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Introduction

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INTRODUCTION

by

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1. Uniformly hyperbolic dynamical systems

The theory of *uniformly hyperbolic dynamical systems* was constructed in the 1960's under the dual leadership of Smale in the USA and Anosov and Sinai in the Soviet Union. It is nowadays almost complete. It encompasses various examples [**35**]: expanding maps, horseshoes, solenoid maps, Plykin attractors, Anosov maps and DA, all of which are *basic pieces*.

We recall standard definitions. Let f be a C^1-diffeomorphism f of a finite dimensional manifold M. A compact f-invariant subset $\Lambda \subset M$ is *uniformly hyperbolic* if the restriction to Λ of the tangent bundle TM splits into two continuous invariant subbundles

$$TM|\Lambda = E^s \oplus E^u,$$

E^s being uniformly contracted and E^u being uniformly expanded.

Then for every $z \in \Lambda$, the sets

$$W^s(z) = \{z' \in M : \lim_{n\to+\infty} d(f^n(z), f^n(z')) = 0\},$$
$$W^u(z) = \{z' \in M : \lim_{n\to-\infty} d(f^n(z), f^n(z')) = 0\}$$

are called *the stable and unstable manifolds* of z. They are immersed manifolds tangent at z to respectively $E^s(z)$ and $E^u(z)$.

The *ϵ-local stable manifold* $W^s_\epsilon(z)$ of z is the connected component of z in the intersection of $W^s(z)$ with a ϵ-neighborhood of z. The *ϵ-local unstable manifold* $W^u_\epsilon(z)$ is defined likewise.

Definition 1.1. — A *basic set* is a compact, f-invariant, uniformly hyperbolic set Λ which is transitive and *locally maximal*: there exists a neighborhood N of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(N)$. A basic set is an *attractor* if the neighborhood N can be chosen in such a way that $\Lambda = \bigcap_{n\geq 0} f^n(N)$. Such a basic set contains the unstable manifolds of its points.

A diffeomorphism whose nonwandering set is a finite union of disjoint basic sets is called *uniformly hyperbolic* or *Axiom A*.

Such diffeomorphisms enjoy nice properties, which are proved in [**35**] and the references therein.

SRB and physical measure. — Let $\alpha > 0$, and let Λ be an attracting basic set for a $C^{1+\alpha}$-diffeomorphism f. Then there exists a unique invariant, ergodic probability μ supported on Λ such that its conditional measures, with respect to any measurable partition of Λ into plaques of unstable manifolds, are absolutely continuous with respect to the Lebesgue measure class (on unstable manifolds). Such a probability is called *SRB* (for Sinai-Ruelle-Bowen). It turns out that a SRB -measure is *physical*: the Lebesgue measure of its basin $B(\mu)$

$$(\mathscr{B}) \qquad B(\mu) = \{z \in M : \frac{1}{n}\sum_{i<n} \delta_{f^i(x)} \rightharpoonup \mu\}$$

is positive. Actually, up to a set of Lebesgue measure 0, $B(\mu)$ is equal to the topological basin of Λ, i.e the set of points attracted by Λ.

Persistence. — A basic set Λ for a C^1-diffeomorphism f is *persistent*: every C^1-perturbation f' of f leaves invariant a basic set Λ' which is homeomorphic to Λ, via a homeomorphism which conjugates the dynamics $f|\Lambda$ and $f'|\Lambda'$.

Coding. — A basic set Λ for a C^1-diffeomorphism f admits a (finite) Markov partition. This implies that its dynamics is semi-conjugated with a subshift of finite type. The semi-conjugacy is 1-1 on a generic set. Its lack of injectivity is itself coded by subshifts of finite type of smaller topological entropy. This enables to study efficiently all the invariant measures of Λ, the distribution of its periodic points, the existence and uniqueness of the maximal entropy measure, and if f is $C^{1+\alpha}$, the Gibbs measures which are related to the geometry of Λ.

1.1. End of Smale's program. — Smale wished to prove the density of Axiom A in the space of C^r-diffeomorphisms. In higher dimensions, obstructions were soon discovered by Shub [**33**]. For surfaces Newhouse showed the non-density of Axiom A diffeomorphisms for $r \geq 2$: he constructed robust tangencies between stable and unstable manifolds of a thick horseshoe [**26**]. Numerical studies by Lorenz [**18**] and Hénon [**14**] explored dynamical systems with hyperbolic features that did not fit in the uniformly hyperbolic theory. In order to include many examples such as the Hénon one, the *non-uniform hyperbolic theory* is still under construction.

2. Non-uniformly hyperbolic dynamical systems

2.1. Pesin theory. — The natural setting for non-uniform hyperbolicity is Pesin theory [**2, 17**], from which we recall some basic concepts. We first consider the simpler settings of invertible dynamics.

Let f be a $C^{1+\alpha}$-diffeomorphism (for some $\alpha > 0$) of a compact manifold M and let μ be an ergodic f-invariant probability measure on M. The Oseledets multiplicative ergodic theorem produces Lyapunov exponents (w.r.t. μ) for the tangent cocycle of f, and an associated μ-a.e f-invariant splitting of the tangent bundle into characteristic subbundles.

Denote by $E^s(z)$ (resp. $E^u(z)$) the sum of the characteristic subspaces associated to the negative (resp. positive) Lyapunov exponents.

The *stable and unstable Pesin manifolds* are defined respectively for μ-a.e. z by

$$W^s(z) = \{z' \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n(z), f^n(z')) < 0\},$$
$$W^u(z) = \{z' \in M : \liminf_{n\to-\infty} \frac{1}{n} \log d(f^n(z), f^n(z')) > 0\}.$$

They are immersed manifolds through z tangent respectively at z to $E^s(z)$ and $E^u(z)$.

The measure μ is *hyperbolic* if 0 is not a Lyapunov exponent w.r.t. μ. Every invariant ergodic measure, which is supported on a uniformly hyperbolic compact invariant set, is hyperbolic.

SRB, physical measures. — An invariant ergodic measure μ is *SRB* if the largest Lyapunov exponent is positive and the conditional measures of μ w.r.t. a measurable partition into plaques of unstable manifolds are μ-a.s. absolutely continuous w.r.t. the Lebesgue class (on unstable manifolds). When μ is SRB and hyperbolic, it is also *physical*: its basin has positive Lebesgue measure.

The paper [**44**] provides a general setting where appropriate hyperbolicity hypotheses allow to construct hyperbolic SRB measures with nice statistical properties.

Coding. — Let μ be a f-invariant ergodic hyperbolic SRB measure. Then there is a partition mod 0 of M into finitely many disjoint subsets $\Lambda_1, \ldots, \Lambda_k$, which are cyclically permuted by f and such that the restriction $f^k_{|\Lambda_1}$ is metrically conjugated to a Bernoulli automorphism.

Of a rather different flavor is Sarig's recent work [**32**]. For a $C^{1+\alpha}$-diffeomorphism of a compact surface of positive topological entropy and any $\chi > 0$, he constructs a countable Markov partition for an invariant set which has full measure w.r.t. any ergodic invariant measure with metric entropy $> \chi$. The semi-conjugacy associated to this Markov partition is finite-to-one.

Non-invertible dynamics. — One should distinguish between the non-uniformly expanding case and the case of general endomorphisms.

In the first setting, a SRB measure is simply an ergodic invariant measure whose Lyapunov exponents are all positive and which is absolutely continuous.

Defining appropriately unstable manifolds and SRB measures for general endomorphisms is more delicate. One has typically to introduce the inverse limit where the endomorphism becomes invertible.

2.2. Case studies. — The paradigmatic examples in low dimension can be summarized by the following table:

Uniformly hyperbolic	Non-uniformly hyperbolic
Expanding maps of the circle	Jakobson's Theorem
Conformal expanding maps of complex tori	Rees' Theorem
Attractors (Solenoid, DA, Plykin...)	Benedicks-Carleson's Theorem
Horseshoes	Non-uniformly hyperbolic horseshoes
Anosov diffeomorphisms	Standard map ?

Let us recall what these theorems state, and the correspondence given by the lines of the table.

Expanding maps of the circle may be considered as the simplest case of uniformly hyperbolic dynamics. The Chebychev quadratic polynomial $P_{-2}(x) := x^2 - 2$ on the invariant interval $[-2, 2]$ has a critical point at 0, but it is still semi-conjugated to the doubling map $\theta \mapsto 2\theta$ on the circle (through $x = 2\cos 2\pi\theta$). For $a \in [-2, -1]$, the quadratic polynomial $P_a(x) := x^2 + a$ leaves invariant the interval $[P_a(0), P_a^2(0)]$ which contains the critical point 0.

***Theorem 2.1* (Jakobson [16]).** — *There exists a set $\Lambda \subset [-2, -1]$ of positive Lebesgue measure such that for every $a \in \Lambda$ the map $P(x) = x^2 + a$ leaves invariant an ergodic, hyperbolic measure which is equivalent to the Lebesgue measure on $[P_a(0), P_a^2(0)]$.*

Actually the set Λ is nowhere dense. Indeed the set of $a \in \mathbb{R}$ such that P_a is Axiom A is open and dense **[13, 20]**.

Let L be a lattice in $\mathbb{C}$ and let c be a complex number such that $|c| > 1$ and $cL \subset L$. Then the homothety $z \mapsto cz$ induces an expanding map of the complex torus $\mathbb{C}/L$. The Weierstrass function associated to the lattice L defines a ramified covering of degree 2 from $\mathbb{C}/L$ onto the Riemann sphere which is a semi-conjugacy from this expanding map to a rational map of degree $|c|^2$ called a *Lattes map*. For any $d \geq 2$, the set Rat_d of rational maps of degree d is naturally parametrized by an open subset of $\mathbb{P}(\mathbb{C}^{2d+2})$.

***Theorem 2.2* (Rees [29]).** — *For every $d \geq 2$, there exists a subset $\Lambda \subset \mathrm{Rat}_d$ of positive Lebesgue measure such that every map $R \in \Lambda$ leaves invariant an ergodic hyperbolic probability measure which is equivalent to the Lebesgue measure on the Riemann sphere.*