which is actually established at the Eulerian

(1) Here $[\rho]_{\alpha,R} := \sup_{x,y\in B_R} \frac{|\rho(x)-\rho(y)|}{|x-y|^\alpha}$ denotes the $C^{0,\alpha}$-semi-norm.

level (i.e., for the solutions of the Benamou-Brenier formulation of optimal transportation, see [21, Th. 8.1] or [3, Chap. 8]), see Proposition 3.3. It is for this result that we need the outcome of McCann's displacement convexity, cf. Lemma 3.2, since it is required for the quasi-orthogonality property (see Step 3 of the proof of Proposition 3.3). Our argument is variational and proceeds by defining a competitor based on the solution of a Poisson equation with suitable flux boundary conditions, and a boundary-layer construction. The boundary-layer construction is carried out in Lemma 2.4; by a duality argument it reduces to a trace estimate (see Lemma 2.3). This part of the proof is reminiscent of arguments from [1].

This entire approach to ε-regularity is guided by De Giorgi's strategy for minimal surfaces (see [16] for instance). Let us notice that because of the natural scaling of the problem, our Campanato iteration operates directly at the $C^{1,\alpha}$-level for T, as opposed to [12, 9], where $C^{0,\alpha}$-regularity is obtained first.

Motivated by applications to the optimal matching problem, we extended in [14] together with M. Huesmann, Proposition 3.5 to arbitrary target measures.

The plan of the paper is the following. In Section 2, we recall some well-known facts about the Poisson equation and then prove Lemma 2.4, the proof of which is based on the trace estimate given by Lemma 2.3. In the final section, we prove Theorem 1.2 and then Theorem 1.1.

Since it simplifies some of the technicalities, we suggest at first reading to consider the simpler case of transportation between sets i.e., $\rho_0 = \chi_E$ and $\rho_1 = \chi_F$ for some sets E and F. A previous version of this paper treating that case is available on our webpages.

Notation

In the paper we will use the following notation. The symbols $\sim$, $\gtrsim$, $\lesssim$ indicate estimates that hold up to a global constant C, which only depends on the dimension d and the Hölder exponent α (if applicable). For instance, $f \lesssim g$ means that there exists such a constant with $f \le Cg$, $f \sim g$ means $f \lesssim g$ and $g \lesssim f$. An assumption of the form $f \ll 1$ means that there exists $\varepsilon > 0$, only depending on the dimension and the Hölder exponent, such that if $f \le \varepsilon$, then the conclusion holds. We write $|E|$ for the Lebesgue measure of a set E. Inclusions will always be understood as holding up to a set of Lebesgue measure zero, that is for two sets E and F, $E \subseteq F$ means that $|E \backslash F| = 0$. When no confusion is possible, we will drop the integration measures in the integrals. For $R > 0$ and $x_0 \in \mathbb{R}^d$, $B_R(x_0)$ denotes the ball of radius R centered in x_0. When $x_0 = 0$, we will simply write B_R for $B_R(0)$. We will also use the notation

$$\fint_{B_R} f := \frac{1}{|B_R|} \int_{B_R} f.$$

For a function ρ defined on a ball B_R we introduce the Hölder semi-norm of exponent $\alpha \in (0, 1)$

$$[\rho]_{\alpha,R} := \sup_{x \neq y \in B_R} \frac{|\rho(x) - \rho(y)|}{|x - y|^\alpha}.$$